§21. Diffusion Phenomena in a Self-Similar Structure with Islands

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The subject in this report is to consider diffusion phenomena in a self-similar structure with magnetic islands for the collisionless limit. In general, when an equilibrium magnetic field has a symmetry, the equations for a magnetic field line can be written as Hamilton's equations with an integrable Hamiltonian. [1] If the magnetic field is perturbed by a three-dimensional fluctuating magnetic field, then magnetic islands appear in the equilibrium. And in the statistical meaning, the structure with islands has a self-similarity, which is a generic feature in the structure described by a Hamiltonian with relatively large perturbation [2]. For example, the island structure can be found in the edge region of a stellarator equilibrium. While, recently the high temperature divertor plasma operation is considered to improve the energy confinement of helical devices. [3] In the operation, the divertor temperature is expected to become as high as several keV. [3] In the edge region of a helical system, the collisionless-plasma can be realized by this operation. Therefore, the diffusion in a self-similar structure with islands for the collisionless limit will be the realistic issue.

The random motion in a fractal medium seems to be very anomalous, but we can analytically understand these properties by using the representation of random walk in fractal space-time [4]. This analytical method is based on the idea that from the particle's viewpoint, space-time seems to have non-integral dimensions, because the random particles are restricted to move on only fractal structure except for islands. According to Ref.[4], the diffusion theory of Rechester and Rosenbluth [5] is developed to the situation with a self-similar structure with islands for the collisionless limit. We use as an example, which is same as one in Ref.[5], a magnetic configuration in cylindrical geometry;

$$\mathbf{B} = \mathbf{B}_z + \mathbf{B}_\theta(r) + \delta\mathbf{B}(r,\theta,z). \tag{1}$$

We assume that the system is periodic in the z direction with period $2\pi R$ in order to model toroidal periodicity and both of the rotational transform $\iota(r)/2\pi$ and the shear $\mathrm{d}(\iota/2\pi)/\mathrm{d}r$ are small. Then the fluctuating magnetic field $\delta\mathbf{B}$ can be written as

$$\delta\mathbf{B} = \sum_{m,n} \delta\mathbf{B}_{m,n}(r) \exp[\, \mathrm{i}(m\theta - n\zeta)\,] + \text{c.c.}, \tag{2}$$

where $\zeta = z/R$. Here, ζ means 'time', because of $\mathbf{B}_z \neq 0$ and $|\mathbf{B}_z| > |\delta\mathbf{B}_z|$ in this model. According to Ref.[4], the random motion in a fractal structure except for islands can be understood as the Brownian motion in fractal space-time $(\tilde{r},\tilde{\zeta})$, where $(\tilde{r},\tilde{\zeta})$ are defined by the Hausdorff length;

$$\tilde{r} \; = \; \lim_{\rho\to 0} H_\rho^\alpha(\text{fractal-space}), \tag{3}$$

$$\tilde{\zeta} \; = \; \lim_{\rho\to 0} H_\rho^\beta(\text{fractal-time}). \tag{4}$$

Here, α and β are the fractal dimensions of space and time, respectively. $H_\rho^\ell(X)$ is the length of the set X divided by N parts $\{X_i\}$ and is given by

$$H_\rho^\ell(X) = \inf\left\{ \sum_{i=1}^{N} {d_i}^\ell \,\middle|\, 0 < d_i \leq \rho, X \subseteq \bigcup_{i=1}^{N} X_i \right\}, \tag{5}$$

where ℓ is the Hausdorff dimension of X, and d_i is a diameter of the i-th part X_i and is measured in real space-time (r,ζ). The diffusion coefficient D_0 in fractal space-time is given as

$$D_0 = \frac{(2\pi)^{2\alpha-\beta}}{2} \left\{ R^2 \sum_{m,n} \left| \frac{\delta B_{m,n}(r)}{B_z} \right|^2 \delta_{nq(r)-m} \right\}^\alpha, \tag{6}$$

where $q(r) = 2\pi/\iota(r)$ is the safety factor. The distribution function in real space-time is given by using the path integral method; [4]

$$f(\Delta r,\zeta) = \frac{1}{C} \exp\left\{ -\frac{(\Delta r)^{2\alpha}}{4D_0\zeta^\beta} \right\}, \tag{7}$$

where $C = \{\Gamma(1/2\alpha)/\alpha\}\left(4D_0\zeta^\beta\right)^{1/2\alpha}$ is the normalizing factor; $\int_{-\infty}^{+\infty} \mathrm{d}(\Delta r)\; f(\Delta r,\zeta) \equiv 1$, and Γ is the gamma function. Thus, the mean square displacement in real space-time is given by

$$\left\langle (\Delta r)^2 \right\rangle = 2D\hat{\lambda}^{\beta/\alpha}, \tag{8}$$

where $\hat{\lambda} = \lambda/R$ and λ is the collisional mean free path in the z direction, and the diffusion coefficient in real space-time, D, is described as

$$D \; = \; \frac{2^{(1+\alpha^2-\beta)/\alpha}\pi^{2-\beta/\alpha}\Gamma(3/2\alpha)}{\Gamma(1/2\alpha)} \times R^2 \sum_{m,n} \left| \frac{\delta B_{m,n}(r)}{B_z} \right|^2 \delta_{nq(r)-m}. \tag{9}$$

References

[1] Hazeltine,R.D. and Meiss,J.D. : *Plasma Confinement* (Addison-Wesley, California, 1992) Chap.9.

[2] Lichtenberg,A.J. and Liberman,M.A. : *Regular and Chaotic Dynamics* (Springer-Verlag, New York, 1992) Chap.3.

[3] Ohyabu,N. et al. : Nucl. Fusion **34** (1994) 387.

[4] Kanno,R. : Physica A **248** (1998) 165.

[5] Rechester,A.B. and Rosenbluth,M.N. : Phys. Rev. Lett. **40** (1978) 38.