## §35. Derivation of the Fractal Path Integral

Kanno, R., Nakajima, N.

By means of the method introduced in ref.[1], we derive a path integral constituted by fractal principal paths. Consider the following definition of a path $x(\tau)$;

$$x(\tau) \equiv \lim_{M\to+\infty} x^{(M)}(\tau) = \lim_{M\to+\infty}\left[x_a + (t_b - t_a)\int_0^\tau d\tau'\ v^{(M)}(\tau')\right] \quad (1)$$

where $\tau \in [0,1]$. The velocity $v^{(M)}(\tau)$ is given by

$$v^{(M)}(\tau) \equiv \frac{x_b - x_a}{t_b - t_a} + \left\{\frac{\pi D}{t_b - t_a}\right\}^{1/2} \times \sum_{n=0}^{M} \sum_{k=0:k\in\{k_p(n)\}}^{q^n-1} C_{n,k}\ Z_{n,k}(q^n\tau), \quad (2)$$

where $C_{n,k}$ is the expansion coefficient. The summation $\sum_{k:k\in\{k_p(n)\}}$ means the selective sum of k from the set $\{k_p(n)\}$ which is defined by

$$\{k_p(n)\} \equiv \{k_p|\text{select the number of } p^n \text{ from a set } \{m|m = [q^n\tau_*]_G \text{ for } \forall\tau_* \in [0,1]\}\}. \quad (3)$$

Here $q$ and $p$ are any natural numbers that should satisfy the condition $1/2 \le \log_q p \le 1$. The selection law is arbitrary. The function $Z_{n,k}(q^n\tau)$ is any orthonormal function, which satisfies the following three conditions;

$$Z_{n,k}(q^n\tau) = \begin{cases} \text{a function of } \tau & \text{for } q^n\tau \in [k, k+1] \\ 0 & \text{for the others,} \end{cases} \quad (4)$$

$$\int_0^1 d\tau\ Z_{n,k}(q^n\tau) = 0, \quad (5)$$

$$\int_0^1 d\tau\ Z_{n,k}(q^n\tau)\cdot Z_{i,j}(q^i\tau) = \begin{cases} 1 & \text{for } n=i \text{ and } k=j \\ 0 & \text{for the others.} \end{cases} \quad (6)$$

Here a function given by

$$G_{n,k}(q^n\tau) = \int_0^\tau d\tau'\ Z_{n,k}(q^n\tau') \qquad \text{for any } \tau \quad (7)$$

represents any generator producing a single-valued continuous fractal curve, and the initiator $I$ is given as $I = x_a + (x_b - x_a)\tau$.

Suppose that the contribution of a path $x(t)$ to the path integral is proportional to $\exp\{-\hat{S}[x(t)]/D\}$, where the function $\hat{S}[x(t)]$ is the modified action defined as $\hat{S}[x(t)] \equiv \{T/(t_b - t_a)\}^{\beta-1}S[x(t)]$, $T$ is the characteristic time, and $\beta$ is a real number:$0 \le \beta \le 2$.

By summing over all paths given by eq.(1), the path integral is derived as follows

$$f(b,a) = \lim_{M\to+\infty}\left\{\frac{1}{\pi DT}\left(\frac{T}{t_b - t_a}\right)^\beta\right\}^{1/2} \times \exp\left\{-\frac{(x_b - x_a)^2}{DT}\left(\frac{T}{t_b - t_a}\right)^\beta\right\} \times \prod_{n=0}^{M}\prod_{k=0:k\in\{k_p(n)\}}^{q^n-1}\int_{-\infty}^{+\infty} d\hat{C}_{n,k} \times \exp\left\{-\pi(\hat{C}_{n,k})^2\right\}, \quad (8)$$

where $\prod_{k:k\in\{k_p(n)\}}$ means the product of $k$ from the set $\{k_p(n)\}$, and $\hat{C}_{n,k} = \{T/(t_b - t_a)\}^{(\beta-1)/2}C_{n,k}$. This path integral is constituted by the only fractal curve produced by a generator $G_{n,k}(q^n\tau)$, so we call the integral (8) the fractal path integral (FPI).

We define here the average of arbitrary functional $F[x(\tau)]$ as follows

$$< F >= \lim_{M\to+\infty}\prod_{n=0}^{M}\prod_{k=0:k\in\{k_p(n)\}}^{q^n-1} \int_{-\infty}^{+\infty} d\hat{C}_{n,k}\ F[\{\hat{C}_{n,k}\}]\ \exp\{-\pi(\hat{C}_{n,k})^2\}. \quad (9)$$

Using this definition of the average of functional, one can investigate the property of principal paths of any FPI.

References

1) Kanno,R. and Ishida,A., J.Phys.Soc.Jpn63(1994)2902.