§29. Diffusion on a Self-similar Structure

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We analyze diffusion phenomena on a self-similar structure with fractal in the time-axis. The behavior of this motion shows the non-Gaussian-type. This behavior can be seen numerically, for example, for the field-line transport in stochastic magnetic fields between magnetic islands[1]. Previous analytical studies attempt to treat the motion in real space-time with integral dimensions. To clarify the relation between the non-Gaussian-type behavior and the fractal nature, we choose a different starting point. We notice that the random particle is able to move on only fractal medium with fractal in the time-axis. This means that from the particle's viewpoint, space-time seems to have non-integral dimensions, *i.e.* we can say that this motion is the random walk in fractal space-time.

It is well known that in real space-time, a fractal structure has infinite or zero length. To measure a finite and non-zero length of any self-similar set X, usually the Hausdorff length H^ℓ is used;

$$H^\ell(X) = \lim_{\rho\to 0} H^\ell_\rho(X) = \text{constant for only } \ell, \tag{1}$$

where ℓ is a real-number and describes the Hausdorff dimension of the set X, and $H^\ell_\rho(X)$ is a length of X divided by N parts $\{X_i\}$ and is given by

$$H^\ell_\rho(X) = \inf\left\{ \sum_{i=1}^{N} {d_i}^\ell \,\middle|\, 0 < d_i \le \rho, X \subseteq \bigcup_{i=1}^{N} X_i \right\}. \tag{2}$$

Here d_i is a diameter of the i-th part X_i and is measured in real-space with integral dimensions. By using the Hausdorff length, arbitrary fractal-space and fractal-time can be measured. Now, we assume that these Hausdorff lengths can be used to specify coordinates $(\tilde{x}, \tilde{t})$, and derivatives and integrals of a function on these coordinates are acceptable. Fractal space-time $(\tilde{x}, \tilde{t})$ is defined by these Hausdorff lengths.

$$\tilde{x} \equiv H^\alpha(x), \qquad \tilde{t} \equiv H^\beta(t). \tag{3}$$

Here, space-time with $\alpha = 1$ and $\beta = 1$ means real space-time.

By following the path integral method developed by Feynman[2], we carry out integrations and finally obtain a result as follows.

$$f(b,a) = \left\{ \frac{1}{4\pi D(\tilde{t}_b - \tilde{t}_a)} \right\}^{1/2} \exp\left\{ -\frac{(\tilde{x}_b - \tilde{x}_a)^2}{4D(\tilde{t}_b - \tilde{t}_a)} \right\}. \tag{4}$$

In real space-time, the transition probability f can be seen to be proportional to $\exp\left\{-z^{2\alpha}/4D\tau^\beta\right\}$ under the expectation that important contributions to the integral of f will occur only for small z and τ, where $z = \tilde{z}^{1/\alpha}$ and $\tau = \tilde{\tau}^{1/\beta}$ are measured in real space-time. Thus, by re-normalizing the probability in real-space, the equation (4) can be rewritten as follows.

$$f(z,\tau) = \frac{1}{B} \exp\left\{ -\frac{z^{2\alpha}}{4D\tau^\beta} \right\}, \tag{5}$$

where B is the normalizing factor. Using eq.(5), we can calculate $R(\tau) \equiv \overline{z^2}$ and $G \equiv \overline{z^4}/(3\overline{z^2}^2) - 1$;

$$R(\tau) = \frac{(4D)^{1/\alpha}\Gamma\left(\frac{3}{2\alpha}\right)}{\Gamma\left(\frac{1}{2\alpha}\right)} \tau^{\beta/\alpha}, \tag{6}$$

$$G = \frac{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\frac{5}{2\alpha}\right)}{3\left[\Gamma\left(\frac{3}{2\alpha}\right)\right]^2} - 1, \tag{7}$$

where Γ is the gamma function. In real space-time, the random walk shows the non-Gaussian process, if $G \neq 0$, *i.e.* $\alpha \neq 1$. On the other hand, for a case of $\beta \neq 1$ and $\alpha = 1$, the process represents the fractional Brownian process. Thus, we have the unified representation of the Brownian process ($\alpha = 1$, $\beta = 1$), the fractional Brownian process ($\alpha = 1$, $\beta \neq 1$), and the non-Gaussian process with fractal nature ($\alpha \neq 1$).

References
1) Zimbardo,G. and Veltri,P., Phys. Rev. E 51(1995)1412.
2) Feynman,R.P., Rev. Mod. Phys. 20(1948)367.