§14. A Spectral Method in Spherical Coordinates with Coordinate Singularity at the Origin

Kageyama, A., Kida, S.

In a spherical coordinate system (r,θ,ϕ), special cares should be taken not to degrade numerical accuracy and efficiency which might originate from coordinate singularities along the axis $(\theta = 0, \pi)$ and at the origin $r = 0$. Although the coordinate singularity along the axis has been studied extensively so far, there are a very few literatures on that at the origin. The purpose of the present work is to provide a new spectral method in spherical coordinates including the origin.

The difficulty in spectral methods with a coordinate singularity relates to an analytical property to be satisfied by infinitely differentiable solutions near the singularity. This is called the pole condition. It is proved that any analytical function $f(r,\theta,\phi)$ is expanded around the origin as

$$f(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} r^l F_{lm}(r) Y_{lm}(\theta,\phi), \qquad (1)$$

where $F_{lm}(r)$ is an even function of r,

$$F_{lm}(-r) = F_{lm}(r), \qquad (2)$$

and

$$|\,F_{lm}(0)\,| < \infty. \qquad (3)$$

Equations (1)–(3) are the pole condition at the origin in the spherical coordinate system when it is expanded in terms of the spherical harmonics in the (θ,ϕ) space [1].

Our spectral method is constructed on the basis of the pole condition described above. Suppose that a function $f(r,\theta,\phi)$ is governed by a differential equation defined in a sphere $0 \le r \le 1$, $0 \le \theta \le \pi$, and $0 \le \phi < 2\pi$. We expand $f(r,\theta,\phi)$ as (1) and the coefficient $F_{lm}(r)$ in terms of Chebyshev polynomials of even order as,

$$F_{lm}(r) = \sum_{n=0}^{N} F_{lmn}\, T_{2n}(r), \qquad (4)$$

which is justified by the property of (2) and $T_{2n}(-r) = T_{2n}(r)$. Here, N is the truncation mode number. Note that the problem of unnecessarily refined resolution near the origin has been avoided automatically in this expansion. By choosing such radial node points as

$$r_j = \cos\left(\frac{j\pi}{2N}\right), \quad j = 0,1,2,\ldots,N, \qquad (5)$$

we can invoke the fast Fourier cosine transformation to calculate the summation in (4).

When the problem to be solved is quadratically nonlinear, a care should be taken in dealing with the nonlinear terms. Suppose a nonlinear term of

$$h(r,\theta,\phi) = f(r,\theta,\phi)\; g(r,\theta,\phi). \qquad (6)$$

According to our algorithm, we expand f, g, and h as

$$\begin{aligned} f(r,\theta,\phi) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} r^l\, F_{lm}(r)\, Y_{lm}(\theta,\phi) \\ &= \sum_{l=0}^{\infty} r^l\, \widetilde{F}_l(r,\theta,\phi), \end{aligned} \qquad (7)$$

$$\begin{aligned} g(r,\theta,\phi) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} r^l\, G_{lm}(r)\, Y_{lm}(\theta,\phi) \\ &= \sum_{l=0}^{\infty} r^l\, \widetilde{G}_l(r,\theta,\phi), \end{aligned} \qquad (8)$$

$$\begin{aligned} h(r,\theta,\phi) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} r^l\, H_{lm}(r)\, Y_{lm}(\theta,\phi) \\ &= \sum_{l=0}^{\infty} r^l\, \widetilde{H}_l(r,\theta,\phi). \end{aligned} \qquad (9)$$

Then, what is required in our method is to calculate $H_{lm}(r)$ from $F_{lm}(r)$ and $G_{lm}(r)$. Notice that numerical errors would be amplified if a spherical expansion of $h(r,\theta,\phi)$ were divided by r^l to obtain $H_{lm}(r)$, since r^l could be too small when l is large and r is small. This problem is resolved if $H_{lm}(r)$ is calculated from $\widetilde{F}_l(r,\theta,\phi)$ and $\widetilde{G}_l(r,\theta,\phi)$. An accurate and fast algorithm to calculate $H_{lm}(r)$ using the fast Fourier transform can be found in [1].

We have applied the present algorithm to a free decay of magnetic field in a sphere to check its validity and accuracy. Consider an electrically conducting solid sphere of radius a with a finite electrical resistivity. A magnetic field is given at an initial time $t = 0$ with an arbitrary distribution. Since the type of boundary condition is not important in the present algorithm, the outer region of the sphere $(r > a)$ is supposed to be a perfect insulator, or a vacuum for simplicity. It is physically evident that the magnetic field decays with time due to the finite resistivity. An analytical expression of decaying magnetic field is provided in [2]. High-accuracy of this method is confirmed [1].

References

1) A. Kageyama and S. Kida, NIFS Report No. 636

2) H.K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids*, (Cambridge University Press, London, 1978), pp. 36-40