§35. Symmetry of Reversal of Swirling Flow Based on Variational Approach with the Aid of Helicity

Itoh, S.-I. (Kyushu U.),
Itoh, K.,
Yoshizawa, A., Yokoi, N., Nishizima, S. (Univ. Tokyo)

Based on the variational approach, the reversal of the swirling flow has been described by the equation

$$\boldsymbol{\Omega}=\frac{1}{\lambda}\nabla\times\boldsymbol{\Omega}\ , \qquad (1)$$

where $\boldsymbol{\Omega}$ is the vorticity of the flow. An axisymmetric solution was obtained, and a reversal of axial flow was demonstrated.
In order to relax the axisymmetric condition, we write

$$\boldsymbol{U}=\left(U_r(r,\theta),\ U_\theta(r,\theta),\ U_z(r,\theta)\right)\ , \qquad (2)$$

$$\boldsymbol{\Omega}=\left(\Omega_r(r,\theta),\Omega_\theta(r,\theta),\Omega_z(r,\theta)\right)\ . \qquad (3)$$

From the vorticity equation (1), we have a general solution as

$$\Omega_z=\Omega_z^{(C)}J_0(\lambda r)+\sum_{n=1}^{\infty}\left(\Omega_{zn}^{(c)}\cos n\theta+\Omega_{zn}^{(s)}\sin n\theta\right)J_n(\lambda r) \qquad (4)$$

$$\Omega_\theta=-\Omega_z^{(C)}J_1(\lambda r)+\frac{1}{\lambda}\sum_{n=1}^{\infty}\left(\Omega_{zn}^{(c)}\cos n\theta+\Omega_{zn}^{(s)}\sin n\theta\right)\frac{d}{dr}J_n(\lambda r) \qquad (5)$$

$$\Omega_r=\frac{1}{\lambda}\sum_{n=1}^{\infty}\left(n\Omega_{zn}^{(c)}\sin n\theta-n\Omega_{zn}^{(s)}\cos n\theta\right)\frac{J_n(\lambda r)}{r} \qquad (6)$$

where $\Omega_{zn}^{(c)}$ and $\Omega_{zn}^{(s)}$ are numerical coefficients.
The flow velocity $\boldsymbol{U}$ may be expressed as

$$U_z=U^{(N)}-\frac{1}{\lambda}\Omega_z^{(C)}J_0(\lambda r)-\frac{1}{\lambda}\sum_{n=1}^{\infty}\left(\Omega_{zn}^{(c)}\cos n\theta+\Omega_{zn}^{(s)}\sin n\theta\right)J_n(\lambda r) \qquad (7)$$

$$U_\theta=\frac{\Omega_z^{(C)}}{\lambda}J_1(\lambda r)-\frac{1}{\lambda^2}\sum_{n=1}^{\infty}\left(\Omega_{zn}^{(c)}\cos n\theta+\Omega_{zn}^{(s)}\sin n\theta\right)\frac{d}{dr}J_n(\lambda r) \qquad (8)$$

$$U_r=-\frac{1}{\lambda^2}\sum_{n=1}^{\infty}\left(n\Omega_{zn}^{(c)}\sin n\theta-n\Omega_{zn}^{(s)}\cos n\theta\right)\frac{J_n(\lambda r)}{r} \qquad (9)$$

In order to simply see effects of nonaxisymmetric components, we retain the lowest-order component as

$$\Omega_{zn}^{(c)}=\Omega_{z1}^{(c)}\delta_{n1}\ ,\ \Omega_{zn}^{(s)}=0\ . \qquad (10)$$

The present solution is not valid in the near-wall region, as is noted in the main text. In the calculation of dissipation the integral $\Phi=\int_V\left(\frac{\partial U_j}{\partial x_i}\right)^2 dV$ is used, and the helicity is given as $\Psi=\int_V \boldsymbol{U}\cdot\boldsymbol{\Omega}dV$.

We limit the volume integral to the range $r\le r_M$ per unit length of a cylinder. After a lengthy mathematical manipulation, we have

$$\Phi=2\pi\Omega_z^{(C)2}r_M^2\,J_0(\lambda r_M)^2+\pi\Omega_{z1}^{(c)2}\left\{r_M^2J_0(\lambda r_M)^2+2\lambda^{-2}\left(1-J_0(\lambda r_M)^2\right)\right\} \qquad (11)$$

$$\Psi=\pi\left(2\Omega_z^{(C)2}+\Omega_{z1}^{(c)2}\right)r_M{}^2\lambda^{-1}J_0(\lambda r_M)^2\ . \qquad (12)$$

From Eqs.(11) and (12), Φ is rewritten as

$$\Phi=\lambda\Psi+2\pi\Omega_{z1}^{(c)2}\lambda^{-2}\left(1-J_0(\lambda r_M)^2\right)\ . \qquad (13)$$

From the eigenvalue condition $J_1(\lambda r_M)=0$, we have

$$J_0(\lambda r_M)=-0.36\ . \qquad (14)$$

In Eq. (13), the second term is positive from Eq. (14). This fact indicates that the minimum-Φ state under the constraint of fixed Ψ is realized for

$$\Omega_{z1}^{(c)}=0\ . \qquad (15)$$

That is, the axisymmetric condition is given as the minimum state of Φ.

Reference

Akira Yoshizawa, Nobumitsu Yokoi, Shoiti Nisizima, Sanae-I. Itoh, Kimitaka Itoh: NIFS-680 (2001)