§11. Pollutionless Approximation of the Linearized MHD Operator Spectrum

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A number of codes for MHD instabilites have developed extensively. Most of these coedes employed piesewise linear functions and step functions in hybrid as the base functions of Ritz-Galerkin approximation, so that the numerical errors in the eigenvalues, ω, scale as N^{-2}, where N is the number of intervals. We inspected the method of the MHD stability calculation with more higher accuracy.

When we assume $\boldsymbol{X} \in (H^1)^M, \boldsymbol{Y} \in L_2^N$, (where H^c is the Sobolev space), we can write the Lagrangian of the linearized MHD equations in the following form,

$$\begin{aligned} L[\boldsymbol{X},\boldsymbol{Y}] = \int_0^a \mathrm{d}x\{&\boldsymbol{X}'^T\cdot\boldsymbol{D}\cdot\boldsymbol{X}' \\ &+\boldsymbol{X}^T\cdot(\boldsymbol{A}_{11}-\lambda\boldsymbol{B}_1)\cdot\boldsymbol{X}+\boldsymbol{Y}^T\cdot(\boldsymbol{A}_{22}-\lambda\boldsymbol{B}_2)\cdot\boldsymbol{Y} \\ &+2[\boldsymbol{X}^T\cdot\boldsymbol{A}_{12}\cdot\boldsymbol{Y}+\boldsymbol{X}^T\cdot\boldsymbol{C}_1\cdot\boldsymbol{X}'+\boldsymbol{Y}^T\cdot\boldsymbol{C}_2^T\cdot\boldsymbol{X}'\} \end{aligned} \tag{1}$$

where the prime denotes the differential with respect to x, $\boldsymbol{X}$ and $\boldsymbol{Y}$ refer the plasma displacement normal to the magnetic surface and in the magnetic surface, respectively. It should be noted that eq.(1) does not include $\boldsymbol{Y}'$. From this feature the eigenvalue problem of the ideal linearized MHD equations has essential spectrum in addition to discrete spectrum. Therefore, making use of a simple method, it is known for the trouble, which is called the spectrum pollution to occur. Descloux-Nassif-Rappaz[1] have verified the mathematical conditons which ensure the efficiency of approximations for the spectrum of a non-compact operator including our problem.

In order to apply the finite element method we subdivide the region [0,a] into N elements. We here introduce the following two kinds of function spaces $X_h \in H_0^{c_1}(0,a)$ and $Y_h \in H^{c_2}(0,a)$,

$X_h = \{f_i | f_i$ is B-spline function of degree (p_1)
which is continuouslyc_1-th$(0 < c_1 \leq p_1)$
differentiable on neighboring finite elements,
$f_i(0) = f_i(a) = 0, i = 1,2,...,n_1\}$

$Y_h = \{f_i | f_i$ is B-spline function of degree (p_2)
which is continuouslyc_2-th$(0 < c_2 \leq p_2)$
differentiable on neighboring finite elements,
$f_i(0) = f_i(a) = 0, i = 1,2,...,n_2\}$

where

$n_1 = (p_1+1)N - c_1(N-1) - 2 = (p_1 - c_1 + 1)N + c_1 - 2$

$n_2 = (p_2+1)N - c_2(N-1) - 2 = (p_2 - c_2 + 1)N + c_2 - 2.$

The function spaces $\boldsymbol{X}_h$ and $\boldsymbol{Y}_h$ are respectively finite dimensional subspaces of $\boldsymbol{X}$ and $\boldsymbol{Y}$ in eq.(1), the functions $\boldsymbol{X}$ and $\boldsymbol{Y}$ in eq.(1) are expanded as

$$\boldsymbol{X}(\psi) = \sum_i^{n_1} \boldsymbol{X}_i S_i(\psi), \boldsymbol{Y}(\psi) = \sum_i^{n_2} \boldsymbol{Y}_i T_i(\psi). \tag{2}$$

We need to adopt adequately parameters for spline functions $p_1, p_2, c_1, and c_2$ in oder to obtain good approximation properties of the spectrum of eq.(1) and, especially, the pollutionless property. We can prove that prameters which satisfy Descloux-Nassif-Rappaz conditons mentiond above are

$$p_1 - p_2 = c_1 - c_2 = 1 \tag{3}$$

The integration in eq.(1) is estimated by using the $q(\geq p_1)$-th order Gaussian integration formula. It is expected that eigenvalues are efficiently calculated with high accuracy by virtue of higher order spline functions.

We applyed our method to several problems. Then we did not observe the spectrum pollution in any case, regardless of degree and continuity of the base functions and the number of Gaussian integration points. Figure 1 shows the numerically calculated spectrum in a case of the cylinder model by using our method as a function of the number of intervals N. In this model, there are discrete eigenvalues of fast waves and continua of Alfvén waves and slow waves. In figure 1, we obtained adequate approximation of the spectrum. The numerical errors in the discrete eigenvalues scale as N^{-2p_1} which is expected by the theory based on the finite element method.

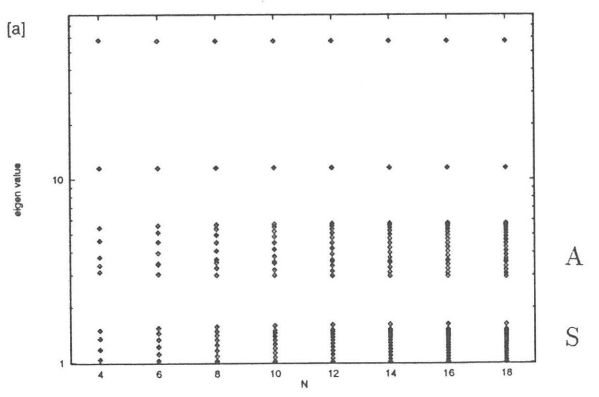

Fig. 1, The numerically calculated spectrum in case of the cylinder model by using our method as a function of the number of intervals N, where the numbers beside the dots denote multiplicity. we selected the parameters;$p_1 = 2, p_2 = 1, c_1 = 2, c_2 = 1$.

References
1) J.Descloux, N.Nassif, J.Rappaz, RAIRO Anal. Numér. 12 (1978) 97