§9. Averaged MHD Equilibrium Equation on Magnetic Coordinates

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Since there is no symmetry in Heliotron configurations such as LHD, various approaches to reduce the 3D problem to 2D one have been developed. Todoroki[1)] developed an averaging MHD equilibrium equation by introducing coordinates (X, Y, ζ) on which the flux function does not depend on the toroidal angle ζ. This averaged equation is a Grad-Shafranov (GS) type equation which is the relation of the 3 flux functions with the metrics. Since any ordering is not needed in the derivation, it has an advantage that it is consistent with the 3D equilibrium. Since (X, Y, ζ) coordinates is not necessarily useful in the stability analysis, the equivalent equation in a magnetic coordinates (ρ, θ, ϕ) is derived here, where ρ is the label of the flux label, θ and ζ are the angle variable in the poloidal and the toroidal directions, respectively. In these coordinates, the equilibrium magnetic field and current are given by

$$\boldsymbol{B} = \nabla\rho \times \nabla\left(\frac{d\chi}{d\rho}\theta - \frac{d\Psi}{d\rho}\zeta + \tilde{\mu}(\rho, \theta, \zeta)\right) \quad (1)$$

and

$$\boldsymbol{J} = \nabla\rho \times \nabla\left(\frac{dI}{d\rho}\theta + \frac{dF}{d\rho}\zeta - \tilde{\nu}(\rho, \theta, \zeta)\right), \quad (2)$$

respectively. Here, $2\pi\chi(\rho)$, $2\pi\Psi(\rho)$, $2\pi I(\rho)$ and $2\pi F(\rho)$ denote the toroidal magnetic flux, the poloidal magnetic flux, the total toroidal current inside the flux surface and the total poloidal current outside the flux surface, respectively. $\tilde{\mu}$ and $\tilde{\nu}$ are periodic functions with respect to θ and ζ. Substituting the contravariant components of (1) and (2) into the force balance equation

$$\nabla P = \boldsymbol{J} \times \boldsymbol{B}, \quad (3)$$

averaging in the ζ direction defined by $\langle f \rangle_\zeta = \oint f d\zeta / \oint d\zeta$ and utilizing the relations of

$$\langle\sqrt{g}J^\zeta\rangle_\zeta = \frac{\partial\langle B_\theta\rangle_\zeta}{\partial\rho} - \frac{\partial\langle B_\rho\rangle_\zeta}{\partial\theta}, \quad (4)$$

$$\langle\sqrt{g}B^\zeta\rangle_\zeta = \left(F - G_{\theta\zeta}\frac{d\Psi}{d\rho}\right)/G_{\zeta\zeta}, \quad (5)$$

$$\langle B_\rho\rangle_\zeta = G_{\rho\theta}\frac{d\Psi}{d\rho} + G_{\rho\zeta}(\sqrt{g}B^\zeta), \quad (6)$$

$$\langle B_\theta\rangle_\zeta = G_{\theta\theta}\frac{d\Psi}{d\rho} + G_{\theta\zeta}(\sqrt{g}B^\zeta) \quad (7)$$

and

$$\langle B_\zeta\rangle_\zeta = G_{\theta\zeta}\frac{d\Psi}{d\rho} + G_{\zeta\zeta}(\sqrt{g}B^\zeta) = F, \quad (8)$$

we can obtain the GS type equation on the flux coordinates,

$$\frac{d\Psi}{d\rho}\left[\frac{\partial}{\partial\rho}\left(G^\perp_{\theta\theta}\frac{d\Psi}{d\rho}\right) - \frac{\partial}{\partial\theta}\left(G^\perp_{\rho\theta}\frac{d\Psi}{d\rho}\right) + \frac{\partial(Fh_\theta)}{\partial\rho} - \frac{\partial(Fh_\rho)}{\partial\theta}\right] = -\langle\sqrt{g}\rangle_\zeta\frac{dP}{d\rho} - T\frac{dF}{d\rho} \quad (9)$$

$$G^\perp_{ij} = G_{ij} - \frac{G_{i\zeta}G_{j\zeta}}{G_{\zeta\zeta}}, \qquad h_i = \frac{G_{i\zeta}}{G_{\zeta\zeta}},$$
$$G_{ij} = \left\langle\frac{g_{ij}}{\sqrt{g}}\right\rangle_\zeta, \quad T = \frac{F}{G_{\zeta\zeta}} - h_\theta\frac{d\Psi}{d\rho}, \quad (10)$$

if and only if the flux coordinates satisfies $\partial\tilde{\mu}/\partial\zeta = 0$ or $\partial\tilde{\nu}/\partial\zeta = 0$.

If we choose coordinates with $\tilde{\mu} = 0$ as a special case, which corresponds to the coordinates with the straight field lines, eq.(9) can be reduced to

$$\frac{d\Psi}{d\rho}\left[\frac{\partial}{\partial\rho}\left(G_{\theta\theta}\frac{d\Psi}{d\rho} + G_{\theta\zeta}\frac{d\chi}{d\rho}\right) - \frac{\partial}{\partial\theta}\left(G_{\rho\theta}\frac{d\Psi}{d\rho} + G_{\rho\zeta}\frac{d\chi}{d\rho}\right)\right] = -\langle\sqrt{g}\rangle_\zeta\frac{dP}{d\rho} - \frac{dF}{d\rho}\frac{d\chi}{d\rho} \quad (11)$$

by using the relation of $\sqrt{g}B^\zeta = d\chi/d\rho$.

References

1) Todoroki, J., J.Phys.Soc.Jpn.,58 (1989) 3979.