§ 8. Modulation Instability in Two-Dimensional Nonlinear Shcrödinger Lattices

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The problem of modulation instability of continuous wave and array soliton solutions within the framework of a two-dimensional continuum-discrete nonlinear Schrödinger lattice model which accounts for dispersion and long-range interactions between elements, is investigated. The linear stability analysis based on an energetic principle and a variational approach, which were originally developed for the continuum nonlinear Schrödinger model, is proposed. Regions of instability are identified and analytical expressions for the corresponding thresholds and the growth rate spectra are calculated.

The basic mathematical model describing two-dimensional lattice with nonlocal nonlinear interacting elements in anomalous dispersion regime has a form of continuum-discrete nonlinear Schrödinger equation

$$i\frac{\partial\psi_{\vec r}}{\partial t}+\frac{\partial^2\psi_{\vec r}}{\partial z^2}+2\psi_{\vec r}|\psi_{\vec r}|^2+\sum_{\vec r'(\vec r'\neq\vec r)}J_{|\vec r'-\vec r|}(\psi_{\vec r'}-\psi_{\vec r})=0, \qquad (1)$$

where $\vec r=(n,m,0)$, $(n=1,2,3...N;\ \ m=1,2,3...M)$ is the discrete lattice vector in a x-y plane, z is the spatial continuous coordinate along the lattice elements and $\psi_{\vec r}=\psi_{n,m}$ is the wave function into the $(n;m)$-th lattice element. The nonlocal interaction term $J_{|\vec r'-\vec r|}$ describes a long-range isotropic coupling between lattice elements and depends on the distance between interacting elements. This interaction model is quite general and enables a mathematical modeling of a variety of discrete dispersive physical systems with long-range interactions. The well known interaction model for 1D DNLS lattice model with a power law dependence on the distance between interacting elements was originally proposed in [1]. In our case, for CDNLS model (1) with regularly spaced 2D lattice with interelement distance equal to 1, the power law dependence can be written in a form

$$J_{|\vec r'-\vec r|}=\frac{1}{|\vec r'-\vec r|^p}. \qquad (2)$$

For the lattice with periodic boundary conditions imposed on the discrete dimensions we can consider a set of lattice independent stationary solutions of Eq. (1), in a form $\psi_{\vec r}=f(z)\exp i\lambda^2 t$. We shall restrict our stability study to two particularly simple and most frequently studied stationary solutions; the first one is a uniform, continuous wave (CW) solution $f_{cw}=\lambda/\sqrt{2}$, while the second one is an array soliton (AS), given by $f_{as}=\lambda/\cosh\lambda z$.

For the case of CW solution the stability analysis gives the following dispersion relation

$$\omega^2=(k^2+4\Sigma)(k^2-2\lambda^2+4\Sigma) \qquad (4)$$

where

$$\Sigma(N,M)=\sum_{n=1}^{N}J_{n,0}\sin^2\left(\frac{k_n n}{2}\right)+\sum_{m=1}^{M}J_{0,m}\sin^2\left(\frac{k_m m}{2}\right)-\sum_{n=1}^{N}\sum_{m=1}^{M}J_{n,m}[\cos(k_n n)\cos(k_m m)-1] \qquad (5)$$

The instability occurs for $\omega^2<0$, which leads to the following instability threshold

$$\lambda^2>\frac{k^2}{2}+2\Sigma(N,M). \qquad (6)$$

The application of the variational method on the stability problem of the array soliton solution leads to the next instability condition

$$\lambda>\lambda_c=\frac{2\sqrt{\Sigma(N,M)}}{\sqrt{3}} \qquad (7)$$

and the growth rate structure

$$\Gamma^2(\mu)=\frac{32}{\pi^2}\mu\left(1-\frac{\mu}{3}\right);\qquad \mu=\frac{4\Sigma}{\lambda^2} \qquad (8)$$

The graphic illustrations of the results (8) and (9) are shown in Fig. 1. and Fig. 2.

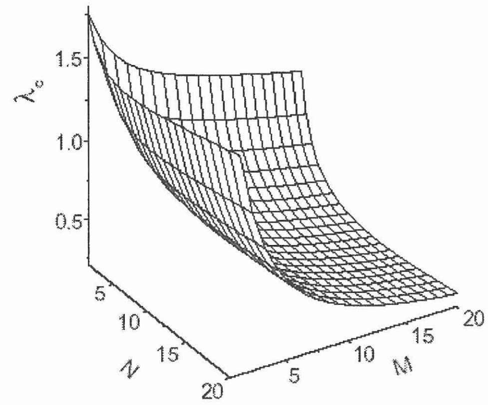

Fig. 1. The instability threshold as a function of the size of the 2D lattice with long-range interactions.

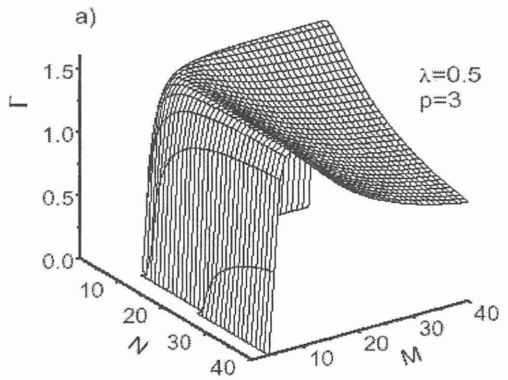

Fig. 2. Growth rates of the instability of the array soliton for 2D lattice with long-range interactions.

References

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2) M. Stepic, Lj. Had ievski and M. Škoric, Phys. Rev. E **65** (2), (2002) 026604.