§21. On Origin and Dynamics of the Discrete NLS Equation

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We investigate soliton-like dynamics in the discrete nonlinear Schroedinger equation (DNLSE) describing the generic 3-element discrete nonlinear system with a dispersion. The DNLSE (1+2) is numerically solved on the $3\times N$ descrete lattice, where N is the variable number introduced through the descretized dispersion term. The DNLS (1+2) equation reads

$$i\frac{\partial\psi_n}{\partial t}+\frac{\partial^2\psi_n}{\partial x^2}+2|\psi_n|^2+(\psi_{n+1}+\psi_{n-1}-2\psi_n)=0,\quad n=1,2,\ldots M \qquad \psi_0=\psi_{M+1}=0 \tag{1}$$

The equation (1) is non-integrable, if $M>2$ (M is the number of elements of DNLS equation) and it is known to possess three conserved quantities: (i) total energy $P=\sum_n\int_{-\infty}^{\infty}|\psi_n|^2dx$, (ii) Hamiltonian H and (iii) momentum in Mx

Compared with the continuum 2D NLS equation, equation (1) exhibits a novel unique feature: existence of multidimensional soliton-like solutions localized in both dimensions, discrete and continual. Another difference from the standard continuum NLS equation is that the DNLS equation (1) shows no singular behaviour. Instead, the quasi-collapse takes place, which is closely related to the collapse phenomenon in two-dimensional NLS; however for DNLS, the solution, instead toward a singularity, evolves to stable multidimensional soliton-like solutions [1]. Accordingly, a practical new method for optical pulse amplification and compresson, based on the quasi-collapse of the optical pulses, in a linearly coupled nonlinear optical fiber array, was proposed in ref. [2].

For the three element DNLS equation ($M=3$) one exact analytical, anti-symmetric soliton-like solution is found, as well as, two approximate symmetric soliton-like solutions, with:
(i) energy concentrated along the central element with linear waves in the side elements

$$\psi_2=\frac{\sqrt{2+\lambda^2}}{\cosh(x\sqrt{2+\lambda^2})}\exp(i\lambda^2t), \tag{2}$$

$$\psi_1=\psi_3=\tfrac{1}{2\sqrt{2+\lambda^2}}[e^{x\sqrt{2+\lambda^2}}\ln(1+e^{-2x\sqrt{2+\lambda^2}}) +e^{-x\sqrt{2+\lambda^2}}\ln(1+e^{2x\sqrt{2+\lambda^2}})]\exp(i\lambda^2t), \tag{3}$$

(ii) energy concentrated along the side elements and a linear wave in the central one

In order to study different dynamical regimes of the soliton dynamics, launched into the central element of the three-element DNLS, we have performed a numerical simulation of the equation (1) using our generalized split-step Fourier method assuming the periodic boundary conditions with respect to x, with continuos monitoring of the conserved quantities: the total energy P and the Hamiltonian H.

In quasi-linear and strongly nonlinear regimes the robustness with respect to the N number is found. However, the intermediate regime exhibiting quasi-periodic and often chaotic dynamics, appears highly sensitive to the number of discrete points, making an exact solving of the DNLS (1+2) equation a dubious task. In this case, the correct solution can be expected only if we consider a generic 2-dimensional discrete system with a given fixed number of elements. However, if our task is to discretize a continuum variable with a number of points, our numerical approach can fail and becomes questionable in the intermediate regime, as seen in the above DNLS (1+2) simulation. In addition, these conclusions can be possibly extended to other finite-difference and spectral numerical schemes with the similar background of the introduced discreteness.

It is to be expected that this situation is not only typical for the DNLS equation but also for other types of discrete nonlinear evolution equations. More generally, this problem is of an additional relevance, because it can possibly emerge in different numerical schemes aims to simulate various continuum nonlinear PDE's in two and three dimensions.

References

[1] A. B. Aceves, G. G. Luther, C. De Angelis, A. M. Rubenchik and S. K. Turitsyn, Phys. Rev. Lett. **75**, 73 (1995).
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