

§17. Applicability of the Direct-interaction Approximation

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The aim of this work is to clarify the validity limit of the direct-interaction approximation (DIA), which was originally proposed ¹⁾ as a closure theory of incompressible turbulence governed by the Navier-Stokes equation, and which is successful in the application to the Lagrangian velocity fields ²⁾. Here, we consider a model equation for N variables X_i ($i = 1, 2, \dots, N$) as

$$\left[\frac{d}{dt} + \nu \right] X_i(t) = \sum_{j,k=1}^N C_{ijk} X_j(t) X_k(t) + F_i(t),$$

where C_{ijk} , ν and F_i are constant coefficients, the viscosity and a Gaussian random force with zero-mean, respectively. This model equation has similar properties to those of the Navier-Stokes equation with an appropriate choice of the coefficients C_{ijk} which satisfy the following two conditions. First, the nonlinear term conserves the energy $\frac{1}{2} \sum_i (X_i)^2$. Second, there is only one direct-interaction, at the most, between an arbitrary pair of modes. For the Navier-Stokes system, the second property holds if a periodic boundary condition is imposed. This second property of *weak nonlinear couplings* is essential for DIA.

The DIA is based upon a decomposition called direct-interaction decomposition

$$X_i(t) = X_{i/i_0 j_0 k_0}^{(0)}(t|t_0) + X_{i/i_0 j_0 k_0}^{(1)}(t|t_0),$$

where $X_{i/i_0 j_0 k_0}^{(0)}$ is an artificial field without the direct-interaction between three modes $\{X_{i_0}, X_{j_0}, X_{k_0}\}$, and t_0 denotes the time when this decomposition is made. By the use of this decomposition, DIA is constructed under the following two assumptions: (i) The three modes $\{X_{i/i_0 j_0 k_0}^{(0)}, X_{j/i_0 j_0 k_0}^{(0)}, X_{k/i_0 j_0 k_0}^{(0)}\}$ are statistically independent of each other because there is no direct-interaction between them. (ii) The magnitude of $X_i^{(1)}$ is much smaller than that of $X_i^{(0)}$ as long as $t - t_0$ is of the order of the autocorrelation time of X_i .

In order to make the applicability limit of the approximation clear, we should clarify the parameter region in which the above two assumptions hold. Intuitively, these assumptions may be satisfied if the number of the degrees of freedom N is large enough to ran-

domize and cancel the contribution of the *indirect* interactions to the correlation between the three modes, and to make the influences of artificially removing a direct-interaction from the system inconspicuous.

We examine the above *a priori* argument by the use of numerical calculations. First, in order to confirm whether the DIA assumption (i) is actually valid for large N or not, we evaluate the triple correlation function

$$R_{ijk}(t - t') = \overline{X_i(t) X_j(t) X_k(t')} / (\overline{X_i(t)^2})^{3/2},$$

where an overbar stands for the long-time average, by the numerical integration of the model equation. The results of the calculations are as follows: Although the triple correlation R_{ijk} takes nontrivial value in the true field X_i irrespective of N , it vanishes in the non-direct-interaction field $X_{i/i_0 j_0 k_0}^{(0)}$ if N is large ($N \gtrsim 20$). Next, as for the second assumption, we evaluate the ensemble average of the magnitude of the direct-interaction field $X_i^{(1)}(t|t_0)$. Then, it is shown to be much smaller than the magnitude of $X_i^{(0)}(t|t_0)$ as long as $t - t_0$ is of the order of the time-scale of the autocorrelation function if N is large.

In conclusion, DIA for a nonlinear system with weak couplings is applicable if the number of the degrees of freedom is sufficiently large. Actually, DIA gives excellent predictions for the autocorrelation function of X_i in the cases of large N (Fig.1). For further detailed arguments, see ref. 3).

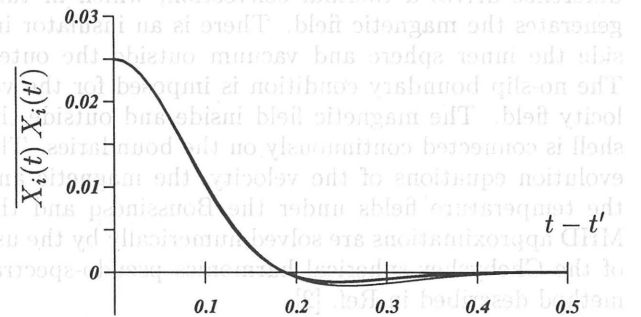


Fig.1 The autocorrelation function of X_i for $(\nu, N) = (0, 40)$. Thick and thin solid lines are respectively due to DIA and a long-time average of direct numerical integrations of the basic equation.

References

- 1) Kraichnan, R. H., J. Fluid Mech. **5** (1959) 497.
- 2) Kida, S. and Goto, S., J. Fluid Mech. **345** (1997) 307.
- 3) Goto, S. and Kida, S., Physica D (in press).