# §24. Passive Material Line Statistics in Turbulence

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This study is intended as the investigation of statistics of material lines passively floating in turbulence of incompressible fluid (Fig.1). The position vector $\boldsymbol{x}_p(t)$ of a point on the line is advected by the local turbulent velocity field $\boldsymbol{u}(\boldsymbol{x}_p, t)$, while the dynamics of $\boldsymbol{u}(\boldsymbol{x}, t)$ are independent of the existence of the line, and governed by the Navier-Stokes equation and the equation of continuity.

A passive material line can be regarded as a set of many successive line segments of sufficiently short length $l^{(i)}(t)$ $(i = 1, 2, \cdots, I)$. Then, *the line average* of a quantity $g^{(i)}(t)$ accompanied with the line is defined by

$$\langle g \rangle_{\text{line}} = \frac{1}{L(t)} \sum_{i=1}^{I} g^{(i)}(t)\, l^{(i)}(t)\,, \tag{1}$$

where $L(t) = \sum_{i=1}^{I} l^{(i)}(t)$ is the total length of the line. On the other hand, *the line-element average* of $g^{(i)}(t)$ is defined by

$$\langle g \rangle_{\text{line-element}} = \frac{1}{I} \sum_{i=1}^{I} g^{(i)}(t)\,. \tag{2}$$

Since Batchelor's pioneer work[1] it has been believed that these two averages are identical in statistically homogeneous turbulence because stretching histories of all the line segments might become statistically equivalent after sufficiently long time.

Since the line average (1) can be rewritten as

$$\langle g \rangle_{\text{line}} = \frac{\langle gl \rangle_{\text{line-element}}}{\langle l \rangle_{\text{line-element}}}\,, \tag{3}$$

if the correlation between $g^{(i)}(t)$ and $l^{(i)}(t)$ vanished as $t \to \infty$, the two averages would be identical. However, this is not true. The present study shows theoretically and numerically that the correlation between the length $l^{(i)}(t)$ of the segment and the quantity $g^{(i)}(t)$ cannot decay in general even after long time, and therefore

$$\langle g \rangle_{\text{line}} \neq \langle g \rangle_{\text{line-element}}\,. \tag{4}$$

As a typical and important example, we consider the stretching rate $\gamma(t)$ of the material line,

$$\gamma(t) = \frac{\mathrm{d}}{\mathrm{d}t} \log L(t)\,, \tag{5}$$

which is nothing but the line average of the stretching rate $\gamma_e^{(i)}(t) = \frac{\mathrm{d}}{\mathrm{d}t} \log l^{(i)}(t)$ of each line segment, i.e., $\gamma = \langle \gamma_e \rangle_{\text{line}}$. It should be stressed that the line-element average $\langle \gamma_e \rangle_{\text{line-element}}$ of the stretching rate underestimates the true value of $\gamma$. This can be understood by looking at the relation between $l^{(i)}(t)$ and $\gamma_e^{(i)}(t)$,

$$l^{(i)}(t) = l^{(i)}(0) \exp\left( \int_0^t \gamma_e^{(i)}(t')\, \mathrm{d}t' \right)\,. \tag{6}$$

It can be shown numerically that $\gamma_e^{(i)}(t)$ has a finite autocorrelation time of $O(5\tau_\eta)$, where $\tau_\eta$ denotes the Kolmogorov time. This finite correlation time implies the positive correlation between $l^{(i)}(t)$ and $\gamma_e^{(i)}(t)$ even in the limit of $t \to \infty$. Note that the time integral in (6) is in an exponential function. If it were not in the exponent, the contribution to the correlation from a finite time period $t - \tau_\eta < t' < t$ would vanish as $t \to \infty$.

In order to confirm the above argument, we carry out direct numerical simulations of passive material lines, and estimate both of the line and the line-element averages of the stretching rate of the lines. Results are plotted in Fig.2. The averages in the statistically stationary state $(t \geq 20\tau_\eta)$ are constants around

$$\langle \gamma_e^{(i)} \rangle_{\text{line}} = 0.17{\tau_\eta}^{-1} > \langle \gamma_e^{(i)} \rangle_{\text{line-element}} = 0.13{\tau_\eta}^{-1}\,. \tag{7}$$

As expected, the line-element average of the stretching rate underestimates the true value.

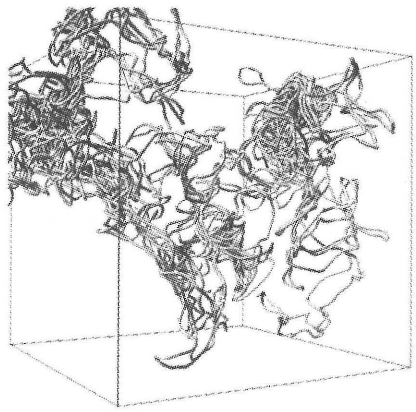

Fig.1 Passive material line in turbulence. A result of direct numerical simulation.

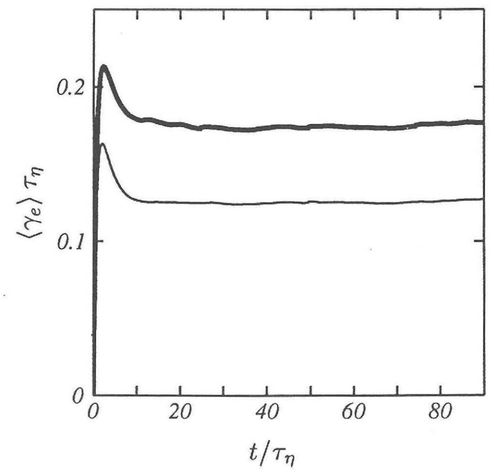


Fig.2 Line average (thick curve) and line-element average (thin curve) of stretching rate of passive material lines.

Reference

1) Bathelor, G. K., Proc. Roy. Soc. London A **213** (1952) 349.