§25. Nonlinear Coupling Density

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There are two independent parameters which characterize nonlinearity of a dynamical system. One is the strength of nonlinearity, which expresses magnitude of the nonlinear term comparing that of the linear one, e.g., the Reynolds number in the Navier-Stokes system. Another parameter is nonlinear coupling density, which will be introduced in the next paragraph. The purpose of this study is to demonstrate that we can construct a closure theory, called direct-interaction approximation (DIA), even for a very strong nonlinear system, on condition that its nonlinear coupling is sparse.

In order to introduce the nonlinear coupling density, we consider a dynamical system $\{X_i | i = 1, 2, \cdots, N\}$ governed by

$$\frac{\mathrm{d}}{\mathrm{d}t} X_i(t) = \mathcal{R} \sum_{j=1}^{N} \sum_{k=1}^{N} C_{ijk}\, X_j(t) X_k(t) - X_i(t) + f_i(t)\,, \tag{1}$$

where constant coefficients C_{ijk} ($\sim O(1)$) satisfy three conditions: $C_{ijk} + C_{jki} + C_{kij} = 0$ (conservation of energy), $C_{ijk} = C_{i+m,j+m,k+m}$ (homogeneity) and $C_{ijj} = 0$ (no self interaction). The coupling density ρ is defined by the number of direct interactions (i.e., the nonlinear interactions which appear explicitly on the right-hand side of (1)), between a pair of modes. In other words, ρ is the number of non-zero components of C_{ijk} for fixed i and j. Thus, this system has three parameters: $\mathcal{R}$ (the strength of nonlinearity), N (the number of degrees of freedom) and ρ ($\leq N-2$).

The purpose of closure theories is to predict statistical quantities, e.g., the autocorrelation function $\mathcal{V}(\tau) = \overline{X_i(t+\tau) X_i(t)}$ (an overbar denotes the ensemble average), based on the governing equation (1) of the system. The DIA[1] was proposed for this purpose, and its main assumption is summarized as "*if there were no direct interaction between a triplet of modes, they would be statistically independent of each other*". It is intuitively expected and numerically confirmed[2] (see below) that this assumption (then, DIA) is valid when the nonlinear coupling is sparse, even if $\mathcal{R}$ is large.

Under the DIA assumption stated above, we can easily derive a closed equation for $\mathcal{V}(\tau)$ in a statistically stationary state as

$$\left[\frac{\mathrm{d}}{\mathrm{d}\tau} + 1\right] \mathcal{V}(\tau) = -\frac{2\mathcal{C}\mathcal{R}}{\mathcal{V}(0)} \int_0^{\tau} \mathrm{d}\tau'\ \mathcal{V}(\tau')^2\, \mathcal{V}(\tau - \tau')\,. \tag{2}$$

Prediction $\mathcal{V}_{\mathrm{DIA}}$ by this closure equation coincides excellently with $\mathcal{V}_{\mathrm{DNS}}$ by direct numerical simulation of (1), when $\rho \sim 1$ and $N = 10^3$ (Fig.1(a), a sparse coupling case). However, the overlap between two curves is much worse, when $\rho = 8$ and $N = 10$ (Fig.1(b), the densest coupling case). These results are consistent with the conjecture. For a quantitative estimation of accuracy of the DIA equation, we define (see the figure caption) a parameter Δ, and plot it in Fig.2 for various combinations of ρ and N in the case of $\mathcal{R} \to \infty$. This figure suggests that DIA is valid when $\rho \ll \sqrt{N}$. The DIA assumption requires smallness, in a statistical sense, of contribution from indirect interactions between the triplet of modes to the correlation between them. The validity condition can be regarded as the condition that such a randomization of indirect interactions does occur; it can be shown that if $\rho \gtrsim \sqrt{N}$, probability of existence of non-randomized indirect interactions gets larger.

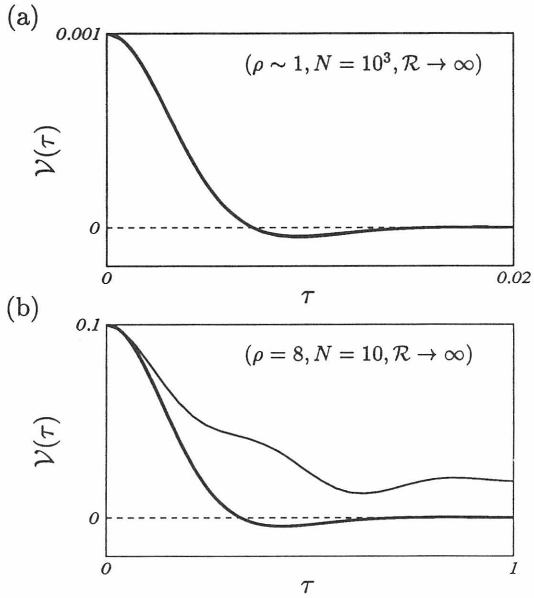

Fig.1 Autocorrelation function, $\mathcal{V}_{\mathrm{DIA}}$ (thick line) and $\mathcal{V}_{\mathrm{DNS}}$ (thin line). (Without any adjustable parameter.)

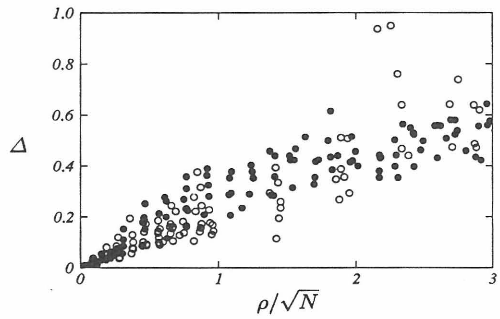

Fig.2 Integral difference parameter,

$$\Delta = \int_0^{\infty} \mathrm{d}\tau \left|\mathcal{V}_{\mathrm{DIA}}(\tau) - \mathcal{V}_{\mathrm{DNS}}(\tau)\right| \Big/ \int_0^{\infty} \mathrm{d}\tau \left|\mathcal{V}_{\mathrm{DNS}}(\tau)\right|.$$

$\circ$, $N = 10^2$; $\bullet$, $N = 10^3$. $\mathcal{R} \to \infty$.

Reference
1) Kraichnan, R.H., Phys. Rev. **109** (1958) 1407.
2) Goto, S. & Kida, S., Physica D **117** (1998) 191; Nonlinearity (submitted).