

§14. Hamiltonian Representation of the Toroidal Helical Magnetic Field Lines System

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We study the toroidal helical magnetic field in the vacuum in the framework of Hamiltonian theory and investigate the equations for the magnetic field lines flow by means of symplectic numerical integration methods. Starting from the general scalar potential satisfying the condition $\nabla^2\Phi = 0$ and regular at $\xi = 0$ (we use toroidal coordinates ξ, η, φ), and selecting only the two harmonics (l, m) and $(l, 0)$, where l is the poloidal multipolarity and m is the number of toroidal periods, we first construct the Taylor expansion for the corresponding vector potential \mathbf{A} in a gauge such that $A_\xi = 0$, using the inverse aspect ratio ϵ , the strength of the vertical field ϵ' and the radial distance ξ as smallness parameters. We also introduce the variable $\theta = \eta + (m/l)\varphi$ and choose $l = 2$. In the obtained expression $\mathbf{A} = \mathbf{A}^0 + \mathbf{A}^1 + \dots$, the coordinates ξ and θ are not canonical coordinates, however the transformation $(1/2)\xi^2 = \psi$ puts the lowest order potential \mathbf{A}^0 in the canonical form $\mathbf{A}^0 = (0, \psi, -(m/2)\psi)$. Then we use perturbation theory based on the Lie transform to change systematically the variables from (ψ, θ) to (Ψ, Θ) in order to obtain the vector potential in canonical form to the desired perturbative order. Using for the transformed vector potential the notation $\mathbf{\Gamma} = \mathbf{\Gamma}^0 + \mathbf{\Gamma}^1 + \dots$, we obtain a transformation such that $\mathbf{\Gamma}^0 = \mathbf{A}^0$ and $\Gamma_\theta^k = 0, \Gamma_\psi^k = 0$ for $k \geq 1$. The form of the transformed vector potential is therefore $\mathbf{\Gamma} = (0, \Psi, \Gamma_\varphi)$, with $\Gamma_\varphi = \Gamma_\varphi(\Psi, \Theta, \varphi)$, and $H = -\Gamma_\varphi$ is the Hamiltonian. The equations for the magnetic field lines are $\dot{\Theta} = \partial H / \partial \Psi, \dot{\Psi} = -\partial H / \partial \Theta$. We develop the calculations until the third perturbative order obtaining an explicit form for the Hamiltonian in terms of elementary functions. For example the first order Hamiltonian is given by $H = (m/2)\Psi + \alpha\epsilon'\Psi \sin(m\varphi - 2\Theta) - \beta\epsilon\Psi \sin(2\Theta)$, where α, β are constant. This Hamiltonian de-

pends on the three variables Θ, Ψ and φ and contrasts with the Hamiltonian obtained in ref. 2, which depends on only two variables and is integrable. It is shown that this fact is related to the choice of a different gauge in the Lie transform. We then use a linear symplectic integration methods (ref. 3) to study the behavior of the magnetic field lines. The use of symplectic methods is particularly important when it is necessary to follow for a long time the phase space trajectory of the system, because the numerical algorithm exactly preserves the Hamiltonian nature of the physical system (in particular artificial damping or excitation do not occur). The possibility to use symplectic integrators is the main motivation that led us to develop a Hamiltonian formulation. The numerical results show the formation of magnetic islands, regular and chaotic regions in the Poincaré section taken at $\varphi = 0$. We conclude that it is possible to study in the context of a simple Hamiltonian model the transition from regular to chaotic behavior of the field lines.

References

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