§10. Formulation of Lagrangian Neoclassical Transport Theory

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Recently, neoclassical transport in the core region of tokamaks again attracts much attention. It is well-known that there appear non-standard guiding-center orbits near the magnetic axis called "potato" orbits[1]. Typical orbit width of potato particles is as large as $(q^2\rho^2R_0)^{1/3}$, where q is the safety factor, ρ is the Larmor radius, and R_0 is the major radius, respectively. The standard neoclassical transport theory[2,3] constructed in the small-orbit-width (SOW) approximation is not applicable to the near-axis region, and the orbital properties of potato particles should be considered in analyzing transport in this region.

Neoclassical transport theory has usually been discussed in Eulerian representation. However, to include orbital properties in the transport theory, Lagrangian formulation[4] was found to be suitable for a collisionless (banana-regime) plasma. In this approach, transport phenomena are described by a reduced drift-kinetic equation in the space of three constants-of-motion (COM) along a collisionless particle orbit in a tokamak.

We apply for the first time Lagrangian formulation to the near-axis region[5] in which the finite-orbit-width (FOW) effect becomes really important. To utilize Lagrangian transport theory, we improve the treatment of like-particle collision term by adding a momentum-restoring term to a simple Lorentz operator. This transport theory reflects quantitatively the properties of all types of particles appearing near the magnetic axis.

The Lagrangian formulation is derived from the drift kinetic equation in an Eulerian representation in the $(\mathbf{x}, \mathcal{E}, \mu)$ space,

$$\frac{\partial}{\partial t} f_a(\mathbf{x}, \mathcal{E}, \mu, t) + \dot{\mathbf{x}} \cdot \frac{\partial f_a}{\partial \mathbf{x}} = C_{ab}, \qquad (1)$$

where $\dot{} = d/dt$, $\mathbf{x}$ is the guiding-center position and C_{ab} is a collision operator. The independent variables in eq. (1) are transformed into three constants of motion in the collisionless limit $(z_1, z_2, z_3) = (\mathcal{E}, \mu, \langle\psi\rangle)$, and the other three variables $(\tilde{z}_4, \tilde{z}_5, \tilde{z}_6)$. Here, $\mathcal{E}$ is the energy of a particle, μ is the magnetic momentun, and $\langle\psi\rangle$ represents the averaged radial position of a particle orbit. The orbit average operator for any function $a(\mathbf{z}, \tilde{\mathbf{z}})$ is defined as

$$\langle a \rangle \equiv \frac{1}{4\pi^2\tau_p} \oint \frac{d\theta}{\dot{\theta}} d\zeta d\phi \; a(\mathbf{z}, \tilde{\mathbf{z}}), \qquad (2)$$

where

$$\tau_p \equiv \oint \frac{d\theta}{\dot{\theta}} \qquad (3)$$

is the poloidal period of an particle orbit. Note that we can use $z_3 = \langle r \rangle$ instead of $\langle\psi\rangle$ when it is convenient. By using the set of variables $(\mathbf{z}, \tilde{\mathbf{z}})$, eq. (1) is transformed into

$$\frac{\partial}{\partial t} f_a(\mathbf{z}, \theta, t) + \dot{\theta}\frac{\partial f_a}{\partial \theta} = C_{ab}, \qquad (4)$$

where the property $\partial/\partial\phi = \partial/\partial\zeta = 0$ is used.

In the collisionless regime, averaging both sides of eq. (4) yields the reduced drift-kinetic equation in the COM space,

$$\frac{\partial \bar{f}}{\partial t} = \frac{1}{J_c}\frac{\partial}{\partial \mathbf{z}} \cdot \left(J_c \left\langle \frac{\partial \mathbf{z}}{\partial \mathbf{v}} \cdot \mathbf{\Gamma}(\bar{f}) \right\rangle \right) = \bar{C}, \qquad (5)$$

where $\bar{f} = f(\mathcal{E}, \mu, \langle\psi\rangle, t)$ and J_c is Jacobian.

Equation (5) is expanded in $\delta_b = \Delta_b/L \ll 1$ and then solved order by order. Here, Δ_b and L are typical potato width and typical gradient scale, respectively. By taking moments of $O(\delta_b^2)$ part of eq. (5) with $\mathcal{E}$ and μ, one obtains the particle and heat flux equations,

$$\frac{\partial \bar{n}_i}{\partial t} + \frac{\lambda_n}{V'}\frac{\partial}{\partial\langle\psi\rangle}(V'\Gamma_i) = 0, \qquad (6)$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}\bar{n}_i T_i\right) + \frac{\lambda_q}{V'}\frac{\partial}{\partial\langle\psi\rangle}\left[V'\left(q_i + \frac{3}{2}\Gamma_i T_i\right)\right] = -\lambda_q e_i \Gamma_i \frac{d\Phi}{d\langle\psi\rangle}, \qquad (7)$$

where

$$\begin{bmatrix} \Gamma_i \\ \frac{q_i}{T_i} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} \frac{d\ln\bar{n}_i}{d\langle\psi\rangle} + \frac{e_i}{T_i}\frac{d\Phi}{d\langle\psi\rangle} \\ \frac{d\ln T_i}{d\langle\psi\rangle} \end{bmatrix}. \qquad (8)$$

These equations are in the comparable form to those of standard Eulerian representation. The difference arose by using $\langle\psi\rangle$ instead of ψ can be seen to be factorized by λ_n and λ_q. These factors tend to be unity away from the axis and go infinity as $\langle\psi\rangle \to 0$. We also emphasize that the transport coefficients A_{jk}, though we do not have enough space to write the definitions here, have a convenient form to calculate numerically. Then, by using Lagrangian formulation, we can evaluate neoclassical transport coefficients near the axis, with including the effect of potato orbits.

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