§27. A Method to Eliminate the Logarithmic Singularity in Fourier Series, Which Appear in Parallel Viscosity Calculations in the Banana Regime

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One of troublesome problems in analytical calculations of the neoclassical transport is a singularity of the drift kinetic equation in the banana regime expansion [1]. Although the singularity corresponding to the characteristics of the collisionless orbits conserving magnetic moments μ is usually integrable, the singularity due to the collision operator finally appears explicitly. However, in most of applications, final expressions of diffusions and viscosities result in integrable problems. In the analytical theories for non-symmetric toroidal plasmas, this problem apparently appears in the bootstrap current calculations [2]. The functions $\partial h/\partial\lambda$, $\mathrm{d}\alpha_{mn}(\lambda)/\mathrm{d}\lambda$ and $\mathrm{d}\beta_{mn}(\lambda)/\mathrm{d}\lambda$ appearing in Ref.[2-4] have a logarithmic singularity at the poloidal and toroidal coordinates of $(\theta,\zeta)=(\theta_\mathrm{M},\zeta_\mathrm{M})$, and at the pitch angle of the circulating/trapped boundary λ=1, which originates in the singularity of the equation at $v_{/\!/}$=0. Since the final expressions of the parallel viscosity forces are given by the $\langle B\int \mathrm{d}^3\mathbf{v}\; v_{/\!/}\; L_j^{(3/2)}\rangle$ moments of the collision operator, this singularity does not severely affect the final results also in this case. However, we should note that, in the method in Ref.[2-4] where this logarithmically singular function is expressed by a Fourier series, infinite number of Fourier modes of $|m|\le\infty$, $|n|\le\infty$ are required to retain some mathematically guaranteed characteristic of the formulas. One of the most important characteristics of these formulas is that the part expressed by this series must vanish in symmetric configurations [2]. For practical applications in which the mode number must be truncated to be finite, a technique to eliminate the singularity before the Fourier expansion to retain the mathematically guaranteed characteristics even in the case of truncated series.

We show here the numerical scheme for the formula in Ref.[4] as an example. The formulas in Ref.[2-3] also can be calculated in analogous manners. By changing the order of $\int \mathrm{d}\lambda$, Σ, $\langle\,\rangle$, and $\int \mathrm{d}\theta_\mathrm{B}\int \mathrm{d}\zeta_\mathrm{B}$, we can obtain another expression for the integral of the function $W(\lambda)$ defined in Ref.[2-4] as

$$\int_0^1 \mathrm{d}\lambda \frac{\lambda W(\lambda)}{\left\langle (1-\lambda B/B_\mathrm{M})^{1/2}\right\rangle} = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\mathrm{d}\theta_\mathrm{B}\int_{-\pi}^{\pi}\mathrm{d}\zeta_\mathrm{B}\, Q(\theta_\mathrm{B},\zeta_\mathrm{B})$$
$$+\frac{4}{3}\frac{1}{2\pi^2}\sum_{\substack{(m,n)\\ \neq(0,0)}}\frac{\cos(m\theta_\mathrm{M}^{(\mathrm{Boozer})}-n\zeta_\mathrm{M}^{(\mathrm{Boozer})})}{\chi' m-\psi' n}\times$$
$$\int_{-\pi}^{\pi}\mathrm{d}\theta_\mathrm{B}\int_{-\pi}^{\pi}\mathrm{d}\zeta_\mathrm{B}\cos(m\theta_\mathrm{B}-n\zeta_\mathrm{B})\left(\frac{1}{2}\frac{B(\theta_\mathrm{B},\zeta_\mathrm{B})}{B_\mathrm{M}}+1\right)\left(1-B(\theta_\mathrm{B},\zeta_\mathrm{B})/B_\mathrm{M}\right)^{1/2}$$
$$\times\left\{Rm+Sn+\frac{\chi'\psi'}{\langle B^2\rangle}\frac{4\pi^2}{V'}\left(\frac{\langle B^2\rangle}{\{B(\theta_\mathrm{B},\zeta_\mathrm{B})\}^2}-1\right)(B_\zeta m+B_\theta n)\right\} \tag{1}$$

$$Q(\theta_\mathrm{B},\zeta_\mathrm{B})\equiv\frac{1}{4\pi^2}\frac{\langle B^2\rangle}{\{B(\theta_\mathrm{B},\zeta_\mathrm{B})\}^2}\sum_{\substack{(m,n)\\ \neq(0,0)}}\frac{\cos(m\theta_\mathrm{B}-n\zeta_\mathrm{B})}{\chi' m-\psi' n}\times$$
$$\int_{-\pi}^{\pi}\mathrm{d}\theta_\mathrm{B}'\int_{-\pi}^{\pi}\mathrm{d}\zeta_\mathrm{B}'\cos(m\theta_\mathrm{B}'-n\zeta_\mathrm{B}')\,F\{B(\theta_\mathrm{B},\zeta_\mathrm{B}),B(\theta_\mathrm{B}',\zeta_\mathrm{B}')\}\left(\frac{B(\theta_\mathrm{B}',\zeta_\mathrm{B}')}{B_\mathrm{M}}\right)^2$$
$$\times\left\{Rm+Sn+\frac{\chi'\psi'}{\langle B^2\rangle}\frac{4\pi^2}{V'}\left(\frac{\langle B^2\rangle}{\{B(\theta_\mathrm{B}',\zeta_\mathrm{B}')\}^2}-1\right)(B_\zeta m+B_\theta n)\right\} \tag{2}$$

using the function $F(B_1,B_2)$ defined by

$$F(B_1,B_2)\equiv\int_0^1 \mathrm{d}\lambda\frac{\lambda\left(1-\lambda B_1/B_\mathrm{M}\right)^{1/2}}{\left\langle(1-\lambda B/B_\mathrm{M})^{1/2}\right\rangle\left(1-\lambda B_2/B_\mathrm{M}\right)^{1/2}}\,. \tag{3}$$

We made the spline function of $F(B_1,B_2)$ using the spline of $\langle(1-\lambda B/B_\mathrm{M})^{1/2}\rangle^2$ as the function of λ and then calculated Eqs.(2) and (1).

It is obvious that the truncation of the Fourier modes in Eqs.(1-2) shown here is better approximation than that in $\mathrm{d}\alpha_{mn}(\lambda)/\mathrm{d}\lambda$ and $\mathrm{d}\beta_{mn}(\lambda)/\mathrm{d}\lambda$ having the singularity. The λ, θ, ζ dependence of $\partial h/\partial\lambda$ in Ref.[2-3] has a singularity at $(\theta,\zeta)=(\theta_\mathrm{M},\zeta_\mathrm{M})$, λ=1 although, $\langle(1-\lambda B/B_\mathrm{M})^{1/2}\rangle$ and $F(B_1,B_2)$ (as the function of $(B_\mathrm{M}-B_1)^{1/2}$ and $(B_\mathrm{M}-B_2)^{1/2}$) calculated here are moderate functions without singularity and with monotonic dependences on the variables. Therefore we made, in numerical examples described in the previous studies[5], the tabulation of $\langle(1-\lambda B/B_\mathrm{M})^{1/2}\rangle$ in λ space with 10 grid points for the range of $0\le\lambda\le1$, and $F(B_1,B_2)$ in $((B_\mathrm{M}-B_1)^{1/2},(B_\mathrm{M}-B_2)^{1/2})$ space with 10×10 points for $0\le(B_\mathrm{M}-B_1)^{1/2},(B_\mathrm{M}-B_2)^{1/2}\le(B_\mathrm{M}-B_\mathrm{min})^{1/2}$. The $F(B_1,B_2)$ can be commonly used for both of the Boozer and the Hamada coordinates. Spline functions are used also to calculate $\int \mathrm{d}\theta'\int \mathrm{d}\zeta'$ integrals in Eq.(2) as the function of $(B_\mathrm{M}-B(\theta,\zeta))^{1/2}$ for each (m,n) modes.

In the axisymmetric and helically symmetric cases [5], the good agreements were obtained by using only 5~6 "symmetric" (i.e., m/n=const) Fourier modes (m,n) to expand the circulating particle distribution. The reduction of the high frequency modulation components in the Fourier expanded part of the perturbation [4], combined with this technique removing the logarithmic singularity in the Fourier series, improves the approximations and extends the application areas of the theories in Ref.[2-3]. It will be useful also for transport calculations in recently designed advanced helical devices [6].

References

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