§26. A Numerically Robust Formula for Bootstrap Current Coefficients in the Banana Regime in Boozer Coordinates

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In many applications of the theory of neoclassical bootstrap currents, an analytical expression derived by Shaing, Carreras, et al.[1] in the Boozer coordinates is often used. This representation was an extension of the previous theory developed in the Hamada coordinates [2-3]. To derive parallel viscosity forces driving the parallel flows, we have to solve the lowest order drift kinetic equation in the banana regime expansion. In general non-symmetric toroidal plasmas, this equation does not satisfy the solubility condition $\oint(\mathbf{v}_{\rm da}\bullet\nabla s/v_{//})dl = 0$ for the trapped particle part in the phase space. On the other hand, the incompressible particle and energy conservation laws enable us to derive the parallel viscosity forces without solving full part of the equation [2-3]. The results of this method, however, depend on the poloidal and toroidal coordinates giving the maximum value of the magnetic field strength $B_{\rm M}$ on the flux surface as shown later. Such coordinates cannot be determined uniquely in symmetric plasmas. The ingenious technique developed in Ref.[2-3] for this problem was the following separation of the lowest order perturbation.

$$f_{\rm a1}^{(0)} = -\frac{m_{\rm a}c}{e_{\rm a}}\frac{\partial f_{\rm aM}}{\partial s}v_{//}\left(\frac{H_1}{B}+\frac{V'}{4\pi^2}\frac{BH_2}{2\chi'\psi'}\right)+\frac{m_{\rm a}c}{e_{\rm a}\chi'\psi'}\frac{V'}{4\pi^2}\frac{\partial f_{\rm aM}}{\partial s}v B_{\rm M}h \tag{1}$$

Here, H_1 expresses a part of the incompressible particle and energy flows, and H_2 is a constant indicating the magnetic flux surface averaged effects of the local structure (i.e., high frequency modulation) of the magnetic field on the trapped particle flow pattern. This constant is defined in Ref.[2-3]. The first term, which is proportional to $v_{//}$, satisfies the solubility condition. The treatment of this part is substantially identical with that in symmetric plasmas. For the remaining part of the perturbation h, the parallel viscosity was derived directly not by calculating the trapped particle distribution but by only calculating circulating particle distribution. The constant H_2 is therefore chosen to satisfy the condition in that the h vanishes in symmetric configurations. The explicit expression for H_1 and the equation for h in the Hamada coordinates were simple as shown in Ref.[2-3].

In contrast to the Hamada coordinates, the equations in the Boozer coordinates are generally complicated. It depends on the explicit expression for H_1. Shaing, Carreras, et al.[1] used the expression, which is mathematically equivalent to that in Ref.[2-3],

$$H_1 = \frac{\psi' B_\zeta - \chi' B_\theta}{2\chi'\psi'} + \frac{B^2}{2\chi'\psi'}\frac{V'}{4\pi^2}\sum_{\substack{(m,n)\\ \neq(0,0)}}\frac{\chi' m+\psi' n}{\chi' m-\psi' n}\,\varepsilon_{mn}^{\rm (Boozer)}\cos(m\theta_{\rm B}-n\zeta_{\rm B}) \tag{2}$$

Another expression of H_1 equivalent to Eq.(2) is given by

$$H_1 = \frac{B^2}{\langle B^2\rangle}\frac{\psi' B_\zeta - \chi' B_\theta}{2\chi'\psi'} + \frac{B^2}{\langle B^2\rangle}\sum_{\substack{(m,n)\\ \neq(0,0)}}\frac{B_\zeta m+B_\theta n}{\chi' m-\psi' n}\,\varepsilon_{mn}^{\rm (Boozer)}\cos(m\theta_{\rm B}-n\zeta_{\rm B}) \tag{3}$$

Here, ε_{mn} in Eqs.(2-3) is Fourier coefficients of $(\langle B^2\rangle/B^2-1)$ [1,4]. The second term in Eq.(3) is obtained as the solution of the equation for the incompressible flow pattern associated with the Pfirsch-Schlüter transport $\mathbf{B}\bullet\nabla(U/B) = (\mathbf{B}\times\nabla s)\bullet\nabla(1/B^2)$[4]. By the procedure in Ref.[2-3] starting from Eq.(3), we obtained an expression for the geometrical factor $G^{\rm (BS)}$ [1-4], which expresses the driving force for the flows,

$$G^{\rm (BS)} = \langle H_1\rangle + \frac{\langle B^2\rangle H_2}{2\chi'\psi'}\frac{V'}{4\pi^2} - \frac{3\langle B^2\rangle}{4\chi'\psi' f_{\rm t}}\frac{V'}{4\pi^2}\int_0^1 d\lambda\,\frac{\lambda W(\lambda)}{\langle(1-\lambda B/B_{\rm M})^{1/2}\rangle}$$

$$\begin{aligned}W(\lambda) = &\sum_{\substack{(m,n)\\ \neq(0,0)}}\frac{Rm+Sn}{\chi' m-\psi' n}\left[\begin{array}{c}-2\dfrac{\partial\alpha_{mn}^{\rm (Boozer)}}{\partial\lambda}\left\langle\dfrac{|v_{//}|}{v}\cos(m\theta_{\rm B}-n\zeta_{\rm B})\right\rangle+\\ f_{\rm c}^{-1}\dfrac{\langle B^2\rangle}{B_{\rm M}^2}\cos(m\theta_{\rm M}^{\rm (Boozer)}-n\zeta_{\rm M}^{\rm (Boozer)})\left\{\dfrac{3}{2}\alpha_{mn}^{\rm (Boozer)}(\lambda=1)+d_{mn}^{\rm (Boozer)}\right\}\end{array}\right]\\ &+\frac{\chi'\psi'}{\langle B^2\rangle}\frac{4\pi^2}{V'}\sum_{\substack{(m,n)\\ \neq(0,0)}}\frac{B_\zeta m+B_\theta n}{\chi' m-\psi' n}\left[\begin{array}{c}-2\dfrac{\partial\beta_{mn}}{\partial\lambda}\left\langle\dfrac{|v_{//}|}{v}\cos(m\theta_{\rm B}-n\zeta_{\rm B})\right\rangle+\\ f_{\rm c}^{-1}\dfrac{\langle B^2\rangle}{B_{\rm M}^2}\cos(m\theta_{\rm M}^{\rm (Boozer)}-n\zeta_{\rm M}^{\rm (Boozer)})\left\{\dfrac{3}{2}\beta_{mn}(\lambda=1)+e_{mn}\right\}\end{array}\right]\end{aligned} \tag{4}$$

Here, $R\equiv\chi'(1-H_2)/2$ and $S\equiv\psi'(1+H_2)/2$, respectively, and $(\theta_{\rm M}, \zeta_{\rm M})$ are the poloidal and toroidal coordinates giving the maximum value of the magnetic field strength on the flux surface as $B(\theta_{\rm M}, \zeta_{\rm M})=B_{\rm M}$. Fourier coefficients $\alpha_{mn}(\lambda)$, $\beta_{mn}(\lambda)$, d_{mn}, e_{mn} are defined in Ref.[1]. When we replace $\chi'\psi'(B_\zeta m+B_\theta n)/(\langle B^2\rangle V'/4\pi^2)$ by $(\chi' m+\psi' n)/2$, Eq.(4) coincides with the result obtained by Shaing, Carreras, et al.[1] who started from Eq.(2). The mathematical equivalence of Eq.(2) and Eq.(3), and that of Eq.(4) and result in Ref.[1] can be easily confirmed by using a relation $B_\zeta\psi' + B_\theta\chi' = \langle B^2\rangle V'/4\pi^2$ and the definition of β_{mn}, e_{mn}. In actual applications, however, these Fourier series must be truncated to be series with finite modes. By the relations $|\chi' m| << |\psi' n|$ and $|B_\zeta m| >> |B_\theta n|$ in many practical cases, it can be understood that the expressions in Ref.[1] contain many non-axisymmetric Fourier components $n\neq0$ that are canceled by each others in Σ, while those in Eqs.(3), (4) are removed from the part using the series expression Σ in advance of the final expression Eq.(4). Therefore Eq.(4) is more useful in many practical applications. This technique was already used in studies in Ref.[5], and the validity is confirmed there by comparing with a direct numerical calculation of the drift kinetic equation.

References

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