Content & Watkins' Account of Natural Axiomatizations

In *Science and Skepticism*, John Watkins ([1984], p. 208-9) canvassed the following set of conditions for defining what counts as a natural axiomatization of a theory:

1. Independence requirement: each axiom in the axiom set must be logically independent of the conjunction of the others.
2. Non-redundancy requirement: no predicate or individual constant may occur inessentially in the axiom set.
3. Segregation requirement: if axioms containing only theoretical predicates can be separately stated, without violating other rules, they should be.
4. Wajsberg's Requirement: an axiom is impermissible if it contains a (proper) component that is a theorem of the axiom set, or becomes one when its variables are bound by the quantifiers that bind them in the axiom.
5. Decomposition requirement: if the axiom set can be replaced by an equivalent one that is more numerous (though still finite) without violating the preceding rules, it should be.

To see the import of Wasjberg's requirement consider that theory (set of sentences) which can be axiomatized by

A1*: (x)(Px→Tx)
A2*: (x)(Px→Tx) → (x)(Tx→Qx).

Wasjberg's requirement stops the set {A1*, A2*} counting as a natural axiomatization since A2* contains as a proper component, namely '(x)(Px→Tx)', which is a theorem of the relevant theory. The desired natural axiomatization here is the set containing the two members

A1: (x)(Px→Tx)
A2: (x)(Tx→Qx).

Unfortunately Wasjberg's requirement bars this from counting as a natural axiomatization. To see this consider the following reaxiomatization, due to Graham Oddie ([1989), Cf. the propositional version of this case on p. 347), of the theory captured by {A1, A2},

A1+: (x)((Px&Qx) →Tx)
A2+: (x)((Px&~Qx)→~Tx)
A3+: (x)((~Px&~Qx) →~Tx)
A4+: (x)((Px&~Qx) →Tx)

{A1, A2} is logically equivalent to {A1+, A2+, A3+, A4+}, so, given the decomposition requirement, {A1, A2} does not count as a natural axiomatization.

Elie Zahar has suggested a revision of Wasjberg's requirement which effectively blocks this type of counterexample. Basically Zaher's requirement is a near equivalent to the following:

4a. Wajsberg-Zaher Requirement: an axiom A is impermissible if there is some logical equivalent A' which contains only propositional functions occurring in A, and A' contains a (proper) component that is a theorem of the axiom set, or becomes one when its variables are bound by the quantifiers that bind them A'.[1]

4a, like 4, blocks A2* as being part of any natural axiomatization of any theory that has A1 as a theorem. Furthermore, 4a, unlike 4, rules out A1+ as part of any natural axiomatization of any theory that has A1 as a theorem. To see this note that A1+ is logically equivalent to

A1++: (x)(Qx→ (Px→Tx))

and A1++ contains only propositional functions occurring in A1+ and A1++ contains a proper component, namely '(Px→Tx)', which when bound by the appropriate quantifier, is a theorem of any axiom set, such as {A1+, A2+, A3+, A4+}, that has A1 as a theorem.

Unfortunately, both the requirements 4 and 4a rule out

C1: (∃x)(Px&Qx)

as being part of any natural axiomatization, since C1 itself contains a component, namely 'Px' which when bound by the quantifier binding it in C1 yields a theorem of any axiom set that has C1 as a theorem. This means, for instance, that the theory whose sole axiom is C1 has no natural axiomatization. Axioms such as C1 are an integral part of many theories. For instance, theories aimed at explaining various astronomical phenomena sometimes contain an axiom to the effect that there exists a body in a such and such a region having thus and thus degree of gravitational pull. Such axioms are needed to explain anomalies in the orbits of various bodies and to explain the apparent phenomena of twin quasars.

Also 4a rules out the following as a natural axiomatization

B1: (x)((Px&Qx) →Tx)
B2: (x)(Tx→Qx)

since B1 is logically equivalent to

B3: (x)(((Px&Qx)→Tx)&((Tx→Qx)v(Tx))

[1] The requirement actually proposed in Zaher [1991] is a good deal more complicated than 4a, especially when applied to certain quantified formulae, and is not extensionally equivalent to 4a. However this does not matter for our purposes since the counter-example given in this paper applies equally to both 4a and Zaher's actual proposal.

which contains a proper component, namely '(Px→Tx)', which when bound by the appropriate quantifier, is a theorem of the axiom set {B1, B2 }.[2]

Before proposing an alternative to 4 and 4a it is helpful to consider exactly what counts as part of the content of a theory. Consider again the theory which may be axiomatized by

A1*: (x)(Px→Tx)
A2*: (x)(Px→Tx) → (x)(Tx→Qx).

Note, that if we count A2* as a content part of this theory then on the evidence of 'Pa&~Ta' we would have to say that part of the theory has been conclusively confirmed since 'Pa&~Ta' deductively entails A2*. In other words, the theory that is naturally axiomatized by

A1: (x)(Px→Tx)
A2: (x)(Tx→Qx)

is partially confirmed by 'Pa&~Ta'! This result seems simply monstrous. The lesson here is that not every consequence of a theory should count as part of the theory's content. The content parts of the theory that may be naturally axiomatized by A1 and A2 (or unnaturally axiomatized by A1* and A2*) should include A1 and A2 but not A2*. In other words, not every axiom of any axiomatization of a theory should count as part of the theory's content.[3]

Here is an alternative account of content

α is part of the content of β=df. α is a logical consequence of β and there is no stronger consequence of β all of whose non-logical vocabulary occurs in α.][4]

We say σ is stronger than α iff σÃαand αÃ/σ.

I propose we replace Wajsberg's requirement with the following

4b Content requirement: Every axiom must be a content part of the theory.

4b eliminates Oddie's counter-example since none of A1+ - A4+ count as content parts of

---

[2] 4a might be altered to avoid this last result. For instance, it might require that A' have no shorter logical equivalent.
[3] For more on this see Gemes [1994].
[4] Close relations of this notion of content are examined in Gemes [1994a] and [1997]. The key difference is that the definitions proposed in those places, unlike the one here, are all closed under logical equivalence.

the relevant theory. For instance, A1+, that is, '(x)((Px&Qx→Tx)',does not count as part of the content of any theory that has '(x)(Px→Tx)' as a theorem since '(x)(Px→Tx)' is stronger than '(x)((Px&Qx→Tx)', and all the non-logical vocabulary in '(x)(Px→Tx)' occurs in '(x)((Px&Qx)→Tx)'. 4b also eliminates A2*, that is, '(x)(Px→Tx) → (x)(Tx→Qx)', as part of any natural axiomatization of any theory that has '(x)(Tx→Qx)' as a theorem since '(x)(Tx→Qx)' is stronger than '(x)(Px→Tx) → (x)(Tx→Qx)' and all of the non-logical vocabulary of '(x)(Tx→Qx)' occurs in '(x)(Px→Tx) → (x)(Tx→Qx)'.

Elsewhere, in Gemes [1994], I have given an alternative account of the notion of natural axiomatization. That account, unlike that of Watkins, would allow the set whose sole member is

A!: (x)(Px→Tx)&(x)(Tx→Qx)

to count as a natural axiomatization. Furthermore, it would allow the set whose sole members are

A1?: (x)(Px→ (Tx&(Rxv~Rx)))
and
A2?: (x)(Tx→Qx)

to count as a natural axiomatization. In other words, the definition of natural axiomatization in Gemes [1994] contained no analog of Watkins' decomposition requirement (which rules out A1!) or of Watkins' non-redundancy requirement (which rules out A1?).[5] However, while Watkins' Non-redundancy requirement rules out A1? it does not rule out

A1%: (x)(Px→ (Tx&(Qxv~Qx).
A2%: (x)(Tx→Qx)

The trouble here is that while 'Q' is non essential to A1% it is essential to other axioms of the theory. Perhaps the non-redundancy requirement should read as follows,

2a. Non-redundancy requirement: no predicate or individual constant may occur inessentially in any axiom.

Gemes [1994] is aimed at providing an account of hypothetico-deductivism and the notion of natural axiomatization utilized there adequately serves that purpose and is somewhat simpler than the notion captured by Watkins conditions. However, for the very

---

5 To be fair, we should note that Gemes ([1994, p.483, note 3) points out the something like Watkins' decomposition requirement would be needed to construct a notion of natural axiomatization serviceable for various ends other than that which is the focus of Gemes [1994].

significant purposes Watkins is pursuing *in Science and Skepticism* the notion of natural axiomatization captured by conditions 1, 2a, 3, 4b, and 5 serves admirably.[6]

[6] Thanks are due to John Watkins who provided valuable input for this piece over e-mail and lunch.

References

Gemes, K. [1997], "A New Theory of Content II: Model Theory and Some Alternatives," forthcoming in *Journal of Philosophical Logic*.

Gemes, K. [1994], "Hypothetico-Deductivism, Content, and The Natural Axiomatization of Theories," *Philosophy of Science*, 60, pp. 477-487.

Gemes, K. [1994a], "A New Theory of Content I: Basic Content," *Journal of Philosophical Logic*, 23, pp. 96-620.

Oddie, G. [1989]: "The Unity of Theories" in F. D'Agostino and I. Jarvie (eds.), *Freedom and Rationality: Essays in Honor of John Watkins*, Dordrecht: Kulwer, 1989, pp.343-68.

Watkins, J. [1984]: *Science and Scepticism*, London: Hutchinson & Co.

Zaher, E. [1989]: Natural Axiomatization: a Revision of 'Wajsberg's Requirement', *British Journal for the Philosophy of Science*, 42, pp. 391-396.

Endnotes

 The requirement actually proposed in Zaher [1991] is a good deal more complicated than 4a, especially when applied to certain quantified formulae, and is not extensionally equivalent to 4a. However this does not matter for our purposes since the counter-example given in this paper applies equally to both 4a and Zaher's actual proposal.
 For more on this see Gemes [1994].
 Close relations of this notion of content are examined in Gemes [1994a] and [1997]. The key difference is that the definitions proposed in those places, unlike the one here, are all closed under logical equivalence.
 To be fair, we should note that Gemes ([1994, p.483, note 3) points out the something like Watkins' decomposition requirement would be needed to construct a notion of natural axiomatization serviceable for various ends other than that which is the focus of Gemes [1994].
 Thanks are due to John Watkins who provided valuable input for this piece over e-mail and lunch.