§37. Complete Set of Eigenfunctions for Vlasov Equation in Multispecies Plasmas

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The complete set of eigenfunctions for the Vlasov equation has been derived only for the electron plasma. The complete set consists of the discrete mode determined from the dispersion relation, and the continuum contribution. The former corresponds to the collective mode, while the latter corresponds to the individual or beam mode. From the view point of particle transport process, the former corresponds to the diffusion process, while the latter to the free streaming or burst and intermittence. The spectral decomposition of the transport equation, thus, gives an important information on physical processes.

Our purpose here is to constructed the complete set of eigenfunctions for the Vlasov equation in mutispecies plasmas including both electron and ion dynamics in the electrostatic approximation.

We consider the Vlasov equation for the j-th species charged particles perturbed distribution function f_j in the electrostatic approximation:

$$\frac{\partial f_j}{\partial t} + v.\nabla f_j + \left(\frac{e}{m}\right)_j E.\frac{\partial f_{0j}}{\partial v} = 0, \qquad (1)$$

where **E**=$-\nabla\phi$ is the perturbed electric field with ϕ being the perturbed scalar potential, e_j and m_j are charge and mass, and f_{oj} is the unperturbed distribution for the j-th species particle. We must solve eq.(1) combining with the Poisson equation:

$$\nabla^2\phi = -4\pi\sum_j e_j \int d^3v f_j(r,v,t). \qquad (2)$$

We assume for the perturbed quantities f_j and ϕ are proportional to the Fourier factor exp(**ik**.**v**). Combining with eq.(2), eq.(1) can be written in the form of the charge density:

$$\frac{\partial\rho}{\partial t} + ikv\rho = -ik\alpha\int_{-\infty}^{\infty} dv\rho(v,t), \qquad (3)$$

where the charge density ρ is defined by

$$\rho(v,t) = \sum_j e_j g_j(v), \qquad (4)$$

and g_j is the integrated distribution function given by

$$g_j(v) = \int f_j(v,v_\perp)dv_\perp. \qquad (5)$$

The coefficient α is defined by

$$\alpha = \sum_j e_j\alpha_j$$

where

$$\alpha_j(v) = -\frac{4\pi e_j}{m_j k^2}\frac{\partial}{\partial v}\int f_{0j}(v,v_\perp)dv_\perp.$$

Since eq.(3) for the charge density ρ is the same form as the equation for the distribution function g treated by Case,[1] we applied the same method. We assume the solution of the form $\rho \sim \exp(-i\omega t)$, and introduce the parameter $\mu = \omega/k$. Then we have the eigenfunction

$$\rho_{\mu_i}(v) \equiv \rho_i(v) = \frac{\alpha(v)}{\mu_i - v}. \qquad (6)$$

for the discrete eigenvalue μ_i determined by the dispersion relation:

$$\varepsilon(\mu) \equiv 1 - \int_{-\infty}^{\infty}\frac{\alpha(v)}{\mu - v}dv = 0. \qquad (7)$$

For $\mu \in \Sigma$, we have the singular eigenfucntion:

$$\rho_\mu(v) = P\frac{\alpha(v)}{\mu - v} + \lambda(\mu)\delta(\mu - v), \qquad (8)$$

where the coefficient λ is given by

$$\lambda(\mu) \equiv 1 - P\int_{-\infty}^{\infty}\frac{\alpha(v)}{\mu - v}dv. \qquad (9)$$

The set of eigenfunctions $\{\rho_{\mu i},\rho_\mu\}$ is complete has been proved by using the orthogonality relations:[2]

$$\langle\rho_\mu,\rho^*_{\mu'}\rangle = 0 \quad \text{for} \quad \mu\neq\mu'$$

$$\langle\rho_\mu,\rho^*_{\mu'}\rangle = C_\mu\delta(\mu-\mu'), \quad \text{for } \mu, \mu' \in\Sigma,$$

where ρ^* is the adjoint eigenfunction and $C_\mu = \varepsilon^+(\mu)\varepsilon^-(\mu)/\alpha(\mu)$.

Making use of the complete set of eigenfunctions, the velocity dependent charge density has been given by[2]

$$\rho(x,v,t) = \sum_j a_j e^{ik(x-\mu_j t)}\frac{\alpha(v)}{\mu_i - v} + \lambda(v)A(v)e^{ik(x-vt)} + P\int\frac{\alpha}{\mu - v}A(\mu)e^{ik(x-\mu t)}d\mu$$

The first discrete mode term indicates singularity at $v=\mu_i$. The second term is the ballistic mode, and the third represents the interaction between the ballistic mode and cloud of particles trapped by the shielded potential. Integrating ρ over v, we have the perturbed scalar potential:

$$\phi(x,t) = \frac{4\pi}{k^2}\rho_0\left\{\sum_j \frac{e^{ik(x-\mu_j t)}}{\varepsilon'(\mu_j)}\frac{\alpha(v_0)}{\mu_j - v_0} + \frac{\lambda(v_0)e^{ik(x-v_0 t)}}{\lambda^2(v_0)+\pi^2\alpha^2(v_0)} + P\int\frac{e^{ik(x-\mu t)}}{\mu - v_0}\frac{\alpha(\mu)d\mu}{\lambda^2(\mu)+\pi^2\alpha^2(\mu)}\right\}$$

The first discrete mode term also has the singularity when the eigenvalue coinsides with the inital beam velocity, $\mu_i = v_0$. Distribution function f_j for the j-th species can be obtained making use of the complete set of the eigenfunctions $\{\rho_{\mu i},\rho_\mu\}$ by the same manner. [2]

1)K.M.Case and P.F.Zweifel, Annals of Physics, 7(1959)349.
2) T.Yamagishi, Report NIFS-578 (1998).