§42. Characteristic Exponents for Particle Transport Processes

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Particle transport process may be regarded as the series of elementary events which consist of collision and subsequent free streaming in a medium. Depending on the scattering law, each elementary event may generate displacements in phase space. The series of the elementary events, therefore, generate the series of the displacements which can be considered as randomvariables. The particle transport problem may be solved if the probability distribution for the total displacement is obtained. Summing this probability distribution for all collision numbers may give the solution of the Boltzmann equation.[1]

Since the displacement is the random variable in stochastic process, this theory of particle random walk is found to be the same as the recent theory of nonlinear dynamical systems in which the random variable is generated by the nonlinear mapping. Some analytical results in the random walk model is also found to be applicable for analytical derivation of the characteristic expoment which is important to determine the stability and to evaluate the fractal dimension in the nonlinear system.[2]

First we present the simplest concrete example in neutron slowing down process. When neutron suffers a collision with atom, its energy may change from E_1 to E_2. In term of lethergy defined by $u=\log(E_1/E_2)$, the probability distribution function for the lethergy displacement has been given by[3]

$$f(u) = \frac{\exp(-u)}{1-\alpha} \quad \text{for } 0<u<q_M, \qquad (1)$$

where $q_M = -\ln\alpha$ with $\alpha=((A-1)/(A+1))^2$ and A being atomic mass. The characteristic function equivalent to the partition function for this case is obtained in the form

$$\chi(k) = \frac{1}{1-\alpha}\,\frac{1-\exp\{-(1+ik)q_M\}}{1+ik}. \qquad (2)$$

The probability distribution after n collisions is obtained by the inverse Fourier transformation:

$$P_n(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \exp(iku)\chi_0(k)\left\{\frac{1-\exp\{-(1+ik)q_M\}}{(1+ik)(1-\alpha)}\right\}^n, \qquad (3)$$

where c_0 is the characteristic function for the neutron source distribution. If the source distribution is monochromatic, $f_0(u)=\delta(u)$, then $c_0=1$, and making use of the residue theorem, Eq.(3) yields the exact solution:

$$P_n(u) = \frac{\exp(-u)}{(1-\alpha)^n(n-1)!}\sum_{j=0}^{m}\binom{n}{j}(-1)^j(u-jq_M)^{n-1}, \qquad (4)$$

where m is the maximum number of j which keep $u-jq_M$ positive. For n>>1, Eq.(4) can be expressed in the Gaussian form:

$$P_n(u) = \frac{\exp(-u)q_M^{n-1}}{(1-\alpha)^n}\sqrt{\frac{6}{n\pi}}\exp\left\{-\frac{6}{nq_M^2}\left(u-\frac{nq_M}{2}\right)^2\right\}. \qquad (5)$$

Applying Eq.(2) and replacing ik by -q, the similarity exponent equivalent to the Helmholtz free energy is given by

$$\lambda_q = \frac{1}{q}\ln\frac{1-\exp\{-(1-q)q_M\}}{(1-q)(1-\alpha)}. \qquad (6)$$

Equation (6) looks singular at q=0 and 1. At these points, λ_q converges and not singular:

$$\lambda_0 = 1-\frac{\alpha^2}{1-\alpha}\ln\alpha \qquad (7)$$

$$\lambda_1 = \ln\left(\frac{-\ln\alpha}{1-\alpha}\right) \qquad (8)$$

In the extreme limits $q\to\pm\infty$, λ_q tends to $\lambda_\infty=-\ln\alpha$ and $\lambda_{-\infty}=0$. The exponent λ_q monotonically varies between these limits as presented in Fig. 1 in which λ_q is shown for three values of the atomic mass A for comparison. For A=1 (hydrogen),

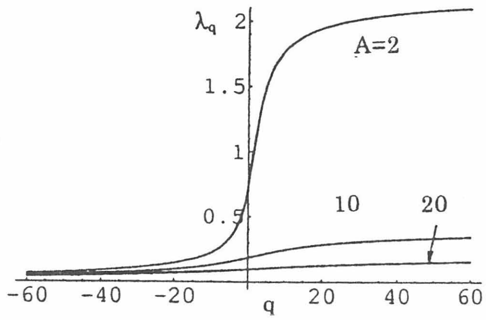

Fig. 1 Variation of similarity exponent versus q

q_M tends to infinity, and λ_q becomes singular and does not show the standard form as in Fig.1.

For the case of two-dimensional random walk, the probability density is given by

$$f(x,\mu) = \frac{\Sigma_s}{2|\mu|}\exp\left(-\frac{\Sigma}{\mu}x\right), \quad (|x|<L) \qquad (9)$$

where Σ, Σ_s and μ are the total and scattering cross sections, and the directional cosine, respectively. For $\mu=1$(one dimensional case) and $\Sigma=\Sigma_s=1$(no absorption), the characteristic function is given in the form:

$$\chi(q) = \frac{1}{2}\left(\frac{1-e^{-(1-q)L}}{1-q}+\frac{1-e^{-(1+q)L}}{1+q}\right), \qquad (10)$$

and the characteristic exponent becomes

$$\lambda_q = \frac{1}{q}\log\left\{\frac{1}{2}\left(\frac{1-e^{-(1-q)L}}{1-q}+\frac{1-e^{-(1+q)L}}{1+q}\right)\right\}, \qquad (11)$$

which also shows the standard form as in Fig.1.

In the case of plasmas, the characteristic function and exponent may be derived making use of the propagator, which is remained to be seen.

1) T.Yamagishi and T.Sekiya : J. Nucl. Sci. Technol. 5 (1968)614.

2) Fujisaka and Inoue; Prog. Theor. Phys. 74 (1985) 20.

3) T.Yamagishi; Memoir of Tokyo Metro. Inst. Technol. 13 (1999)45.