Transport processes and entropy production in toroidal plasmas with gyrokinetic electromagnetic turbulence

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Transport processes and resultant entropy production in magnetically confined plasmas are studied in detail for toroidal systems with gyrokinetic electromagnetic turbulence. The kinetic equation including the turbulent fluctuations are double averaged over the ensemble and the gyrophase. The entropy balance equation is derived from the double-averaged kinetic equation with the nonlinear gyrokinetic equation for the fluctuating distribution function. The result clarifies the spatial transport and local production of the entropy due to the classical, neoclassical and anomalous transport processes, respectively. For the anomalous transport process due to the electromagnetic turbulence as well as the classical and neoclassical processes, the kinetic form of the entropy production is rewritten as the thermodynamic form, from which the conjugate pairs of the thermodynamic forces and the transport fluxes are identified. The Onsager symmetry for the anomalous transport equations is shown to be valid within the quasilinear framework. The complete energy balance equation, which takes account of the anomalous transport and exchange of energy due to the fluctuations, is derived from the ensemble-averaged kinetic equation. The intrinsic ambipolarity of the anomalous particle fluxes is shown to hold for the self-consistent turbulent electromagnetic fields satisfying Poisson's equation and Ampère's law. © *1996 American Institute of Physics.* [S1070-664X(96)03206-5]

I. INTRODUCTION

Plasma transport of particles and heat in magnetically confined toroidal systems consists of classical, neoclassical, and anomalous (or turbulent) processes. Both the classical and neoclassical transport[1–3] are caused by Coulomb collisions of particles, while the anomalous transport[4] results from turbulent fluctuations driven by various instabilities existing in confined plasmas.

On the collisional transport, the classical process involves particle gyromotion while the neoclassical process is concerned with guiding-center drift motion in toroidal magnetic configurations. For the entropy production due to the classical transport, its kinetic form defined by the collision operator is equivalent to its thermodynamic form written as an inner product of thermodynamic forces and their conjugate transport fluxes.[3,5] Also, due to the spatial locality of the process, the Onsager symmetry[6] of the classical transport matrix is directly derived from the self-adjointness of the linearized collision operator.[3,5] On the other hand, since the long mean-free path of the guiding-center motion is involved in the neoclassical process, the neoclassical fluxes are defined through magnetic surface average, and the neoclassical transport matrix contains parameters relating to both the collisionality and the magnetic geometry. Thus, only by taking a magnetic surface average of the kinetic form of the neoclassical entropy production, we can derive the thermodynamic form, from which conjugate pairs of the thermodynamic forces and the neoclassical fluxes are rigorously identified.[5] Then, the Onsager symmetry is shown to be robustly valid for the neoclassical transport equations connecting the conjugate pairs even in nonaxisymmetric magnetic configurations.[5]

Compared to the classical and neoclassical processes, it is rather difficult to analyze the anomalous transport process because of its nonlinearity even for more simplified configurations, and extensive theoretical and experimental studies have been performed so far.[4] However, most theoretical works on the anomalous transport have been done separately from the neoclassical transport theory, except for works by Shaing,[7,8] Balescu,[9] and by Sugama and Horton,[10] which synthesize both the neoclassical and anomalous transport theories. These synthesized theories depend on how to formulate the neoclassical and anomalous parts of the total transport fluxes. In the works by Shaing[7,8] and by Balescu,[9] the separation of variables into the average and fluctuating parts is done at the level of the fluid momentum balance equations, while, in the theory by Sugama and Horton[10] as well as in the present work, the plasma kinetic equation is divided into the ensemble-averaged part and the fluctuating part, and the fluctuation-particle interaction operator $\mathscr{D}_a$ [defined by Eq. (6) in the next section] plays an important role in the linkage of these two parts. Shaing and Balescu define the anomalous fluxes from the fluctuating parts of the fluid variables and use Shaing's ansatz[7–9] for the kinetic distribution function including mixture of the potential fluctuations and the averaged flow. Owing to the use of the fluctuation–particle interaction operator and the standard drift or gyrokinetic equation[11–15] without Shaing's ansatz in our formula-

tion, we can define the anomalous fluxes in more compact forms analogous to the definition of the classical fluxes in terms of the collision term, and we succeed to define the entropy production rate kinetically even for the anomalous transport processes, which is not considered in the theories by Shaing and by Balescu. From the anomalous entropy production rate, we can clearly specify the conjugate pairs of the thermodynamic forces and the anomalous transport fluxes, which are shown to be connected with each other by the Onsager symmetric quasilinear transport matrix. In the present work, we extend the formalism by Sugama and Horton[10] to more general toroidal systems with nonaxisymmetric magnetic configurations[16–18] and gyrokinetic electromagnetic fluctuations, and give the complete description of the entropy and energy balance including all the transport processes, and examine the Onsager symmetry of the transport equations for each process.

In terms of the ordering parameter $\delta \equiv \rho_a/L$ (ρ_a: the thermal gyroradius, L: the equilibrium scale length), the gyrokinetic ordering employed here for the turbulent fluctuations is written as

$$\frac{\hat{f}_a}{f_a} \sim \frac{e_a \hat{\phi}}{T_a} \sim \frac{|\hat{\mathbf{B}}|}{|\mathbf{B}|} \sim \frac{k_\parallel}{k_\perp} \sim \frac{\omega}{\Omega_a} \sim \delta. \tag{1}$$

Here $\hat{f}_a/f_a$, $e_a\hat{\phi}/T_a$, and $|\hat{\mathbf{B}}|/|\mathbf{B}|$ are normalized fluctuations of the distribution function, the electrostatic potential, and the magnetic field strength, respectively, where the caret ($\hat{}$) represents the fluctuating part. The subscript a denotes the particle species and $\Omega_a = e_a B/m_a c$ is the gyrofrequency of the particle with the mass m_a and the charge e_a. The characteristic parallel (perpendicular) wave number and frequency for the turbulence are denoted by $k_\parallel (k_\perp)$ and ω, respectively. Assuming that $k_\parallel \sim L^{-1}$, we find from Eq. (1) that $k_\perp \rho_a \sim 1$ and $\omega \sim \omega_{*a} \sim \omega_{Ta} \sim v_{Ta}/L$, where ω_{*a}, ω_{Ta}, and v_{Ta} are the drift frequency, the transit frequency, and the thermal velocity, respectively.

According to the same formulation as in Ref. 10, the magnetic-surface-averaged radial particle and heat fluxes are given up to $\mathcal{O}(\delta^2)$ by

$$\begin{aligned}
\langle \mathbf{\Gamma}_a \cdot \nabla V \rangle &= \langle \mathbf{\Gamma}_a^{\rm cl} \cdot \nabla V \rangle + \langle \mathbf{\Gamma}_a^{\rm PS} \cdot \nabla V \rangle + \langle \mathbf{\Gamma}_a^{\rm bp} \cdot \nabla V \rangle \\
&\quad + \langle \mathbf{\Gamma}_a^{\rm na} \cdot \nabla V \rangle + \langle \mathbf{\Gamma}_a^{(E)} \cdot \nabla V \rangle + \langle \mathbf{\Gamma}_a^{\rm anom} \cdot \nabla V \rangle, \\
\langle \mathbf{q}_a \cdot \nabla V \rangle &= \langle \mathbf{q}_a^{\rm cl} \cdot \nabla V \rangle + \langle \mathbf{q}_a^{\rm PS} \cdot \nabla V \rangle + \langle \mathbf{q}_a^{\rm bp} \cdot \nabla V \rangle \\
&\quad + \langle \mathbf{q}_a^{\rm na} \cdot \nabla V \rangle + \langle \mathbf{q}_a^{\rm anom} \cdot \nabla V \rangle,
\end{aligned} \tag{2}$$

where $\langle \cdot \rangle$ denotes the magnetic surface average and the volume V inside the magnetic surface is used as a radial variable. Here the superscript ''cl,'' ''PS,'' ''bp,'' ''na,'' and ''anom'' represents the classical, Pfirsch–Schlüter, banana-plateau, nonaxisymmetric, and anomalous fluxes, respectively. In Appendix A, their definitions are given. Since, in the gyrokinetic ordering, the ensemble-averaged drift kinetic equation is not affected up to $\mathcal{O}(\delta)$,[10] all the neoclassical (Pfirsch–Schlüter, banana-plateau, and nonaxisymmetric) fluxes as well as the classical fluxes are given in terms of the thermodynamic forces by the same transport equations as in Ref. 5 for the cases with no turbulence. The anomalous particle and heat fluxes are defined by

$$\begin{aligned}
\mathbf{\Gamma}_a^{\rm anom} &\equiv \frac{c}{e_a B} (\mathbf{K}_{a1} \times \mathbf{n}), \\
\frac{1}{T_a} \mathbf{q}_a^{\rm anom} &\equiv \frac{c}{e_a B} (\mathbf{K}_{a2} \times \mathbf{n}),
\end{aligned} \tag{3}$$

respectively, where $\mathbf{K}_{a1}$ and $\mathbf{K}_{a2}$ are the anomalous forces given by

$$\begin{aligned}
\mathbf{K}_{a1} &\equiv \int d^3v \ \mathscr{D}_a m_a \ \mathbf{v}, \\
\mathbf{K}_{a2} &\equiv \int d^3v \ \mathscr{D}_a m_a \mathbf{v} \left(x_a^2 - \frac{5}{2} \right).
\end{aligned} \tag{4}$$

Here $x_a^2 \equiv m_a v^2/2T_a$ is the normalized kinetic energy and $\mathscr{D}_a$ is the fluctuation–particle interaction term contained in the ensemble-averaged kinetic equation [see Eq. (6) in Sec. II].

It is shown in Ref. 10 that the relative effects of the anomalous forces $\mathbf{K}_{aj}$ ($j=1,2$) on the parallel viscosities and accordingly on the neoclassical transport are measured by Δ^2/δ if we represent the order of the normalized fluctuations by $\hat{f}_a/f_a \sim e_a\hat{\phi}/T_a \sim \cdots \sim \Delta$ instead of the gyrokinetic ordering in Eq. (1). Then, the neoclassical banana-plateau fluxes and the bootstrap current are significantly modified by this coupling to the anomalous forces in the case where $\Delta \sim \delta^{1/2}$, as assumed in Ref. 10. On the other hand, in the present work, this modification of the expressions for the neoclassical transport fluxes does not occur in the dominant or lowest order since the gyrokinetic ordering $\Delta \sim \delta$ in Eq. (1) assures that the coupling effect is smaller by the order of $\Delta^2/\delta \sim \delta \ll 1$. Thus, the condition for the validity of the additive expressions for the neoclassical and anomalous transport without their coupling is estimated by $\Delta \ll \delta^{1/2}$.

From a microscopic point of view, for a single realization in the ensemble of the turbulent systems, the collision is the only irreversible process producing the entropy. The irreversibility or the positive entropy production due to the turbulent process is observed macroscopically by taking the ensemble average or coarse graining. Besides the ensemble average, the gyrophase average is also utilized to coarse grain the microscopic phase space, when the gyrokinetic fluctuations with frequencies much lower than the gyrofrequency are considered. Then, as shown later, the fluctuation–particle interaction operator $\mathscr{D}_a$ defines the anomalous transport fluxes and describes completely the entropy production due to the gyrokinetic electromagnetic turbulence. The entropy production allows the identification of the conjugate pairs of the anomalous fluxes and the forces.

It is a formidable task to give analytically the complete expressions for the anomalous transport equations that give the anomalous fluxes as complicated nonlinear functions of the forces. In addition to the nonlinear gyrokinetic equation,[15] the Poisson's equation and the Ampère's law are required for a self-consistent description of the fluctuations in the particle distributions. Such additional constraints are important for establishing the properties of the anomalous transport without obtaining directly the anomalous transport equations. The intrinsic ambipolarity for the anomalous particle fluxes will be derived from these properties. Then, we will find that the radial electric field in the axisymmetric

configuration is not determined by the ambipolarity condition, even in the presence of the anomalous transport.

Within the quasilinear framework, the anomalous transport coefficients are given as functionals of the turbulence spectrum. Thus, the quasilinear transport equations are also considered as implicitly nonlinear with respect to the forces. Since the Onsager symmetry is relevant to the linear thermodynamic transport equations,[6] its direct validity is questionable for the anomalous transport equations, even in the quasilinear version. However, as in Ref. 10, we will find that the Onsager symmetric matrix connects the conjugate pairs of the anomalous (quasilinear) fluxes and the forces in the gyrokinetic electromagnetic turbulence.

The rest of this work is organized as follows. In Sec. II, the Hazeltine recursion technique[19,20] is applied to the ensemble-averaged kinetic equation in order to obtain the gyrophase-averaged kinetic equation including effects of the electromagnetic fluctuations. The resultant equation contains $\mathcal{O}(\delta^2)$ terms, which are not contained in the conventional drift kinetic equation but necessary for a proper description of the entropy variation due to the classical and anomalous transport processes. In Sec. III, using the kinetic definition of entropy, the entropy balance equation is derived from the gyrophase-averaged kinetic equation obtained in Sec. II and the nonlinear gyrokinetic equation for the fluctuating distribution function. There, the spatial transport and local production of the entropy due to the classical, neoclassical, and anomalous transport processes are clearly described. For each transport process, the kinetic form of the entropy production is rewritten in the thermodynamic form, from which the conjugate pairs of the thermodynamic forces and the transport fluxes are identified. The entropy production due to the anomalous transport process is shown to balance with the collisional dissipation for the fluctuating microscopic distribution function, which agrees with the argument by Krommes and Hu on the entropy paradox.[21] In Sec. IV, the Onsager symmetry for the anomalous transport equations is shown to be valid within the quasilinear framework. Using the Krook collision model, the detailed expressions for the quasilinear anomalous transport coefficients are derived and the results, especially on the magnetic fluctuation effects on the anomalous transport, are compared to those in previous works. In Sec. V, from the ensemble-averaged kinetic equation, we obtain the complete energy balance equation for the cases, in which all of the classical, neoclassical, and anomalous transport processes are involved. That equation elucidates how the fluctuation effects on the energy balance should be expressed, which has been somewhat obscure in previous literatures. In Sec. VI, Poisson's equation, and Ampère's law are used for the self-consistent turbulent electromagnetic fields, and the intrinsic ambipolarity of the anomalous particle fluxes is shown to hold for the self-consistent fields. Finally, the conclusions and a discussion are given in Sec. VII.

II. GYROPHASE AVERAGE OF ENSEMBLE-AVERAGED KINETIC EQUATION

We start from an ensemble-averaged kinetic equation for species a:

$$\frac{\partial f_a}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla} f_a+\frac{e_a}{m_a}\left(\mathbf{E}+\frac{1}{c}\,\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial f_a}{\partial \mathbf{v}}=\langle C_a\rangle_{\mathrm{ens}}+\mathscr{D}_a, \tag{5}$$

where C_a is a collision term and $\mathscr{D}_a$ is a fluctuation–particle interaction term defined by

$$\mathscr{D}_a=-\frac{e_a}{m_a}\left\langle\left(\hat{\mathbf{E}}+\frac{1}{c}\,\mathbf{v}\times\hat{\mathbf{B}}\right)\cdot\frac{\partial\hat{f}_a}{\partial\mathbf{v}}\right\rangle_{\mathrm{ens}},$$
$$\hat{\mathbf{E}}=-\boldsymbol{\nabla}\hat{\phi}-\frac{1}{c}\frac{\partial\hat{\mathbf{A}}}{\partial t},\quad \hat{B}=\boldsymbol{\nabla}\times\hat{\mathbf{A}}. \tag{6}$$

Here $\langle\cdot\rangle_{\mathrm{ens}}$ denotes the ensemble average and we divided the distribution function (the electromagnetic fields) into the ensemble-average part f_a ($\mathbf{E},\mathbf{B},\Phi,\mathbf{A}$) and the fluctuating part $\hat{f}_a(\hat{\mathbf{E}},\hat{\mathbf{B}},\hat{\phi},\hat{\mathbf{A}})$.

Equation (5) is derived by taking the ensemble average of the kinetic equation containing the Coulomb collision term with both the electromagnetic fields and the distribution function regarded as turbulent or stochastic variables. It is shown in Ref. 22 that the averaged kinetic equation for the one-body distribution function $f(1)\equiv f(\mathbf{x}_1,\mathbf{v}_1,t)$ with the collision term and the collective interaction term similar to Eq. (5) also follows from an appropriate truncation of the Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy describing a turbulent plasma. In the second equation in the BBGKY hierarchy, the particle discreteness source term

$$S(1,2)\equiv-(e^2/mr_{12}^3)\mathbf{r}_{12}\cdot(\partial/\partial\mathbf{v}_1-\partial/\partial\mathbf{v}_2)f(1)f(2)(\mathbf{r}_{12}\equiv\mathbf{r}_1-\mathbf{r}_2)$$

causes the collisional interaction part of the two-body correlation function $g_2(1,2)$. Substitution of this part of g_2 into the first equation in the BBGKY hierarchy for the one-body distribution function $f(1)$ gives the collision operator. The residual part of g_2 resulting from plasma unstable modes describes the collective interaction and gives the term corresponding to our $\mathscr{D}_a$ when it is substituted into the one-body equation. Thus, the collision term C_a and the fluctuation–particle interaction term $\mathscr{D}_a$ in Eq. (5) follow from the corresponding parts of the two-body correlation g_2 produced by the discreteness source term $S(1,2)$ and by unstable modes in a turbulent plasma, respectively.

Hereafter, we derive the gyrophase-averaged kinetic equation from Eq. (5) by applying the recursion technique proposed by Hazeltine.[19,20] For this purpose, let us introduce the phase space variables $(\mathbf{x}',\epsilon,\mu,\xi)$ which is defined in terms of $(\mathbf{x},\mathbf{v})$ as

$$\mathbf{x}'=\mathbf{x},\quad \epsilon=\frac{1}{2}\,m_a v^2+e_a\Phi,\quad \mu=\frac{m_a v_\perp^2}{2B},$$
$$\mathbf{v}_\perp/v_\perp=\mathbf{e}_1\cos\xi+\mathbf{e}_2\sin\xi, \tag{7}$$

where $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{n}\equiv\mathbf{B}/B)$ are unit vectors that form a right-handed orthogonal system at each point, and $\mathbf{v}=v_\parallel\mathbf{n}+\mathbf{v}_\perp$ with $v_\parallel=\mathbf{v}\cdot\mathbf{n}$. The differential operator on the left-hand side of Eq. (5) is written as

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} + \frac{e_a}{m_a}\left(\mathbf{E} + \frac{1}{c}\,\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial}{\partial \mathbf{v}}$$

$$\equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}' + \dot{\epsilon}\,\frac{\partial}{\partial \epsilon} + \dot{\mu}\,\frac{\partial}{\partial \mu} + \dot{\xi}\,\frac{\partial}{\partial \xi}, \tag{8}$$

where, in the last line, the partial differentials are taken with $(t,\mathbf{x}',\epsilon,\mu,\xi)$ as independent variables, and $\boldsymbol{\nabla}' \equiv \partial/\partial \mathbf{x}'$ is defined. The fast gyrofrequency Ω_a is contained in $\dot{\xi}$ as $\dot{\xi} = -\Omega_a + \delta\dot{\xi}(\delta\dot{\xi}/\Omega_a \sim \delta)$, and we subtract the fast gyrofrequency from d/dt to define the following operator:

$$\mathscr{L} \equiv \frac{d}{dt} + \Omega_a\,\frac{\partial}{\partial \xi}. \tag{9}$$

Then, rewriting Eq. (5) in the phase space variables $(\mathbf{x}',\epsilon,\mu,\xi)$ and separating it into the average and oscillating parts with respect to the gyrophase angle ξ, we obtain

$$\overline{\mathscr{L}(\bar{f}_a + \tilde{f}_a)} = \langle \bar{C}_a \rangle_{\rm ens} + \bar{\mathscr{D}}_a, \tag{10}$$

$$\Omega_a\,\frac{\partial \tilde{f}_a}{\partial \xi} = \widetilde{\mathscr{L} f_a} - \langle \tilde{C}_a \rangle_{\rm ens} - \tilde{\mathscr{D}}_a, \tag{11}$$

where the average and oscillating parts in ξ are represented for an arbitrary function $F(\xi)$ as

$$\bar{F} \equiv \frac{1}{2\pi}\oint d\xi\, F, \quad \tilde{F} \equiv F - \bar{F}. \tag{12}$$

From Eq. (11), the gyrophase-dependent part of the ensemble-averaged distribution function is given to the lowest order in δ by

$$\tilde{f}_a^{(1)} = \frac{1}{\Omega_a}\int^{\xi} d\xi\, \widetilde{\mathscr{L} f_a}, \tag{13}$$

where the integration constant related to $\int^{\xi} d\xi$ is uniquely determined by the condition $\overline{\tilde{f}_a^{(1)}} = 0$. Substituting Eq. (13) into Eq. (10) with $\mathscr{D}_a$ dropped gives Hazeltine's original drift kinetic equation[19,20] for the case of no turbulence. Since, here, we are concerned with entropy productions due to both collisional and turbulent dissipations, we need to retain $\mathscr{D}_a$ in Eqs. (10) and (11) and also calculate $\tilde{f}_a$ up to $O(\delta^2)$. The solution of Eq. (11) up to $O(\delta^2)$ is written as $\tilde{f}_a = \tilde{f}_a^{(1)} + \tilde{f}_a^{(2)}$, where the $O(\delta^2)$ part is given by

$$\tilde{f}_a^{(2)} = \frac{1}{\Omega_a}\int^{\xi} d\xi [\widetilde{\mathscr{L} f_a^{(1)}} - C_a^L(\tilde{f}_a^{(1)}) - \tilde{\mathscr{D}}_a] \equiv \tilde{f}_a^H + \tilde{f}_a^C + \tilde{f}_a^A. \tag{14}$$

Here $\tilde{f}_a^C$ and $\tilde{f}_a^A$ causes classical and turbulent (or anomalous) dissipation terms, respectively, when they are substituted into Eq. (10). On the other hand, $\tilde{f}_a^H$ gives only the higher-order small corrections to the drift orbit, and it is neglected hereafter since it is not related to any dissipations. Then, using Eqs. (10) and (14), the double-averaged kinetic equation over the statistical ensemble and the gyrophase angle ξ is written as

$$\overline{\mathscr{L}(\bar{f}_a + \tilde{f}_a^{(1)})} \equiv \left(\frac{\partial}{\partial t} + \mathbf{v}_{\rm gc}\cdot\boldsymbol{\nabla}' + \dot{\epsilon}_{\rm gc}\,\frac{\partial}{\partial \epsilon} + \dot{\mu}_{\rm gc}\,\frac{\partial}{\partial \mu}\right)\bar{f}_a$$

$$\equiv \left(\frac{d\bar{f}_a}{dt}\right)_{\rm gc} = \langle \bar{C}_a \rangle_{\rm ens} + \bar{\mathscr{D}}_a - \overline{\mathscr{L}(\tilde{f}_a^C + \tilde{f}_a^A)}, \tag{15}$$

where detailed expressions for the guiding center motion $(\mathbf{v}_{\rm gc}, \dot{\epsilon}_{\rm gc}, \dot{\mu}_{\rm gc})$ are given in Ref. 19. The terms in the last line, except for the first one, are $O(\delta^2)$ and not included in Ref. 19, although they are necessary for deriving complete expressions for the collisional and anomalous entropy productions and transport.

III. ENTROPY BALANCE EQUATION

Let us define the kinetic form of the entropy per unit volume for species a in terms of the ensemble-averaged distribution function f_a as

$$S_a \equiv -\int d^3v\; f_a \ln f_a$$

$$= -\int d^3v\; \bar{f}_a \ln \bar{f}_a + \mathscr{O}(\delta^2), \tag{16}$$

where $\tilde{f}_a/\bar{f}_a = \mathscr{O}(\delta)$ is used. If we use the total distribution function $f_a + \hat{f}_a$ to define the entropy, only the collisional processes produce that entropy while it is preserved, even in the presence of the turbulent transport for the collisionless case. By using the definition of Eq. (16) in which the microscopic turbulent processes are coarse grained, we can formulate the positive definite entropy production caused by the turbulent or anomalous transport.

In order to obtain the entropy balance equation, let us multiply Eq. (15) by $-(\ln \bar{f}_a + 1)$ and integrate it over the velocity space. First, we consider the contribution from collisions to the entropy balance equation, which is represented by

$$\dot{S}_a^C \equiv -\int d^3v (\ln \bar{f}_a + 1)(\langle \bar{C}_a \rangle_{\rm ens} - \overline{\mathscr{L}\tilde{f}_a^C})$$

$$\simeq -\int d^3v (\ln f_{aM}) C_a(f_a) - \int d^3v\; \frac{\bar{f}_a^{(1)}}{f_{aM}}\, C_a^L(\bar{f}_a^{(1)})$$

$$+ \int d^3v (\ln \bar{f}_a + 1)\overline{\mathscr{L}\tilde{f}_a^C}$$

$$\equiv \int d^3v\; \frac{m_a v^2}{2T_a}\, C_a(f_a) + \sigma_a^{\rm ncl} + \dot{S}_a^{\rm cl}. \tag{17}$$

Here the first term on the right-hand side is rewritten as

$$\int d^3v\; \frac{m_a v^2}{2T_a}\, C_a(f_a) = \frac{1}{T_a}\,(Q_a + \mathbf{u}_a\cdot\mathbf{F}_{a1}), \tag{18}$$

where $Q_a \equiv \int d^3v\, \frac{1}{2} m_a (\mathbf{v}-\mathbf{u}_a)^2 C_a(f_a)$ and $\mathbf{F}_{a1} \equiv \int d^3v\; m_a \mathbf{v} C_a(f_a)$ represent the collisional heat and momentum generation rates, respectively. The energy conservation in collisions requires

$$\sum_a (Q_a + \mathbf{u}_a \cdot \mathbf{F}_{a1}) = 0. \tag{19}$$

The entropy variation $\sigma_a^{\rm ncl}$ caused by the $O(\delta)$ deviation $\bar{f}_a^{(1)}$ from f_{aM} is related to the neoclassical transport processes, as shown in detail in Ref. 5. Taking the species summation of the flux surface average of $\sigma_a^{\rm ncl}$ multiplied by T_a, we obtain the thermodynamic form of the entropy production due to the neoclassical transport:

$$\sum_a T_a \langle \sigma_a^{\rm ncl} \rangle = \sum_a (J_{a1}^{\rm ncl} X_{a1} + J_{a2}^{\rm ncl} X_{a2}) + J_E X_E, \tag{20}$$

where the thermodynamic forces (X_{a1}, X_{a2}, X_E) and the neoclassical transport fluxes $(J_{a1}^{\rm ncl}, J_{a2}^{\rm ncl}, J_E)$ are defined in Appendix A. In Ref. 5 it is shown from the self-adjointness and the positive definiteness of the linearized collision operator that these fluxes and forces are connected to each other by the neoclassical transport matrix with the Onsager symmetry and that $\Sigma_a T_a \langle \sigma_a^{\rm ncl} \rangle \geqslant 0$ holds. After some calculations, we find that the entropy variation $\dot{S}_a^{\rm cl}$ defined from $\tilde{f}_a^C$ is rewritten as

$$\dot{S}_a^{\rm cl} = -\boldsymbol{\nabla} \cdot \mathbf{J}_{Sa}^{\rm cl} + \sigma_a^{\rm cl}, \tag{21}$$

where $\mathbf{J}_{Sa}^{\rm cl}$ is the entropy flux due to the classical flows $\mathbf{u}_a^{\rm cl} \equiv \boldsymbol{\Gamma}_a^{\rm cl}/n_a$ and $\mathbf{q}_a^{\rm cl}$ (see Appendix A for their definitions), and is given by

$$\mathbf{J}_{Sa}^{\rm cl} = S_a \mathbf{u}_a^{\rm cl} + \frac{1}{T_a} \mathbf{q}_a^{\rm cl}, \tag{22}$$

and $\sigma_a^{\rm cl}$ is given in the thermodynamic form of the entropy production due to the classical particle and heat transport as

$$\sigma_a^{\rm cl} = \frac{1}{T_a} (J_{a1}^{\rm cl} X_{a1} + J_{a2}^{\rm cl} X_{a2}). \tag{23}$$

Thus, $\dot{S}_a^{\rm cl}$ consists of the entropy transport term and the entropy production term, both of which result from the classical particle and heat transport. The classical particle and heat fluxes $(J_{a1}^{\rm cl}, J_{a2}^{\rm cl})$ defined in Appendix A are also related to the thermodynamic forces, (X_{a1}, X_{a2}) with the classical transport matrix with the Onsager symmetry. It is also shown that $\Sigma_a T_a \sigma_a^{\rm cl} \geqslant 0$.

Next, let us consider the contribution from turbulent fluctuations to the entropy balance equation, which is represented by

$$\dot{S}_a^A \equiv -\int d^3v (\ln \bar{f}_a + 1)(\bar{\mathscr{D}}_a - \overline{\mathscr{L}\tilde{f}_a^A}). \tag{24}$$

In order to rewrite $\dot{S}_a^A$ in a physically understandable form, the information for the fluctuating part $\hat{f}_a$ of the distribution function in the turbulent electromagnetic fields is required. We assume that any fluctuating field $\hat{F}$ oscillates rapidly in the directions perpendicular to the magnetic field lines, with a characteric scale length $\lambda_\perp \sim \rho_a$. Then, it is useful to put fluctuating functions in the WKB (or eikonal) form:

$$\hat{F}(t, \mathbf{x}', \epsilon, \mu, \xi) = \hat{F}(t, \mathbf{x}', \epsilon, \mu, \xi; \mathbf{k}_\perp) \exp\left(i \int^{\mathbf{x}'} \mathbf{k}_\perp \cdot d\mathbf{x}' \right), \tag{25}$$

where the rapid spatial variation in the perpendicular directions is included through the eikonal $\int^{\mathbf{x}'} \mathbf{k}_\perp \cdot d\mathbf{x}'$. The fluctuating part of the distribution function is divided into the adiabatic and nonadiabatic parts as

$$\hat{f}_a(\mathbf{k}_\perp) = -\frac{e_a \hat{\phi}(\mathbf{k}_\perp)}{T_a} f_{aM} + \hat{h}_a(\mathbf{k}_\perp) e^{iL_a(\mathbf{k}_\perp)}, \tag{26}$$

where $L_a(\mathbf{k}_\perp) \equiv \mathbf{k}_\perp \cdot \mathbf{v} \times \mathbf{n}/\Omega_a$. The nonadiabatic part of the distribution function satisfies the following nonlinear gyrokinetic equation.[15]

$$\begin{aligned}
&\left[\frac{\partial}{\partial t} + i(\omega_E + \omega_{Da}) + v_\parallel \mathbf{n} \cdot \boldsymbol{\nabla}' \right] \hat{h}_a(\mathbf{k}_\perp) \\
&= \frac{e_a}{T_a} f_{aM} \left[\frac{\partial}{\partial t} + i(\omega_E + \omega_{*a}^T) \right] \hat{\phi}_a(\mathbf{k}_\perp) \\
&\quad + \frac{c}{B} \sum_{\mathbf{k}_\perp' + \mathbf{k}_\perp'' = \mathbf{k}_\perp} [\mathbf{n} \cdot (\mathbf{k}_\perp' \times \mathbf{k}_\perp'')] \hat{\phi}_a(\mathbf{k}_\perp') \hat{h}_a(\mathbf{k}_\perp'') \\
&\quad + \oint \frac{d\xi}{2\pi} e^{-iL_a(\mathbf{k}_\perp)} C_a^L [f_a(\mathbf{k}_\perp)],
\end{aligned} \tag{27}$$

where $\omega_E \equiv \mathbf{k}_\perp \cdot (c\mathbf{E} \times \mathbf{n}/B)$, $\omega_{Da} \equiv \mathbf{k}_\perp \cdot (\mathbf{v}_{a\,\nabla B} + \mathbf{v}_{a\,\rm curv})$ ($\mathbf{v}_{a\,\nabla B}$: $\boldsymbol{\nabla} B$-drift velocity, $\mathbf{v}_{a\,\rm curv}$: curvature drift velocity), $\omega_{*a}^T \equiv \omega_{*a}[1 + \eta_a(x_a^2 - \frac{3}{2})]$ $[\omega_{*a} \equiv (cT_a/e_a B)\mathbf{k}_\perp \cdot (\mathbf{n} \times \boldsymbol{\nabla} \ln n_a)$, $\eta_a \equiv d \ln T_a / d \ln n_a]$, $\hat{A}_\parallel(\mathbf{k}_\perp) \equiv \mathbf{n} \cdot \hat{\mathbf{A}}(\mathbf{k}_\perp)$, and $\hat{B}_\parallel(\mathbf{k}_\perp) \equiv i\mathbf{n} \cdot \mathbf{k}_\perp \times \hat{\mathbf{A}}(\mathbf{k}_\perp)$. Here the gyrophase-averaged potential for the turbulent electromagnetic fields is defined by

$$\begin{aligned}
\hat{\phi}_a(\mathbf{k}_\perp) &\equiv \overline{e^{-iL_a(\mathbf{k}_\perp)} \left(\hat{\phi}(\mathbf{k}_\perp) - \frac{1}{c} \mathbf{v} \cdot \hat{\mathbf{A}}(\mathbf{k}_\perp) \right)} \\
&= J_0\left(\frac{k_\perp v_\perp}{\Omega_a} \right) \left(\hat{\phi}(\mathbf{k}_\perp) - \frac{v_\parallel}{c} \hat{A}_\parallel(\mathbf{k}_\perp) \right) \\
&\quad + J_1\left(\frac{k_\perp v_\perp}{\Omega_a} \right) \frac{v_\perp}{c} \frac{\hat{B}_\parallel(\mathbf{k}_\perp)}{k_\perp}.
\end{aligned} \tag{28}$$

Appendix B shows that the entropy change $\dot{S}_a^A$ given by Eq. (24) is rewritten as

$$\dot{S}_a^A = -\boldsymbol{\nabla} \cdot \mathbf{J}_{Sa}^A + \sigma_a^A + \mathscr{R}_a^A. \tag{29}$$

Here the anomalous entropy flux $\mathbf{J}_{Sa}^A$ is given by

$$\mathbf{J}_{Sa}^A = S_a \mathbf{u}_a^A + \frac{1}{T_a} \mathbf{q}_a^A, \tag{30}$$

where $\mathbf{u}_a^A \equiv \boldsymbol{\Gamma}_a^A/n_a$ and $\mathbf{q}_a^A$ are defined by Eqs. (34). The anomalous entropy production σ_a^A is written in the thermodynamic form as

$$\sigma_a^A = J_{a1}^A X_{a1}^A + J_{a2}^A X_{a2}^A + J_{a3}^A X_{a3}^A, \tag{31}$$

and the residual term is defined by

$$\begin{aligned}
\mathscr{R}_a^A &= \int d^3v \sum_{\mathbf{k}_\perp} \left(\frac{\partial}{\partial t} + v_\parallel \mathbf{n} \cdot \boldsymbol{\nabla}' \right) \\
&\quad \times \frac{\langle |\hat{f}_a(\mathbf{k}_\perp)|^2 - |\hat{h}_a(\mathbf{k}_\perp)|^2 \rangle_{\rm ens}}{2 f_{aM}},
\end{aligned} \tag{32}$$

which vanishes when the magnetic surface average $\langle\cdot\rangle$ and the quasisteady state ordering $\partial\langle\cdot\rangle_{\rm ens}/\partial t=\mathcal{O}(\delta^2)$ are used. In Eq. (31), we defined conjugated pairs of the thermodynamic forces,

$$X^A_{a1}\equiv\frac{X_{a1}}{T_a}\equiv-\frac{\partial\ln p_a}{\partial V}-\frac{e_a}{T_a}\frac{\partial\Phi}{\partial V},$$
$$X^A_{a2}\equiv\frac{X_{a2}}{T_a}\equiv-\frac{\partial\ln T_a}{\partial V},\quad X^A_{a3}\equiv\frac{1}{T_a},\tag{33}$$

and the anomalous fluxes

$$J^A_{a1}\equiv\mathbf{\Gamma}^A_a\cdot\nabla V\equiv\int d^3v\sum_{\mathbf{k}_\perp}\langle\hat{h}^*_a(\mathbf{k}_\perp)\hat{\mathbf{v}}_{da}(\mathbf{k}_\perp)\rangle_{\rm ens}\cdot\nabla V,$$

$$J^A_{a2}\equiv\frac{1}{T_a}\mathbf{q}^A_a\cdot\nabla V$$
$$\equiv\int d^3v\left(x_a^2-\frac{5}{2}\right)\sum_{\mathbf{k}_\perp}\langle\hat{h}^*_a(\mathbf{k}_\perp)\hat{\mathbf{v}}_{da}(\mathbf{k}_\perp)\rangle_{\rm ens}\cdot\nabla V,\tag{34}$$

$$J^A_{a3}\equiv e_a\int d^3v\sum_{\mathbf{k}_\perp}\left\langle\hat{h}^*_a(\mathbf{k}_\perp)\frac{\partial\hat{\phi}_a(\mathbf{k}_\perp)}{\partial t}\right\rangle_{\rm ens},$$

where the guiding center velocity due to the turbulent electromagnetic fields $\hat{\mathbf{v}}_{da}(\mathbf{k}_\perp)$ is defined by

$$\hat{\mathbf{v}}_{da}(\mathbf{k}_\perp)\equiv-i\frac{c}{B}\hat{\phi}_a(\mathbf{k}_\perp)\mathbf{k}_\perp\times\mathbf{n}$$
$$=J_0\left(\frac{k_\perp v_\perp}{\Omega_a}\right)[\hat{\mathbf{v}}_E(\mathbf{k}_\perp)+v_\parallel\hat{\mathbf{n}}(\mathbf{k}_\perp)]$$
$$+\left(\frac{2\Omega_a}{k_\perp v_\perp}\right)J_1\left(\frac{k_\perp v_\perp}{\Omega_a}\right)\hat{v}_{a\,\nabla B}(\mathbf{k}_\perp).\tag{35}$$

Here $\hat{\mathbf{v}}_E(\mathbf{k}_\perp)\equiv-i(c/B)\phi(\mathbf{k}_\perp)\mathbf{k}_\perp\times\mathbf{n}$ is the electric drift velocity due to the fluctuating electrostatic potential $\hat{\phi}$, $\hat{\mathbf{n}}\equiv(i/B)\hat{A}_\parallel(\mathbf{k}_\perp)\mathbf{k}_\perp\times\mathbf{n}$ denotes the perturbation of the magnetic field direction due to $\hat{A}_\parallel$ ($\hat{A}_\parallel$ gives the magnetic field fluctuation in the direction perpendicular to the equilibrium field), and $\hat{\mathbf{v}}_{a\,\nabla B}(\mathbf{k}_\perp)\equiv-i(c\mu/e_aB)\hat{B}_\parallel(\mathbf{k}_\perp)\mathbf{k}_\perp\times\mathbf{n}$ represents the ∇B drift velocity due to the parallel magnetic field fluctuation $\hat{B}_\parallel$. The magnitudes of the drift velocities due to the perpendicular and parallel magnetic fluctuations are estimated as $v_\parallel\hat{\mathbf{n}}\sim v_{Ta}\hat{B}_\perp/B\sim\delta v_{Ta}$ and $\hat{\mathbf{v}}_{a\,\nabla B}\sim k_\perp\rho_a v_{Ta}\hat{B}_\parallel/B\sim k_\perp\rho_a\,\delta v_{Ta}$, and thus the latter is negligible compared to the former in the long-wavelength limit $k_\perp\rho_a\ll1$. In Eq. (35), $J_0(k_\perp v_\perp/\Omega_a)$ and $(2\Omega_a/k_\perp v_\perp)J_1(k_\perp v_\perp/\Omega_a)$ give finite gyroradius effects, and they both reduce to the unity in the long-wavelength limit.

The radial particle flux $J^A_{a1}=\mathbf{\Gamma}^A_a\cdot\nabla V$ is conjugate to the force X^A_{a1} consisting of the radial pressure gradient and the radial electric field; $J^A_{a2}=\mathbf{q}^A_a\cdot\nabla V/T_a$ is the radial heat flux divided by the temperature conjugate to the radial temperature gradient X^A_{a2}. The flux J^A_{a3} is conjugate to the force $X_{a3}=1/T_a$ and represents the heating of the particles due to the electromagnetic fluctuations. It should be noted that $\mathbf{\Gamma}^A_a$ is the same as the anomalous particle flux $\mathbf{\Gamma}^{\rm anom}_a$ defined in Eq. (3), while $\mathbf{q}^A_a$ is different from the anolamous flux $\mathbf{q}^{\rm anom}_a$ defined in Eq. (3). The difference between $\mathbf{q}^A_a$ and $\mathbf{q}^{\rm anom}_a$ is given by

$$\frac{1}{T_a}(\mathbf{q}^A_a-\mathbf{q}^{\rm anom}_a)\cdot\nabla V$$
$$=-\int a^3v\,\frac{\langle\hat{f}^2_a\rangle_{\rm ens}}{2f_{aM}}\mathbf{v}\cdot\nabla V$$
$$\simeq-\int d^3v[\langle(f_a+\hat{f}_a)\ln\langle f_a+\hat{f}_a)\rangle_{\rm ens}$$
$$-f_a\ln f_a]\mathbf{v}\cdot\nabla V,\tag{36}$$

which is regarded as a residual microscopic entropy flux.

Using the gyrokinetic equation (27), the anomalous entropy production σ^A_a defined in Eq. (31) is rewritten as

$$\sigma^A_a=\int d^3v\sum_{\mathbf{k}_\perp}\left(\frac{\partial}{\partial t}+v_\parallel\mathbf{n}\cdot\nabla'\right)\frac{\langle|\hat{h}_a(\mathbf{k}_\perp)|^2\rangle_{\rm ens}}{2f_{aM}}$$
$$-\int d^3v\,\frac{1}{f_{aM}}\sum_{\mathbf{k}_\perp}\langle\hat{f}^*_a(\mathbf{k}_\perp)C^L_a[\hat{f}_a(\mathbf{k}_\perp)]\rangle_{\rm ens}.\tag{37}$$

If we use the quasisteady-state ordering $\partial\langle\cdot\rangle_{\rm ens}/\partial t=\mathcal{O}(\delta^2)$ and take the magnetic surface average of Eq. (37), we obtain the balance between the anomalous entropy production driven by the turbulent transport and the collisional dissipation of the fluctuating distribution function:

$$\langle\sigma^A_a\rangle=-\int d^3v\,\frac{1}{f_{aM}}\sum_{\mathbf{k}_\perp}\langle\langle\hat{f}^*_a(\mathbf{k}_\perp)C^L_a[f_a(\mathbf{k}_\perp)]\rangle\rangle,\tag{38}$$

where $\langle\langle\cdot\rangle\rangle$ represents a double average over the magnetic surface and the ensemble. This balance equation is equivalent to Eq. (41) in Ref. 21, where Krommes and Hu discussed the problem of the ''entropy paradox.'' Equation (38) shows that, if there are no collisions, then the anomalous entropy production from the turbulent transport should vanish. However, Krommes and Hu recognized the critical difference between the limiting behavior of a system with negligibly small collisional dissipation and the behavior in the no collision case, and argued that the dissipation plays an important role, even in the limit of vanishing dissipation. Here, following their argument ''*forcing determines dissipation*'' [see Eq. (43) in Ref. 22], we consider that, in the collisionless limit, the turbulent transport and accordingly the anomalous entropy production $\langle\sigma^A_a\rangle=\Sigma^3_{l=1}\langle J^A_{al}\rangle X^A_{al}$ achieve nonzero steady-state values independent of collisions, although the fluctuating distribution $\hat{f}_a$ adjusts itself such that the balance equation (38) holds. From Eq. (38) and the positive definiteness of the collision operator C^L_a [see Eq. (10) in Ref. 5], we obtain the following inequality for the anomalous entropy production similar to those for the classical and neoclassical entropy productions,

$$\sum_a T_a\langle\sigma_a^A\rangle=\sum_a\sum_{m=1}^{3} T_a\langle J_{am}^A\rangle X_{am}^A$$

$$=-\sum_a T_a\int d^3v\,\frac{1}{f_{aM}}\sum_{\mathbf{k}_\perp}\langle\langle\hat{f}_a^*(\mathbf{k}_\perp)$$

$$\times C_a^L[f_a(\mathbf{k}_\perp)]\rangle\rangle\geqslant 0. \tag{39}$$

Up to this point, we have derived the physically understandable expressions of the entropy variations due to the classical, neoclassical, and anomalous transport processes separately from their kinetic definitions. The entropy production rates for all the transport processes or their magnetic surface averages have been shown to be written in the thermodynamic form, i.e., as the sum of the products of the thermodynamic forces and the conjugate transport fluxes. Using Eqs. (15)–(18), (21), (22), (24), (29), and (30), we obtain the equation describing the temporal variation of the magnetic surface average of the entropy density $\langle S_a\rangle$ for species a,

$$\frac{\partial S_a}{\partial t}+\frac{\partial\langle\mathbf{J}_{Sa}^{\rm tot}\cdot\nabla V\rangle}{\partial V}=\langle\sigma_a^{\rm tot}\rangle, \tag{40}$$

where

$$\langle\sigma_a^{\rm tot}\rangle\equiv\langle\sigma_a^{\rm cl}\rangle+\langle\sigma_a^{\rm ncl}\rangle+\langle\sigma_a^A\rangle+\frac{1}{T_a}\langle Q_a\rangle+\frac{1}{T_a}\langle\mathbf{u}_a\cdot\mathbf{F}_{a1}\rangle \tag{41}$$

and

$$\langle\mathbf{J}_{Sa}^{\rm tot}\cdot\nabla V\rangle\equiv S_a\langle\mathbf{u}_a\cdot\nabla V\rangle+\frac{1}{T_a}\langle\mathbf{q}_a'\cdot\nabla V\rangle. \tag{42}$$

The last term in Eq. (41) is rewritten as

$$\langle\mathbf{u}_a\cdot\mathbf{F}_{a1}\rangle=\frac{\langle Bu_{\|a}\rangle\langle BF_{\|a1}\rangle}{\langle B^2\rangle}-(\langle J_{a1}^{\rm cl}\rangle+J_{a1}^{\rm PS})X_{a1}, \tag{43}$$

which can be expressed as a second-order form of the thermodynamic forces in the same way as $\langle\sigma_a^{\rm cl}\rangle$ and $\langle\sigma_a^{\rm ncl}\rangle$ since $\langle Bu_{\|a}\rangle$ and $\langle BF_{\|a1}\rangle=\langle\mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle-n_ae_a\langle BE_\|\rangle$, as well as $J_{a1}^{\rm cl}$ and $J_{a1}^{\rm PS}$, are given by linear forms of the thermodynamic forces. It should be noted that, to the lowest order in δ, S_a is a magnetic surface quantity like n_a and T_a and is given by $S_a=-\int d^3v\,f_{aM}\ln f_{aM}=-n_a\{\ln[n_a(m_a/2\pi T_a)^{3/2}]-\frac{3}{2}\}=\langle S_a\rangle$. The radial components of the flow $\mathbf{u}_a$ and the heat flux $\mathbf{q}_a'$ are given by

$$\begin{aligned}
n_a\langle\mathbf{u}_a\cdot\nabla V\rangle&\equiv\langle\boldsymbol{\Gamma}_a\cdot\nabla V\rangle\\
&=\langle J_{a1}^{\rm cl}\rangle+J_{a1}^{\rm PS}+J_{a1}^{\rm bp}+J_{a1}^{\rm na}+J_{a1}^{(E)}+\langle J_{a1}^A\rangle,\\
\langle\mathbf{q}_a'\cdot\nabla V\rangle&\equiv\langle[\mathbf{q}_a+(\mathbf{q}_a^A-\mathbf{q}_a^{\rm anom})]\cdot\nabla V\rangle\\
&=\langle J_{a2}^{\rm cl}\rangle+J_{a2}^{\rm PS}+J_{a2}^{\rm bp}+J_{a2}^{\rm na}+\langle J_{a2}^A\rangle,
\end{aligned} \tag{44}$$

where the residual anomalous heat flux $(\mathbf{q}_a^A-\mathbf{q}_a^{\rm anom})$ given by Eq. (36) is added to the heat flux $\mathbf{q}_a$ given by Eq. (2) to define the total heat flux $\mathbf{q}_a'$. Definitions of the classical fluxes $J_{aj}^{\rm cl}$, the Pfirsch–Schlüter fluxes $J_{aj}^{\rm PS}$, the banana-plateau fluxes $J_{aj}^{\rm bp}$, the nonaxisymmetric fluxes $J_{aj}^{\rm na}$, and the particle flux $J_{a1}^{(E)}$ due to the inductive electric field are given in Appendix A. We obtain from Eqs. (19) and (41),

$$\sum_a T_a\langle\sigma_a^{\rm tot}\rangle=\sum_a T_a(\langle\sigma_a^{\rm cl}\rangle+\langle\sigma_a^{\rm ncl}\rangle+\langle\sigma_a^A\rangle)\geqslant 0. \tag{45}$$

IV. ONSAGER SYMMETRY FOR ANOMALOUS TRANSPORT EQUATIONS

Here we examine whether the Onsager symmetry is valid or not for the anomalous transport matrix, which connects conjugate pairs of the thermodynamic forces and anomalous fluxes defined in the previous section. We treat the spectrum of the turbulent electromagnetic fields as given arbitrarily for the moment (the conditions for the self-consistent turbulent fields are discussed in the next section) and consider the anomalous transport matrix as a functional of the fluctuation spectra. It is still difficult to derive the rigorous expression for the response of the distribution function to the fluctuating fields by solving the nonlinear gyrokinetic equation (27), so we neglect the nonlinear term in Eq. (27) and use a linear response relation to give "*quasilinear*" transport fluxes.

The quasilinear anomalous transport equations are written as

$$\begin{bmatrix} J_{a1}^A\\ J_{a2}^A\\ J_{a3}^A\end{bmatrix}=\sum_b\begin{bmatrix}(L^A)_{11}^{ab} & (L^A)_{12}^{ab} & (L^A)_{13}^{ab}\\ (L^A)_{21}^{ab} & (L^A)_{22}^{ab} & (L^A)_{23}^{ab}\\ (L^A)_{31}^{ab} & (L^A)_{32}^{ab} & (L^A)_{33}^{ab}\end{bmatrix}\begin{bmatrix}X_{a1}^A\\ X_{a2}^A\\ X_{a3}^A\end{bmatrix}, \tag{46}$$

where the anomalous transport coefficients $(L^A)_{mn}^{ab}$ are defined by Eq. (C4) in Appendix C. The anomalous transport coefficients $(L^A)_{mn}^{ab}$ are functionals of the spectra of the electromagnetic fluctuations $\hat{\boldsymbol{\phi}}(\mathbf{k}_\perp)\equiv[\hat{\phi}(\mathbf{k}_\perp),\hat{A}_\|(\mathbf{k}_\perp),\hat{B}_\|(\mathbf{k}_\perp)/k_\perp]$ and that it also contains $\mathbf{B}$ as a parameter:

$$(L^A)_{mn}^{ab}=(L^A)_{mn}^{ab}[\mathbf{B},\{\hat{\boldsymbol{\phi}}\}], \tag{47}$$

where the spectra of the electromagnetic fluctuations $\{\hat{\boldsymbol{\phi}}\}$ are assumed to be given *a priori*. It is shown in Appendix C that the quasilinear anomalous transport coefficients satisfy the following Onsager symmetry:

$$T_a\langle(L^A)_{mn}^{ab}[\mathbf{B},\{\hat{\boldsymbol{\phi}}(t)\}]\rangle=T_b\langle(L^A)_{nm}^{ba}[-\mathbf{B},\{\hat{\boldsymbol{\phi}}(-t)\}]\rangle, \tag{48}$$

where T_a and T_b appear because we defined the conjugate pairs of the forces $\mathbf{X}$ and the fluxes $\mathbf{J}$ from the entropy production σ by $\sigma_a=\mathbf{J}_a\cdot\mathbf{X}_a$ for the anomalous transport, but by $\Sigma_aT_a\sigma_a=\mathbf{J}\cdot\mathbf{X}$ for the classical and neoclassical transport. Here it is noted that, as in the neoclassical case, the Onsager symmetry of the quasilinear anomalous transport coefficients is valid for their magnetic-surface-averaged values instead of their spatially local values. The global dependence of the anomalous fluxes arises from the fluctuations with large wavelengths $k_\|^{-1}\sim L$ along the magnetic field lines.

In order to obtain detailed expressions of the quasilinear anomalous transport coefficients, we assume that the parallel correlation length for the fluctuations is short enough for inhomogeneities of the equilibrium quantities to be ignorable, $k_\|\gg L^{-1}$. Then, we use the Fourier transform for spatial variation of the fluctuations along the magnetic field lines so that $\mathbf{n}\cdot\nabla$ is replaced with $ik_\|$, where the parallel wave number $k_\|$ is assumed to satisfy $L^{-1}\ll k_\|\ll k_\perp$. Similarly, the Fou-

rier transform is used for temporal variation of the fluctuations to replace $\partial/\partial t$ with $-i\omega$. Furthermore, for simplicity, we employ the Krook collision operator model $C_a^L(\hat{h}_a e^{iL_a}) = -\nu_a h_a e^{iL_a}$ where ν_a is the collision frequency and the contribution of $\hat{h}_b$ $(b \neq a)$ to C_a^L is neglected. (The validity limits due to the Krook model will be discussed later.) Then the coefficients connecting the fluxes to the forces for different species vanish, and we have

$$(L^A)^{ab}_{mn} = (L^A)^a_{mn}\delta_{ab} \quad (m,n=1,2,3). \tag{49}$$

Here the quasilinear anomalous transport coefficients $(L^A)^a_{mn}$ are given by

$$\begin{aligned}
(L^A)^a_{mn} &= \left(\frac{c}{B}\right)^2 \int d^3v\ f_{aM}\left(x_a^2-\frac{5}{2}\right)^{m+n-2} \sum_{\mathbf{k}_\perp,k_\parallel,\omega} \langle|\hat{\phi}_a(\mathbf{k}_\perp,k_\parallel,\omega)|^2\rangle_{\rm ens}\Delta_a(\mathbf{k}_\perp,k_\parallel,\omega)|(\mathbf{k}_\perp\times\mathbf{n})\cdot\nabla V|^2 \quad (m,n=1,2),\\
(L^A)^a_{m3} &= (L^A)^a_{3m} = \frac{ce_a}{B}\int d^3v\ f_{aM}\left(x_a^2-\frac{5}{2}\right)^{m-1} \sum_{\mathbf{k}_\perp,k_\parallel,\omega} \langle|\hat{\phi}_a(\mathbf{k}_\perp,k_\parallel,\omega)|^2\rangle_{\rm ens}\Delta_a(\mathbf{k}_\perp,k_\parallel,\omega)\omega(\mathbf{k}_\perp\times\mathbf{n})\cdot\nabla V \quad (m=1,2),\\
(L^A)^a_{33} &= e_a^2\int d^3v\ f_{aM} \sum_{\mathbf{k}_\perp,k_\parallel,\omega} \langle|\hat{\phi}_a(\mathbf{k}_\perp,k_\parallel,\omega)|^2\rangle_{\rm ens}\Delta_a(\mathbf{k}_\perp,k_\parallel,\omega)\omega^2,
\end{aligned} \tag{50}$$

where the function $\Delta_a(\mathbf{k}_\perp,k_\parallel,\omega)$ is defined by

$$\Delta_a(\mathbf{k}_\perp,k_\parallel,\omega) \equiv \nu_a[(\omega-\omega_E-\omega_{Da}-k_\parallel v_\parallel)^2+\nu_a^2]^{-1}, \tag{51}$$

which, in the limit $\nu_a \to +0$, reduces to $\pi\delta(\omega-\omega_E-\omega_{Da}-k_\parallel v_\parallel)$. We can directly confirm by Eq. (50) the positive definiteness of $\sigma_a^A = \Sigma_{m,n}(L^A)^a_{mn}X^A_{am}X^A_{an}$ and the symmetry properties of $(L^A)^a_{mn}[\mathbf{B},\{\hat{\boldsymbol{\phi}}\}]$ with respect to the transformations $[\mathbf{B},\{\hat{\boldsymbol{\phi}}(t)\}] \to [-\mathbf{B},\{\hat{\boldsymbol{\phi}}(-t)\}]$ and $(m,n)\to(n,m)$:

$$\begin{aligned}
(L^A)^a_{mn}[\mathbf{B},\{\hat{\boldsymbol{\phi}}(t)\}] &= (L^A)^a_{mn}[-\mathbf{B},\{\hat{\boldsymbol{\phi}}(-t)\}]\\
&= (L^A)^a_{nm}[\mathbf{B},\{\hat{\boldsymbol{\phi}}(t)\}].
\end{aligned} \tag{52}$$

We see that, for $k_\parallel \gg L^{-1}$, the Onsager symmetry holds for the local values of the quasilinear transport coefficients as in the case of the classical transport coefficients. These expressions of the anomalous transport coefficients with the Onsager symmetry are consistent with results of previous works in Ref. 10 and Ref. 23, where only the electrostatic fluctuations are considered. [Note that the conjugate pairs of the forces and the anomalous fluxes (and accordingly the transport coefficients) given in the present work are slightly different from those in Ref. 10 and Ref. 23, although the transport equations for the former pairs are consistently transformed into those for the latter.]

Now let us examine the effects of the magnetic fluctuations in more detail. Two physically distinct types of magnetic fluctuations $\hat{A}_\parallel$ and $\hat{B}_\parallel$ are contained in the anomalous transport coefficients through $\hat{\phi}_a$. The effects of $\hat{A}_\parallel$, which give the fluctuating magnetic field perpendicular to the equilibrium field line, have been thoroughly investigated in the literatures.[24–26] On the other hand, the fluctuations of the parallel component $\hat{B}_\parallel$, which gives the fluctuating ∇B drift velocity $\hat{\mathbf{v}}_{a\,\nabla B}$ as shown in Eq. (35), have scarcely been taken into account. This is because $\hat{B}_\parallel/B \sim \beta(e_a\hat{\phi}/T_a)$ for low $\beta(\equiv 8\pi p/B^2)$ plasmas[27] and because the parallel $(\hat{B}_\parallel)$ effects are negligible for fluctuations with low wave numbers $k_\perp\rho_a \ll 1$, as is expected from a factor $J_1(k_\perp v_\perp/\Omega_a)$ multiplied by $\hat{B}_\parallel$ in Eq. (28). On the other hand, both the parallel and perpendicular effects are comparable to each other for gyrokinetic fluctuations with $k_\perp\rho_a \sim 1$ and $\beta \sim 1$. In order to show these magnetic fluctuation effects on the anomalous transport more clearly, we assume that the temporal variation of the fluctuations is very slow $\omega \sim 0$ and that the wave number spectral functions are written as

$$\begin{aligned}
&\langle|k_\perp\hat{A}_\parallel(\mathbf{k}_\perp,k_\parallel)|^2\rangle_{\rm ens}, \langle|\hat{B}_\parallel(\mathbf{k}_\perp,k_\parallel)|^2\rangle_{\rm ens}\\
&\quad \propto \exp\left[-\frac{1}{2}(k_\perp^2\lambda_\perp^2+k_\parallel^2\lambda_\parallel^2)\right],
\end{aligned} \tag{53}$$

where $\lambda_\perp$ and $\lambda_\parallel$ denote perpendicular and parallel correlation lengths, respectively. Here we do not consider the electrostatic fluctuations and the cross correlation $\langle k_\perp\hat{A}_\parallel^*(\mathbf{k}_\perp,k_\parallel)\hat{B}_\parallel(\mathbf{k}_\perp,k_\parallel)\rangle_{\rm ens}$ for simplicity. Then the contribution of the perpendicular magnetic fluctuations to the particle diffusion coefficient $(L^A)^a_{11}$ is given by

$$\begin{aligned}
(L_{B\perp})^a_{11} &= \frac{n_a v_{Ta} D_{B\perp}|\nabla V|^2}{\sqrt{\pi}(1+2\rho_{Ta}^2/\lambda_\perp^2)^{1/2}}\\
&\quad \times \begin{cases} v_{Ta}/(\sqrt{2}\nu_a\lambda_\parallel), & \text{for } \nu_a \gg v_{Ta}/\lambda_\parallel,\\ 1, & \text{for } \nu_a \ll v_{Ta}/\lambda_\parallel. \end{cases}
\end{aligned} \tag{54}$$

Here $D_{B\perp}$ represents the perpendicular diffusion coefficient of the magnetic field line defined by

$$D_{B\perp} \equiv \int_0^\infty dl\ \frac{\langle\hat{\mathbf{B}}_\perp[\mathbf{x}(l)]\cdot\hat{\mathbf{B}}_\perp(\mathbf{x})\rangle_{\rm ens}}{2B^2}, \tag{55}$$

where $\mathbf{x}(l)$ denotes the position at a distance l from $\mathbf{x}$ along the magnetic field line. In the limit of $\rho_{Ta}/\lambda_\perp \to +0$, the results of Eq. (54) reduce to those in Ref. 26, when the particle motion is within a time scale in the range $t_{\min} < t < t_{\max}$, where $t_{\min} \equiv \nu_a^{-1}\min\{1,(\nu_a\lambda_\parallel/v_{Ta})\}$ and $t_{\max} \equiv \nu_a^{-1}\max\{1,(\nu_a\lambda_\parallel/v_{Ta})^2\}$. By using both the Langevin equation and the Fokker–Planck equation, Balescu *et al.*[26] confirmed that the guiding center motion in the stochastic magnetic field shows

a ballistic behavior for a shorter time scale $t<t_{\min}$ and a subdiffusive behavior for a longer time scale $t>t_{\max}$. Thus, the diffusion coefficient in Eq. (54) obtained by the gyrokinetic equation with the Krook collision model is considered to correctly describe the particle transport only for a time scale $t_{\min}<t<t_{\max}$. Equation (54) shows that the finite gyroradius effect reduces the diffusivity by a factor $(1+2\rho_{Ta}^2/\lambda_\perp^2)^{-1/2}$, which is in agreement with the result obtained by the stochastic Vlasov equation in Ref. 28.

The contribution of the parallel magnetic fluctuations to $(L^A)_{11}^a$ is given by

$$(L_{B\|})_{11}^a=\frac{n_a v_{Ta} D_{B\|}|\nabla V|^2}{\sqrt{\pi}}\frac{\rho_{Ta}^2/\lambda_\perp^2}{(1+2\rho_{Ta}^2/\lambda_\perp^2)^{3/2}} \times\begin{cases} v_{Ta}/(\sqrt{2}\nu_a\lambda_\|), & \text{for } \nu_a\gg v_{Ta}/\lambda_\|, \\ \log(v_{Ta}/\nu_a\lambda_\|), & \text{for } \nu_a\ll v_{Ta}/\lambda_\|, \end{cases} \tag{56}$$

where

$$D_{B\|}\equiv\int_0^\infty dl\,\frac{\langle\hat{B}_\|[\mathbf{x}(l)]\hat{B}_\|(\mathbf{x})\rangle_{\rm ens}}{B^2}. \tag{57}$$

The anomalous diffusion described by Eqs. (56) and (57) results from the fluctuating magnetic drift $\hat{\mathbf{v}}_{a\,\nabla B}$ and accordingly the resultant quasilinear diffusion coefficient $(L_{B\|})_{11}^a$ is proportional to the velocity correlation $\langle\hat{\mathbf{v}}_{a\,\nabla B}\hat{\mathbf{v}}_{a\,\nabla B}\rangle_{\rm ens}$ and therefore to $\langle\hat{B}_\|\hat{B}_\|\rangle_{\rm ens}$. We see that $(L_{B\|})_{11}^a$ vanishes in the both limits of $\rho_{Ta}/\lambda_\perp\to+0$ and $+\infty$, while it has a maximum value at $\rho_{Ta}/\lambda_\perp=1$. Equation (56) shows that $(L_{B\|})_{11}^a$ monotonically increases with decreasing the collision frequency ν_a, even for $\nu_a\ll v_{Ta}/\lambda_\|$, which is a contrast to $(L_{B\perp})_{11}^a$ independent of ν_a for the same collision frequency region. However, it should be recalled that a time scale for validity of Eq. (56), obtained by using the Krook model also has an upper limit, since the velocity-space diffusion, which is described not by the Krook model but by the Fokker–Planck collision operator, deforms the propagator and causes the subdiffusion in a longer time scale.[26]

V. ENERGY BALANCE EQUATION

For transport analyses of toroidal plasmas, particle and energy balance equations are used generally in the magnetic-surface-averaged forms. The fluctuation term in the ensemble-averaged kinetic equation (5) conserves the particle number and gives neither source nor sink terms in the continuity equation derived by taking the zeroth moment of the kinetic equation. Then, the magnetic-surface-averaged continuity equation has a well-known form:

$$\frac{\partial n_a}{\partial t}+\frac{\partial\langle\mathbf{\Gamma}_a\cdot\nabla V\rangle}{\partial V}=0, \tag{58}$$

where it should be noted that anomalous particle flux is also included in the total particle flux $\langle\mathbf{\Gamma}_a\cdot\nabla V\rangle$ as given by Eq. (44).

The energy balance equation is similarly derived from the kinetic equation and it is written in the magnetic-surface-averaged form as

$$\frac{3}{2}\frac{\partial p_a}{\partial t}=-\frac{\partial}{\partial V}\left\langle\left(\mathbf{q}_a+\frac{5}{2}p_a\mathbf{u}_a\right)\cdot\nabla V\right\rangle+\langle\mathbf{u}_a\cdot\nabla V\rangle\frac{\partial p_a}{\partial V} +\langle\mathbf{u}_a\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle+\langle Q_a\rangle+\langle H_a\rangle, \tag{59}$$

where $H_a\equiv\int d^3v\,\mathscr{D}_a\frac{1}{2}m(\mathbf{v}-\mathbf{u}_a)^2$ denotes the anomalous heat generation due to the fluctuations. The viscous heating term $\langle\mathbf{u}_a\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle$ in Eq. (59) can be written in various forms:

$$\begin{aligned}\langle\mathbf{u}_a\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle&=\frac{u_a^\theta}{B^\theta}\langle\mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle+J_{a1}^{\rm na}X_{a1}\\ &=\frac{\langle Bu_{\|a}\rangle\langle\mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle}{\langle\mathbf{B}^2\rangle}+(J_{a1}^{\rm bp}+J_{a1}^{\rm na})X_{a1}\\ &=\frac{n_a e_a\langle Bu_{\|a}\rangle\langle BE_{\|a}\rangle}{\langle B^2\rangle}+(\langle J_{a1}^{\rm cl}\rangle+J_{a1}^{\rm PS}+J_{a1}^{\rm bp} +J_{a1}^{\rm na})X_{a1}+\langle\mathbf{u}_a\cdot\mathbf{F}_{a1}\rangle.\end{aligned} \tag{60}$$

The magnetic-surface-averaged anomalous heat generation $\langle H_a\rangle$ is written as

$$\langle H_a\rangle=-\frac{\partial}{\partial V}\langle(\mathbf{q}_a^A-\mathbf{q}_a^{\rm anom})\cdot\nabla V\rangle+\langle J_{a1}^A\rangle X_{a1}+\langle J_{a3}^A\rangle. \tag{61}$$

Substituting Eqs. (60) and (61) into Eq. (59), we obtain

$$\begin{aligned}\frac{3}{2}\frac{\partial p_a}{\partial t}&=-\frac{\partial}{\partial V}\left\langle\left(\mathbf{q}_a'+\frac{5}{2}p_a\mathbf{u}_a\right)\cdot\nabla V\right\rangle\\ &\quad-e_a n_a\langle\mathbf{u}_a\cdot\nabla V\rangle\frac{\partial\Phi}{\partial V}-J_{a1}^{(E)}X_{a1}+\langle Q_a\rangle\\ &\quad+\langle\mathbf{u}_a\cdot\mathbf{F}_{a1}\rangle+\frac{n_a e_a\langle Bu_{\|a}\rangle\langle BE_\|\rangle}{\langle B^2\rangle}+\langle J_{a3}^A\rangle\\ &=-\frac{\partial}{\partial V}\left\langle\left(\mathbf{q}_a'+\frac{5}{2}p_a\mathbf{u}_a\right)\cdot\nabla V\right\rangle+\frac{\langle Bu_{\|a}\rangle\langle\mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle}{\langle B^2\rangle}\\ &\quad+\frac{1}{n_a}(\langle J_{a1}^{\rm cl}\rangle+J_{a1}^{\rm PS}+J_{a1}^{(E)})\frac{\partial p_a}{\partial V}\\ &\quad-e_a(J_{a1}^{\rm bp}+J_{a1}^{\rm na}+\langle J_{a1}^A\rangle)\frac{\partial\Phi}{\partial V}+\langle Q_a\rangle+\langle J_{a3}^A\rangle,\end{aligned} \tag{62}$$

where $\mathbf{u}_a$ and $\mathbf{q}_a'$ are written as Eq. (44). On the right-hand side of Eq. (62), the parallel flow $\langle Bu_{\|a}\rangle$ and the parallel viscosity $\langle\mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle$ can be written in terms of the thermodynamic forces in the similar way as the other classical and neoclassical transport fluxes. From the species summation of the momentum balance equations multiplied by the flow velocities, we have the following relation:

$$\sum_a\left(\frac{\langle Bu_{\|a}\rangle\langle\mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle}{\langle B^2\rangle}+\frac{1}{n_a}(\langle J_{a1}^{\rm cl}\rangle+J_{a1}^{\rm PS})\frac{\partial p_a}{\partial V}-e_a(J_{a1}^{\rm bp} +J_{a1}^{\rm na}+\langle J_{a1}^A\rangle)\frac{\partial\Phi}{\partial V}+\langle Q_a\rangle\right)=\frac{\langle BJ_\|\rangle\langle BE_\|^{(A)}\rangle}{\langle B^2\rangle}. \tag{63}$$

Equation (63) shows that the species summation of the heating terms on the right-hand side of Eq. (62), except for the

anomalous heating $\langle J_{a3}^A \rangle$, originates from the Ohmic power input due to the inductive electric field. [The term $(J_{a1}^{(E)}/n_a)(\partial p_a/\partial V)$ on the right-hand side of Eq. (62) does not appear in Eq. (63) since its species summation cancels out with the residual Ohmic heating terms $\langle \mathbf{J}_\perp \cdot \mathbf{E}_\perp^{(A)} \rangle$ and $\langle J_\parallel E_\parallel^{(A)} \rangle - \langle BJ_\parallel \rangle \langle BE_\parallel^{(A)} \rangle / \langle B^2 \rangle$.]

Shaing *et al.*[29] presented energy balance equations for toroidal plasmas, including nonaxisymmetric systems from the neoclassical theory, although they did not give a clear theoretical foundation to treat the energy balance in the cases where the anomalous transport exists. Equation (62), which was rigorously derived based on the gyrokinetic ordering from the kinetic equation taking account of the electromagnetic turbulence, shows completely how the fluctuations affect the energy balance. The anomalous particle flow $\mathbf{u}_a^A \equiv \mathbf{u}_a^{\rm anom}$ and the anomalous heat flux $\mathbf{q}_a^A = \mathbf{q}_a^{\rm anom} + (\mathbf{q}_a^A - \mathbf{q}_a^{\rm anom})$ are naturally included in the radial derivative term of the total energy flux. We also find that the product of the anomalous radial current and the radial electric field $-e_a \langle J_{a1}^A \rangle \partial \Phi / \partial V$as well as the anomalous heating $\langle J_{a3}^A \rangle$ should be added into the energy balance equation. For the self-consistent fluctuations not driven externally, both $\Sigma_a e_a J_{a1}^A$ and $\Sigma_a J_{a3}^A$ vanish [see Eqs. (69) and (71)]. Then, $-e_a \langle J_{a1}^A \rangle \partial \Phi / \partial V$ and $\langle J_{a3}^A \rangle$ cause the anomalous energy exchange between different species of particles although they give no net heating of total particles. The summation of these two terms is written as

$$
\begin{aligned}
&-e_a \langle J_{a1}^A \rangle \frac{\partial \Phi}{\partial V} + \langle J_{a3}^A \rangle \\
&\quad = e_a \int d^3v \sum_{\mathbf{k}_\perp} \left\langle \left\langle \hat{h}_a^*(\mathbf{k}_\perp) \left(\frac{\partial}{\partial t} + i\omega_E \right) \hat{\phi}_a(\mathbf{k}_\perp) \right\rangle \right\rangle \\
&\quad = e_a \int d^3v \sum_{\mathbf{k}_\perp} \left\langle \left\langle e^{-i\omega_E t} \hat{h}_a^*(\mathbf{k}_\perp) \frac{\partial}{\partial t} [e^{i\omega_E t} \hat{\phi}_a(\mathbf{k}_\perp)] \right\rangle \right\rangle .
\end{aligned}
\tag{64}
$$

The radial electric field $-\partial \Phi / \partial V$ enters the nonlinear gyrokinetic equation (27) and Eq. (64) only in the form of the Doppler shift $(\partial/\partial t + i\omega_E)$, and does not appear explicitly in the self-consistent conditions given in the next section [see Eqs. (66)–(68)]. Thus, for the solutions of Eqs. (27) and (66)–(68), $e^{i\omega_E t} \hat{h}_a(\mathbf{k}_\perp)$ and $e^{i\omega_E t} \hat{\phi}_a(\mathbf{k}_\perp)$ are independent of the radial electric field. Then, the radial electric field does not affect the magnitude of the anomalous heating effect given by Eq. (64)

The entropy balance equation (40) can be derived also from substituting the continuity equation (58) and the energy balance equation into the temporal variation of the entropy,

$$
\frac{\partial S_a}{\partial t} = \left(\frac{S_a}{n_a} - \frac{5}{2} \right) \frac{\partial n_a}{\partial t} + \frac{3}{2T_a} \frac{\partial p_a}{\partial t},
\tag{65}
$$

although the correspondence between the kinetic and thermodynamic forms of the entropy productions due to the classical, neoclassical, and anomalous transport processes is better understood by the derivation in Sec. III.

VI. SELF-CONSISTENT ELECTROMAGNETIC FLUCTUATIONS AND AMBIPOLARITY CONDITION

Up to this point, the turbulent fluctuations have been general prescribed fields. When the fluctuations are local, self-consistent fields not driven by external sources, there are additional properties of the transport that we now derive.

Here we impose the self-consistent constraints on the turbulent fields, which are given by Poisson's equation:

$$
(k_\perp^2 + \lambda_{\rm D}^{-2}) \hat{\phi}(\mathbf{k}_\perp) = 4\pi \sum_a e_a \int d^3v \ \hat{h}_a(\mathbf{k}_\perp) J_0 \left(\frac{k_\perp v_\perp}{\Omega_a} \right),
\tag{66}
$$

and the parallel and perpendicular components of Ampère's law:

$$
k_\perp^2 \hat{A}_\parallel(\mathbf{k}_\perp) = \frac{4\pi}{c} \sum_a e_a \int d^3v \ v_\parallel \hat{h}_a(\mathbf{k}_\perp) J_0 \left(\frac{k_\perp v_\perp}{\Omega_a} \right),
\tag{67}
$$

$$
-k_\perp \hat{B}_\parallel(\mathbf{k}_\perp) = \frac{4\pi}{c} \sum_a e_a \int d^3v \ v_\perp \hat{h}_a(\mathbf{k}_\perp) J_1 \left(\frac{k_\perp v_\perp}{\Omega_a} \right),
\tag{68}
$$

where the Debye length $\lambda_{\rm D} \equiv (4\pi \Sigma_a n_a e_a^2 / T_a)^{-1/2}$ is used. The use of the Ampère's law is justified since the displacement current is neglected due to the gyrokinetic ordering.

Substituting Eqs. (66)–(68) into the definition of the anomalous fluxes, we find that the anomalous particle fluxes are intrinsically ambipolar:

$$
\sum_a e_a \mathbf{\Gamma}_a^A = 0.
\tag{69}
$$

It is proved from the momentum conservation by collisions $\Sigma_a \mathbf{F}_{a1} = 0$ and the charge neutrality condition $\Sigma_a n_a e_a = 0$ that, even if particles of different species belong to different collisional regimes, the ambipolarity condition is automatically and separately satisfied by the classical $(J_{a1}^{\rm cl})$, Pfirsch–Schlüter $(J_{a1}^{\rm PS})$, and banana-plateau $(J_{a1}^{\rm bp})$ parts of multispecies particle fluxes, which is called the principle of detailed ambipolar balance.[2,30,31] Then, only the nonaxisymmetric particle fluxes $J_{a1}^{\rm na}$ are nonambipolar and the nonintrinsic ambipolarity condition is written as

$$
\sum_a e_a \langle \mathbf{\Gamma}_a \cdot \nabla V \rangle = \sum_a e_a J_{a1}^{\rm na} = 0,
\tag{70}
$$

which is used to determine the radial electric field $-\partial \Phi / \partial V$ in the nonaxisymmetric systems.

Equations (66)–(68) and Eq. (34) with the quasisteady-state ordering $\partial \langle \cdot \rangle_{\rm ens} / \partial t = \mathcal{O}(\delta^2)$ show that the species summation of the anomalous heating J_{a3}^A vanishes:

$$
\sum_a J_{a3}^A = 0.
\tag{71}
$$

The self-consistent fluctuations cause no net heating of the total particles, since the source of the anomalous heating is the energy of the fluctuating electromagnetic fields, which cannot be a stationary energy supplier unless the fluctuations are externally driven.

Furthermore, we find from Eqs. (26), (36), and the Ampère's law that the species summation of $(\mathbf{q}_a^A - \mathbf{q}_a^{\rm anom})$ is written as

$$\sum_a (\mathbf{q}_a^A - \mathbf{q}_a^{\rm anom})\cdot\nabla V = \sum_a e_a \int d^3v \langle \hat{f}_a \hat{\phi}\rangle_{\rm ens} \mathbf{v}\cdot\nabla V$$

$$= \frac{c}{4\pi}\langle (\hat{\mathbf{E}}\times\hat{\mathbf{B}})\cdot\nabla V\rangle_{\rm ens}. \qquad (72)$$

Thus, $\Sigma_a(\mathbf{q}_a^A - \mathbf{q}_a^{\rm anom})$ corresponds to the Poynting energy flux of the fluctuating electromagnetic fields, which is not included in the heat flux $\mathbf{q}_a^{\rm anom}$. [The definition of the heat flux $\mathbf{q}_a$ in Eq. (2), which contains $\mathbf{q}_a^{\rm anom}$, is $\mathbf{q}_a \equiv \int d^3v\ f_a \frac{1}{2} m_a |\mathbf{v}-\mathbf{u}_a|^2 (\mathbf{v}-\mathbf{u}_a)$ and takes account of the heat flux due to the particles only.]

VII. CONCLUSIONS AND DISCUSSION

In this work, we have investigated the entropy production mechanisms due to all transport processes in the magnetically confined toroidal plasmas with the gyrokinetic electromagnetic turbulence. The kinetic equation double averaged over the turbulent fluctuations and the gyrophase was derived up to $\mathcal{O}(\delta^2)$. The kinetic equation is employed as the foundation on which the entropy productions by the classical, neoclassical, and anomalous transport processes are kinetically defined. The recursive technique was used to derive the double-averaged kinetic equation, from which the entropy balance equation (40) was obtained. We showed the correspondence between the kinetic and thermodynamic forms of the entropy productions and identified the conjugated pairs of the forces and fluxes for all the transport processes. For the fluctuating part of the kinetic distribution function, we used the nonlinear gyrokinetic equation (27) derived by the recursive technique instead of the noncanonical Hamiltonian formalism[32–34] since the latter is for the total distribution function and is not clearly given for the case with collisions. The collisions are not only the cause of the classical and neoclassical entropy productions but also in essence required for the balance between the anomalous entropy production and the microscopic dissipation in the stationary state as shown in Eq. (38). It would be interesting to monitor the spatiotemporal variation of the anomalous entropy production given by Eq. (31) in the gyrokinetic simulations and examine the validity of the minimum entropy production for the turbulent stationary states.

It was shown that the anomalous transport equations satisfy the Onsager symmetry within the quasilinear framework. For the gyrokinetic electromagnetic fluctuations with parallel wave numbers $k_\parallel \sim L^{-1}$, the magnetic surface average must be taken in order to show the positive definiteness of the anomalous entropy production [see Eq. (39)] and the Onsager symmetry of the quasilinear anomalous transport coefficients [see Eq. (48)]. This need for the surface average implies that microscopic phenomena (or individual realizations in the ensemble) occurring over the distance of $\mathcal{O}(L)$ along the magnetic field lines should be coarse grained, not only by the ensemble average but also by the magnetic surface average to be viewed as an irreversible macroscopic (or thermodynamic) process.

In the strong turbulence regime defined by short lifetimes of the fluctuation components, the proof of the Onsager symmetry for the anomalous transport matrix breaks down due to the nonresonant nature of the fluctuation–particle interactions. Krommes and Hu[35] claim that, instead of the conventional Onsager symmetry for transport equations near thermal equilibria, the generalized Onsager symmetry is valid for transport near the turbulent steady states. The generalized Onsager symmetry is relevant to the incremental transport equations that connect the small deviations of the forces and fluxes from their steady-state values, although the anomalous transport equations considered here and in many other works relate the total anomalous fluxes at the steady state to the total forces. The transport equations for the total anomalous fluxes are generally nonlinear with respect to the forces, even for the quasilinear case, and it is beyond the scope of this work to obtain them for the strong turbulence.

Using the Krook collision model, we derived for $k_\parallel L \ll 1$ the locally symmetric quasilinear transport matrix as a functional of the gyrokinetic electromagnetic turbulence spectra. As for the magnetic fluctuations, the contribution of the perpendicular magnetic field fluctuations to the anomalous transport decreases monotonically with increasing the ratio of the thermal gyroradius to the characteristic perpendicular fluctuation length $\rho/\lambda_\perp$. In contrast, the parallel magnetic fluctuations' contribution, which has not been considered in previous works, becomes negligible at the both limits $\rho/\lambda_\perp \to +0, +\infty$, and comparable to the perpendicular contribution at $\rho/\lambda_\perp = \mathcal{O}(1)$. At low plasma beta, the parallel magnetic fluctuations' effect is small since $\hat{B}_\parallel/B \sim \beta(e_a\hat{\phi}/T_a)$.

The complete energy balance equation (62) derived from the ensemble-averaged kinetic equation (5) shows how the turbulence effects should be included. The anomalous heat flux $\mathbf{q}_a^A$ occurring in the entropy and energy balance equations contains the contribution of the residual microscopic entropy flux given by Eq. (36). Besides the anomalous particle and heat fluxes included in the radial derivative term of the total energy flux, the energy balance is modified by the turbulence through the product of the anomalous radial current and the radial electric field $-e_a\langle J_{a1}^A\rangle \partial\Phi/\partial V$ and the anomalous heating $\langle J_{a3}^A\rangle$. These anomalous terms can cause a large energy exchange between electrons and ions.[29]

The self-consistent turbulent electromagnetic fields satisfy Poisson's equation (66) and Ampère's law given by Eqs. (67) and (68), from which the intrinsic ambipolarity of the anomalous particle fluxes is derived. Then, as in the conventional neoclassical theory, the radial electric field is determined by the ambipolarity condition for the neoclassical nonaxisymmetric particle fluxes, although it is not for the axisymmetric system in which all the particle fluxes are intrinsically ambipolar. We also find for the self-consistent fluctuations that the species summation of the anomalous heating vanishes and that the residual anomalous heat fluxes sum up to the Poynting energy flux of the turbulent electromagnetic fields.

In some operational regions of tokamak plasmas such as high-confinement modes (H modes)[36] and reversed shear configurations,[37] there have been observed transport barriers with significant reduction of anomalous transport to the level of neoclassical transport. Generally, large radial electric field shear (or sheared $\mathbf{E}\times\mathbf{B}$ flow) is considered as a cause of such a reduction of the transport level. In the present work as well as in the conventional neoclassical theory, the $\mathbf{E}\times\mathbf{B}$ flow velocities have been assumed to be $\mathcal{O}(\delta v_T)$ (v_T: the ion thermal velocity, $\delta=\rho/L$). However, this assumption is not suitable to describing the effects of the large radial electric field shear since the radial electric field is undetermined for tokamak plasmas due to the intrinsic ambipolarity of particle fluxes in axisymmetric systems. In the H-mode theory by Shaing *et al.*,[38] the drift kinetic equation with large flows but without fluctuations are used to obtain the neoclassical viscosities and accordingly the ambipolarity condition as the constraint on the radial electric field, which is different from that given in the present work. Such pure neoclassical models as by Shaing *et al.* do not treat interactions between the $\mathbf{E}\times\mathbf{B}$ background flows and the fluctuations through the Reynolds stress,[39,40] but they are considered as another important factor of transition processes occurring in the transport barriers for a self-consistent description of the radial electric field, fluctuations, and transport. As an important task for understanding the transport barrier physics, extension of our theory to that including the large radial electric field and the Reynolds stress is now under investigation.

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APPENDIX A: DEFINITIONS OF CLASSICAL AND NEOCLASSICAL TRANSPORT FLUXES

The radial components of the classical particle and heat fluxes are defined by

$$J_{a1}^{\mathrm{cl}}\equiv\boldsymbol{\Gamma}_a^{\mathrm{cl}}\cdot\boldsymbol{\nabla}V\equiv\frac{c}{e_aB}\,(\mathbf{F}_{a1}\times\mathbf{n})\cdot\boldsymbol{\nabla}V,$$
$$J_{a2}^{\mathrm{cl}}\equiv\frac{1}{T_a}\,\mathbf{q}_a^{\mathrm{cl}}\cdot\boldsymbol{\nabla}V\equiv\frac{c}{e_aB}\,(\mathbf{F}_{a2}\times\mathbf{n})\cdot\boldsymbol{\nabla}V, \tag{A1}$$

respectively, where the friction forces $\mathbf{F}_{a1}$ and $\mathbf{F}_{a2}$ are given by

$$\mathbf{F}_{a1}=\int d^3v\;m_a\mathbf{v}C_a\,,$$
$$\mathbf{F}_{a2}=\int d^3v\;m_a\mathbf{v}\left(x_a^2-\frac{5}{2}\right)C_a\,. \tag{A2}$$

The fluxes J_{a1}^{cl} and J_{a2}^{cl} are conjugate to the thermodynamic forces X_{a1} and X_{a2}, respectively, which are defined in terms of radial gradients of the pressure, electrostatic potential, and temperature as

$$X_{a1}\equiv-\frac{1}{n_a}\frac{\partial p_a}{\partial V}-e_a\frac{\partial\Phi}{\partial V},\quad X_{a2}\equiv-\frac{\partial T_a}{\partial V}. \tag{A3}$$

The neoclassical particle and heat fluxes are given by

$$J_{a1}^{\mathrm{ncl}}\equiv\langle\boldsymbol{\Gamma}_a^{\mathrm{ncl}}\cdot\boldsymbol{\nabla}V\rangle\equiv J_{a1}^{\mathrm{PS}}+J_{a1}^{\mathrm{bp}}+J_{a1}^{\mathrm{na}},$$
$$J_{a2}^{\mathrm{ncl}}\equiv\frac{1}{T_a}\,\langle\mathbf{q}_a^{\mathrm{ncl}}\cdot\boldsymbol{\nabla}V\rangle\equiv J_{a2}^{\mathrm{PS}}+J_{a2}^{\mathrm{bp}}+J_{a2}^{\mathrm{na}}\,. \tag{A4}$$

Here the Pfirsch–Schlüter (J_{aj}^{PS}), the banana-plateau (J_{aj}^{bp}), and nonaxisymmetric (J_{aj}^{na}) parts are defined by

$$J_{a1}^{\mathrm{PS}}\equiv\langle\boldsymbol{\Gamma}_a^{\mathrm{PS}}\cdot\boldsymbol{\nabla}V\rangle\equiv-\frac{c}{e_aB^\theta}\left\langle\frac{F_{\parallel a1}}{B}\left(B_\zeta-\langle B_\zeta\rangle\frac{B^2}{\langle B^2\rangle}\right)\right\rangle,$$
$$J_{a1}^{\mathrm{bp}}\equiv\langle\boldsymbol{\Gamma}_a^{\mathrm{bp}}\cdot\boldsymbol{\nabla}V\rangle\equiv-\frac{c}{e_aB^\theta}\frac{\langle B_\zeta\rangle}{\langle B^2\rangle}\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_a\rangle,$$
$$J_{a1}^{\mathrm{na}}\equiv\langle\boldsymbol{\Gamma}_a^{\mathrm{na}}\cdot\boldsymbol{\nabla}V\rangle\equiv\frac{c}{e_aB^\theta B^\zeta}\langle\mathbf{B}_t\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_a\rangle,$$
$$J_{a2}^{\mathrm{PS}}\equiv\frac{1}{T_a}\langle\mathbf{q}_a^{\mathrm{PS}}\cdot\boldsymbol{\nabla}V\rangle$$
$$\equiv-\frac{c}{e_aB^\theta}\left\langle\frac{F_{\parallel a2}}{B}\left(B_\zeta-\langle B_\zeta\rangle\frac{B^2}{\langle B^2\rangle}\right)\right\rangle,$$
$$J_{a2}^{\mathrm{bp}}\equiv\frac{1}{T_a}\langle\mathbf{q}_a^{\mathrm{bp}}\cdot\boldsymbol{\nabla}V\rangle\equiv-\frac{c}{e_aB^\theta}\frac{\langle B_\zeta\rangle}{\langle B^2\rangle}\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Theta}_a\rangle,$$
$$J_{a2}^{\mathrm{na}}\equiv\frac{1}{T_a}\langle\mathbf{q}_a^{\mathrm{na}}\cdot\boldsymbol{\nabla}V\rangle\equiv\frac{c}{e_aB^\theta B^\zeta}\langle\mathbf{B}_t\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Theta}_a\rangle, \tag{A5}$$

where the viscosities $\boldsymbol{\pi}_a$ and Θ_a are defined by

$$\boldsymbol{\pi}_a\equiv(p_{\parallel a}-p_{\perp a})\left(\mathbf{nn}-\frac{1}{3}\,\mathsf{I}\right)$$
$$\equiv\int d^3v\;m_a\left(v_\parallel^2-\frac{1}{2}\,v_\perp^2\right)\bar{f}_{a1}\left(\mathbf{nn}-\frac{1}{3}\,\mathsf{I}\right),$$
$$\Theta_a\equiv(\Theta_{\parallel a}-\Theta_{\perp a})\left(\mathbf{nn}-\frac{1}{3}\,\mathsf{I}\right)$$
$$\equiv\int d^3v\;m_a\left(v_\parallel^2-\frac{1}{2}\,v_\perp^2\right)\left(x_a^2-\frac{5}{2}\right)\bar{f}_{a1}\left(\mathbf{nn}-\frac{1}{3}\,\mathsf{I}\right). \tag{A6}$$

The inductive electric field $\mathbf{E}^{(A)}\equiv-c^{-1}\,\partial\mathbf{A}/\partial t$ also produces the radial particle flux $J_{a1}^{(E)}\equiv\langle\boldsymbol{\Gamma}_a^{(E)}\cdot\boldsymbol{\nabla}V\rangle$, defined by

$$J_{a1}^{(E)}\equiv n_ac\left\langle\frac{\mathbf{E}^{(A)}\times\mathbf{n}}{B}\cdot\boldsymbol{\nabla}V\right\rangle$$
$$-\frac{n_ac}{B^\theta}\left\langle\frac{E_\parallel^{(A)}}{B}\left(B_\zeta-\langle B_\zeta\rangle\frac{B^2}{\langle B^2\rangle}\right)\right\rangle. \tag{A7}$$

Here we have used the Hamada coordinates (V,θ,ζ) with the normalization $\oint d\theta=\oint d\zeta=1$ to define the contravariant and covariant components of the magnetic field $B^\theta\equiv\mathbf{B}\cdot\boldsymbol{\nabla}\theta$, $B^\zeta\equiv\mathbf{B}\cdot\boldsymbol{\nabla}\zeta$, $B_\theta\equiv\mathbf{B}\cdot\partial\mathbf{x}/\partial\theta$, $B_\zeta\equiv\mathbf{B}\cdot\partial\mathbf{x}/\partial\zeta$, and the toroidal magnetic field $\mathbf{B}_t\equiv B^\zeta\,\partial\mathbf{x}/\partial\zeta$. Another pair of the flux and the force is (J_E,X_E), where J_E is defined in terms of the total parallel current $J_\parallel\equiv\Sigma_a n_ae_au_{\parallel a}$ as

$$J_E \equiv \frac{\langle BJ_\parallel \rangle}{\langle B^2 \rangle^{1/2}}, \tag{A8}$$

and X_E is given by the parallel electric field $E_\parallel$ as

$$X_E \equiv \frac{\langle BE_\parallel \rangle}{\langle B^2 \rangle^{1/2}}. \tag{A9}$$

APPENDIX B: DERIVATION OF EQ. (29)

Here we find how to derive Eq. (29) from Eq. (24), or in other words, how to rewrite the kinetic form of the anomalous entropy production by the thermodynamic form.

Using integration by parts, Eq. (24) is rewritten as

$$\begin{aligned}
\dot{S}_a^A &= \int d^3v \left[-\frac{e_a}{m_a} \left\langle \hat{f}_a \left(\hat{\mathbf{E}} + \frac{1}{c} \mathbf{v} \times \hat{\mathbf{B}} \right) \right\rangle_{\text{ens}} \cdot \frac{\partial \ln f_{aM}}{\partial \mathbf{v}} \right. \\
&\quad \left. + (\ln f_{aM} + 1) \frac{d\tilde{f}_a^A}{dt} \right] \\
&= \frac{e_a}{T_a} \int d^3v \langle \hat{f}_a^{(1)} \hat{\mathbf{E}}^{(2)} + \hat{f}_a^{(2)} \hat{\mathbf{E}}^{(1)} \rangle_{\text{ens}} \cdot \mathbf{v} \\
&\quad + \boldsymbol{\nabla} \cdot \left(\int d^3v (\ln f_{aM} + 1) \tilde{f}_a^A \mathbf{v}_\perp \right) \\
&\quad - \int d^3v \, \tilde{f}_a^A \mathbf{v}_\perp \cdot \frac{\partial \ln f_{aM}}{\partial \mathbf{x}'}.
\end{aligned} \tag{B1}$$

Here it should be noted that, since $\dot{S}_a^A = \mathscr{O}(\delta^2)$, its calculation requires the fluctuating distribution function $\hat{f}_a$ and the turbulent electric field $\hat{\mathbf{E}}$ up to $\mathscr{O}(\delta^2)$:

$$\begin{aligned}
\hat{f}_a &= \hat{f}_a^{(1)} + \hat{f}_a^{(2)} + \mathscr{O}(\delta^3), \\
\hat{\mathbf{E}} &= \hat{\mathbf{E}}^{(1)} + \hat{\mathbf{E}}^{(2)} + \mathscr{O}(\delta^3).
\end{aligned} \tag{B2}$$

The lowest-order parts of $\hat{f}_a$ and $\hat{\mathbf{E}}$ are $\mathscr{O}(\delta)$ and their Fourier amplitudes for the perpendicular wave number vector $\mathbf{k}_\perp$ are written as

$$\begin{aligned}
\hat{f}_a^{(1)}(\mathbf{k}_\perp) &= -\frac{e_a \hat{\phi}(\mathbf{k}_\perp)}{T_a} f_{aM} + \hat{h}_a(\mathbf{k}_\perp) e^{iL_a(\mathbf{k}_\perp)}, \\
\hat{\mathbf{E}}^{(1)}(\mathbf{k}_\perp) &= -i\mathbf{k}_\perp \hat{\phi}(\mathbf{k}_\perp).
\end{aligned} \tag{B3}$$

For the $\mathscr{O}(\delta^2)$ parts $\hat{f}_a^{(2)}$ and $\hat{\mathbf{E}}^{(2)}$, we have

$$\begin{aligned}
&\Omega_a e^{iL_a(\mathbf{k}_\perp)} \frac{\partial}{\partial \xi} [e^{-iL_a(\mathbf{k}_\perp)} \hat{f}_a^{(2)}(\mathbf{k}_\perp)] \\
&= \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} + \frac{e_a}{m_a} \mathbf{E} \cdot \frac{\partial}{\partial \mathbf{v}} \right) \hat{f}_a^{(1)}(\mathbf{k}_\perp) \\
&\quad + \frac{e_a}{m_a} \left[\hat{\mathbf{E}}^{(1)}(\mathbf{k}_\perp) + \frac{1}{c} \mathbf{v} \times \hat{\mathbf{B}}^{(1)}(\mathbf{k}_\perp) \right] \cdot \frac{\partial f_{a1}}{\partial \mathbf{v}} \\
&\quad + \frac{e_a}{m_a} \left[\hat{\mathbf{E}}^{(2)}(\mathbf{k}_\perp) + \frac{1}{c} \mathbf{v} \times \hat{\mathbf{B}}^{(2)}(\mathbf{k}_\perp) \right] \cdot \frac{\partial f_{a0}}{\partial \mathbf{v}} \\
&\quad + \frac{e_a}{m_a} \sum_{\mathbf{k}_\perp' + \mathbf{k}_\perp'' = \mathbf{k}_\perp} \left[\hat{\mathbf{E}}^{(1)}(\mathbf{k}_\perp') + \frac{1}{c} \mathbf{v} \times \hat{\mathbf{B}}^{(1)}(\mathbf{k}_\perp') \right] \\
&\quad \cdot \frac{\partial \hat{f}_a^{(1)}(\mathbf{k}_\perp'')}{\partial \mathbf{v}} + C_a^L [f_a^{(1)}(\mathbf{k}_\perp)],
\end{aligned} \tag{B4}$$

$$\hat{\mathbf{E}}^{(2)}(\mathbf{k}_\perp) = -\boldsymbol{\nabla} \hat{\phi}(\mathbf{k}_\perp) - \frac{1}{c} \frac{\partial \hat{\mathbf{A}}(\mathbf{k}_\perp)}{\partial t}.$$

Using Eqs. (B3), (B4), and

$$\Omega_a e^{iL_a(\mathbf{k}_\perp)} \frac{\partial e^{-iL_a(\mathbf{k}_\perp)}}{\partial \xi} = -i\mathbf{k}_\perp \cdot \mathbf{v}, \tag{B5}$$

we obtain

$$\begin{aligned}
\int d^3v \langle \hat{f}_a^{(1)} \hat{\mathbf{E}}^{(2)} \rangle_{\text{ens}} \cdot \mathbf{v} &= \int d^3v \sum_{\mathbf{k}_\perp} \left\langle \hat{h}_a^*(\mathbf{k}_\perp) e^{-iL_a(\mathbf{k}_\perp)} \mathbf{v} \right. \\
&\quad \left. \cdot \left(-\boldsymbol{\nabla} \hat{\phi}(\mathbf{k}_\perp) - \frac{1}{c} \frac{\partial \hat{\mathbf{A}}(\mathbf{k}_\perp)}{\partial t} \right) \right\rangle_{\text{ens}}
\end{aligned} \tag{B6}$$

and

$$\begin{aligned}
&\int d^3v \langle \hat{f}_a^{(2)} \hat{\mathbf{E}}^{(1)} \rangle_{\text{ens}} \cdot \mathbf{v} \\
&= \Omega_a \int d^3v \sum_{\mathbf{k}_\perp} \langle \hat{f}_a^{(2)*}(\mathbf{k}_\perp) \hat{\phi}(\mathbf{k}_\perp) \rangle_{\text{ens}} e^{iL_a(\mathbf{k}_\perp)} \frac{\partial e^{-iL_a(\mathbf{k}_\perp)}}{\partial \xi} \\
&= -\Omega_a \int d^3v \sum_{\mathbf{k}_\perp} \left\langle \hat{\phi}^*(\mathbf{k}_\perp) e^{iL_a(\mathbf{k}_\perp)} \right. \\
&\quad \left. \times \frac{\partial}{\partial \xi} [e^{-iL_a(\mathbf{k}_\perp)} \hat{f}_a^{(2)}(\mathbf{k}_\perp)] \right\rangle_{\text{ens}} \\
&= -\int d^3v \sum_{\mathbf{k}_\perp} \left\langle \hat{\phi}^*(\mathbf{k}_\perp) \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \right) \hat{f}_a^{(1)}(\mathbf{k}_\perp) \right\rangle_{\text{ens}}.
\end{aligned} \tag{B7}$$

Noting that

$$\tilde{f}_a^A \mathbf{v}_\perp = \frac{\partial}{\partial \xi} (\tilde{f}_a^A \mathbf{v}_\perp \times \mathbf{n}) - \frac{\partial \tilde{f}_a^A}{\partial \xi} \mathbf{v}_\perp \times \mathbf{n}, \tag{B8}$$

$$\frac{\partial \tilde{f}_a^A}{\partial \xi} = -\frac{1}{\Omega_a} \tilde{\mathscr{D}}_a = \frac{c}{B} \frac{\partial}{\partial \mathbf{v}} \cdot \left\langle \hat{f}_a^{(1)} \left(\hat{\mathbf{E}}^{(1)} + \frac{1}{c} \mathbf{v} \times \hat{\mathbf{B}}^{(1)} \right) \right\rangle_{\text{ens}}, \tag{B9}$$

and

$$\frac{\partial \ln f_{aM}}{\partial \mathbf{x}'} = -\frac{1}{T_a} \left[X_{a1} + X_{a2} \left(x_a^2 - \frac{5}{2} \right) \right] \boldsymbol{\nabla} V, \tag{B10}$$

the last two terms on the right-hand side of Eq. (B1) are rewritten as

$$\begin{aligned}
&\boldsymbol{\nabla} \cdot \left(\int d^3v (\ln f_{aM} + 1) \tilde{f}_a^A \mathbf{v}_\perp \right) \\
&= -\boldsymbol{\nabla} \cdot \left[\frac{c}{B} \int d^3v (\ln f_{aM} + 1)(\mathbf{v} \times \mathbf{n}) \frac{\partial}{\partial \mathbf{v}} \right. \\
&\quad \left. \cdot \left\langle \hat{f}_a^{(1)} \left(\hat{\mathbf{E}}^{(1)} + \frac{1}{c} \mathbf{v} \times \hat{\mathbf{B}}^{(1)} \right) \right\rangle_{\text{ens}} \right]
\end{aligned}$$

$$= -\nabla\cdot\mathbf{J}^A_{Sa} - \nabla\cdot\left(\frac{m_a c}{T_a B}\int d^3v \langle \hat{f}^{(1)}_a \hat{\mathbf{E}}^{(1)}\rangle_{\rm ens}\cdot\mathbf{v}(\mathbf{v}\times\mathbf{n})\right) \tag{B11}$$

and

$$-\int d^3v\ \tilde{f}^A_a \mathbf{v}_\perp\cdot\frac{\partial \ln f_{aM}}{\partial \mathbf{x}}$$
$$= -\frac{c}{T_a B}\int d^3v(\mathbf{v}\times\mathbf{n}\cdot\nabla V)\left[X_{a1}+X_{a2}\left(x_a^2-\frac{5}{2}\right)\right]\frac{\partial}{\partial\mathbf{v}}\cdot\left\langle \hat{f}^{(1)}_a\left(\hat{\mathbf{E}}^{(1)}+\frac{1}{c}\mathbf{v}\times\hat{\mathbf{B}}^{(1)}\right)\right\rangle_{\rm ens}$$
$$= \frac{1}{T_a}(J^A_{a1}X_{a1}+J^A_{a2}X_{a2}) + \frac{m_a c X_{a2}}{T_a^2 B}\times\int d^3v\langle \hat{f}^{(1)}_a\hat{\mathbf{E}}^{(1)}\rangle_{\rm ens}\cdot\mathbf{v}(\mathbf{v}\times\mathbf{n}\cdot\nabla V). \tag{B12}$$

Here, the anomalous entropy, particle, and heat fluxes are defined by

$$\mathbf{J}^A_{Sa} = -\frac{c}{B}\int d^3v(\ln f_{aM}+1)\left\langle \hat{f}^{(1)}_a\left(\hat{\mathbf{E}}^{(1)}+\frac{1}{c}\mathbf{v}\times\hat{\mathbf{B}}^{(1)}\right)\right\rangle_{\rm ens}\times\mathbf{n},$$
$$J^A_{a1} = \frac{c}{B}\int d^3v\left\langle \hat{f}^{(1)}_a\left(\hat{\mathbf{E}}^{(1)}+\frac{1}{c}\mathbf{v}\times\hat{\mathbf{B}}^{(1)}\right)\right\rangle_{\rm ens}\times\mathbf{n}\cdot\nabla V, \tag{B13}$$
$$J^A_{a2} = \frac{c}{B}\int d^3v\left(x_a^2-\frac{5}{2}\right)\times\left\langle \hat{f}^{(1)}_a\left(\hat{\mathbf{E}}^{(1)}+\frac{1}{c}\mathbf{v}\times\hat{\mathbf{B}}^{(1)}\right)\right\rangle_{\rm ens}\times\mathbf{n}\cdot\nabla V,$$

respectively, which are shown to be equivalent to the definitions in Eqs. (30) and (34) from the following relation:

$$\frac{c}{B}\left\langle\overline{\hat{h}^*_a(\mathbf{k}_\perp)e^{-iL_a(\mathbf{k}_\perp)}\left(\hat{\mathbf{E}}^{(1)}(\mathbf{k}_\perp)+\frac{1}{c}\mathbf{v}\times\hat{\mathbf{B}}^{(1)}(\mathbf{k}_\perp)\right)}\right\rangle_{\rm ens}\times\mathbf{n}$$
$$= -i\frac{c}{B}\langle\hat{h}^*_a(\mathbf{k}_\perp)\hat{\phi}_a(\mathbf{k}_\perp)\rangle_{\rm ens}(\mathbf{k}_\perp\times\mathbf{n})$$
$$= \langle\hat{h}^*_a(\mathbf{k}_\perp)\hat{\mathbf{v}}_{da}(\mathbf{k}_\perp)\rangle_{\rm ens}. \tag{B14}$$

Finally, substituting Eqs. (B6), (B7), (B11), and (B12) into Eq. (B1), we obtain

$$\dot{S}^A_a = -\nabla\cdot\mathbf{J}^A_{Sa} + \frac{1}{T_a}(J^A_a X_{a1}+J^A_{a2}X_{a2}+J^A_{a3}) + \frac{\partial}{\partial t}\left[\int d^3v\sum_{\mathbf{k}_\perp}\frac{\langle|\hat{f}^{(1)}_a(\mathbf{k}_\perp)^2-|\hat{h}_a(\mathbf{k}_\perp)|^2\rangle}{2f_{aM}}\right] + \nabla\cdot\left[\int d^3v\sum_{\mathbf{k}_\perp}\frac{\langle|f^{(1)}_a(\mathbf{k}_\perp)|^2-|\hat{h}_a(\mathbf{k}_\perp)|^2\rangle}{2f_{aM}}v_\parallel\mathbf{n}\right], \tag{B15}$$

which is the same one as Eq. (29).

APPENDIX C: PROOF OF ONSAGER SYMMETRY OF QUASILINEAR ANOMALOUS TRANSPORT MATRIX

Here, the Onsager symmetry given in Eq. (48) for the quasilinear anomalous transport matrix is proved by the technique similar to those in Ref. 2 and Ref. 5. This proof is valid for the linearized Landau collision operator without assuming the Krook model. The solution h_a of the gyrokinetic equation (27) with the nonlinear term neglected is written as a linear function of the thermodynamic forces X^A_{am}:

$$\hat{h}_a(\mathbf{k}_\perp) = \sum_b\sum_{m=1}^3\hat{G}_{abm}(\mathbf{k}_\perp)X^A_{bm}, \tag{C1}$$

where $\hat{G}_{abm}(\mathbf{k}_\perp)$ $(m=1,2,3)$ are the Green's functions that satisfy

$$\left(\frac{\partial}{\partial t}+i(\omega_E+\omega_{Da})+v_\parallel\mathbf{n}\cdot\nabla'\right)\hat{G}_{abm}(\mathbf{k}_\perp) - \overline{e^{-iL_a(\mathbf{k}_\perp)}\sum_{a'}C^L_{aa'}[\hat{G}_{abm}(\mathbf{k}_\perp)e^{iL_a(\mathbf{k}_\perp)},\hat{G}_{a'bm}(\mathbf{k}_\perp)e^{iL_{a'}(\mathbf{k}_\perp)}]} = \delta_{ab}f_{bM}\hat{W}_{bm}(\mathbf{k}_\perp), \tag{C2}$$

with

$$\begin{bmatrix}\hat{W}_{a1}(\mathbf{k}_\perp)\\ \hat{W}_{a2}(\mathbf{k}_\perp)\\ \hat{W}_{a3}(\mathbf{k}_\perp)\end{bmatrix} \equiv \begin{bmatrix} -i\frac{c}{B}(\mathbf{k}_\perp\times\mathbf{n}\cdot\nabla V)\hat{\phi}_a(\mathbf{k}_\perp)\\ -i\frac{c}{B}(\mathbf{k}_\perp\times\mathbf{n}\cdot\nabla V)(x_a^2-\frac{5}{2})\hat{\phi}_a(\mathbf{k}_\perp)\\ e_a\frac{\partial\hat{\phi}_a(\mathbf{k}_\perp)}{\partial t}\end{bmatrix}. \tag{C3}$$

Substituting Eq. (C1) into Eqs. (34), we obtain the quasilinear anomalous transport equations (46), with the anomalous transport coefficients given by

$$(L^A)^{ab}_{mn} = \int d^3v\sum_{\mathbf{k}_\perp}\langle\hat{W}_{am}(\mathbf{k}_\perp)\hat{G}^*_{abn}(\mathbf{k}_\perp)\rangle_{\rm ens} \quad (m,n=1,2,3). \tag{C4}$$

Here $\hat{G}_{abm}$ are functions of the perpendicular wave number vector $\mathbf{k}_\perp$, the position $\mathbf{x}'$ (which is used to represent the spatial dependence along the magnetic field line), the time t,

and the velocity space variables ($\epsilon \equiv \frac{1}{2} m_a v^2 + e_a \Phi$, $\mu \equiv m_a v_\perp^2 / 2B$, $\sigma \equiv \mathbf{v}\cdot\mathbf{n}/|\mathbf{v}\cdot\mathbf{n}|$). We also note that $\hat{G}_{abm}$ depends on equilibrium parameters contained in Eq. (C2) such as the equilibrium (or ensemble-averaged) magnetic field $\mathbf{B} = B\mathbf{n}$, and that they are functionals of the spectra of the electromagnetic fluctuations $\hat{\boldsymbol{\phi}}(\mathbf{k}_\perp) \equiv [\hat{\phi}(\mathbf{k}_\perp), \hat{A}_\parallel(\mathbf{k}_\perp), \hat{B}_\parallel(\mathbf{k}_\perp)/k_\perp]$:

$$\hat{G}_{abm} = \hat{G}_{abm}[\mathbf{k}_\perp, \mathbf{x}', t; \epsilon, \mu, \sigma; \mathbf{B}, \{\hat{\boldsymbol{\phi}}\}]. \tag{C5}$$

Then, Eq. (C4) shows that the anomalous transport coefficients $(L^A)^{ab}_{mn}$ are also functionals of the fluctuation spectra and that it also contains $\mathbf{B}$ as a parameter:

$$(L^A)^{ab}_{mn} = (L^A)^{ab}_{mn}[\mathbf{B}, \{\hat{\boldsymbol{\phi}}\}], \tag{C6}$$

where the spectra of the electromagnetic fluctuations $\{\hat{\boldsymbol{\phi}}\}$ are assumed to be given *a priori*.

Now let us divide the fluctuations $\hat{\boldsymbol{\phi}}$ into even and odd parts with respect to the time reversal:

$$\begin{aligned} &\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}_+ + \hat{\boldsymbol{\phi}}_-, \\ &\hat{\boldsymbol{\phi}}_+(t) = \hat{\boldsymbol{\phi}}_+(-t), \quad \hat{\boldsymbol{\phi}}_-(t) = -\hat{\boldsymbol{\phi}}_-(-t). \end{aligned} \tag{C7}$$

According to this division, $\hat{W}_{am}$ and $\hat{G}_{abm}$ are divided as

$$\begin{aligned} &\hat{W}_{am} = \hat{W}_{am}[\{\hat{\boldsymbol{\phi}}_+\}] + \hat{W}_{am}[\{\hat{\boldsymbol{\phi}}_-\}] \equiv \hat{Y}_{am} + \hat{Z}_{am}, \\ &\hat{G}_{abm} = \hat{G}_{abm}[\{\hat{\boldsymbol{\phi}}_+\}] + \hat{G}_{abm}[\{\hat{\boldsymbol{\phi}}_-\}] \equiv \hat{H}_{abm} + \hat{I}_{abm}. \end{aligned} \tag{C8}$$

The functions on the right-hand side of Eq. (C8) satisfy

$$\begin{aligned} &\hat{Y}_{am}[\{\hat{\boldsymbol{\phi}}(t)\}] = \hat{Y}_{am}[\{\hat{\boldsymbol{\phi}}(-t)\}], \\ &Z_{am}[\{\hat{\boldsymbol{\phi}}(t)\}] = -\hat{Z}_{am}[\{\hat{\boldsymbol{\phi}}(-t)\}], \\ &\hat{H}_{abm}[\{\hat{\boldsymbol{\phi}}(t)\}] = \hat{H}_{abm}[\{\hat{\boldsymbol{\phi}}(-t)\}], \\ &\hat{I}_{abm}[\{\hat{\boldsymbol{\phi}}(t)\}] = -\hat{I}_{abm}[\{\hat{\boldsymbol{\phi}}(-t)\}]. \end{aligned} \tag{C9}$$

Then, Eq. (C2) also separates into the two parts:

$$\begin{aligned} &\left[\frac{\partial}{\partial t} + i(\omega_E + \omega_{Da}) + v_\parallel \mathbf{n}\cdot\boldsymbol{\nabla}'\right]\hat{H}_{abm} \\ &\quad - \overline{e^{-iL_a} \sum_{a'} C^L_{aa'} [\hat{H}_{abm} e^{iL_a}, \hat{H}_{a'bm} e^{iL_{a'}}]} \\ &\quad = \delta_{ab} f_{bM} \hat{Y}_{bm}, \\ &\left[\frac{\partial}{\partial t} + i(\omega_E + \omega_{Da}) + v_\parallel \mathbf{n}\cdot\boldsymbol{\nabla}'\right]\hat{I}_{abm} \\ &\quad - \overline{e^{-iL_a} \sum_{a'} C^L_{aa'} [\hat{I}_{abm} e^{iL_a}, \hat{I}_{a'bm} e^{iL_{a'}}]} \\ &\quad = \delta_{ab} f_{bM} \hat{Z}_{bm}. \end{aligned} \tag{C10}$$

Next, we consider the transformation $(t, \mathbf{B}) \to (-t, -\mathbf{B})$. Noting that $\hat{Y}_{am}$ and $\hat{Z}_{am}$ are odd and even with respect to this transformation, respectively:

$$\begin{aligned} &\hat{Y}_{am}(t, \mathbf{B}) = -\hat{Y}_{am}(-t, -\mathbf{B}), \\ &\hat{Z}_{am}(t, \mathbf{B}) = \hat{Z}_{am}(-t, -\mathbf{B}), \end{aligned} \tag{C11}$$

we find that Eqs. (C10) are separated into even and odd parts as

$$\begin{aligned} &\left[\frac{\partial}{\partial t} + i(\omega_E + \omega_{Da}) + v_\parallel \mathbf{n}\cdot\boldsymbol{\nabla}'\right]\hat{H}^-_{abm} \\ &\quad - \overline{e^{-iL_a} \sum_{a'} C^L_{aa'} [\hat{H}^+_{abm} e^{iL_a} \hat{H}^+_{a'bm} e^{iL_{a'}}]} = 0, \\ &\left[\frac{\partial}{\partial t} + i(\omega_E + \omega_{Da}) + v_\parallel \mathbf{n}\cdot\boldsymbol{\nabla}'\right]\hat{H}^+_{abm} \\ &\quad - \overline{e^{-iL_a} \sum_{a'} C^L_{aa'} [\hat{H}^-_{abm} e^{iL_a} \hat{H}^-_{a'bm} e^{iL_{a'}}]} \\ &\quad = \delta_{ab} f_{bM} \hat{Y}_{bm}, \\ &\left[\frac{\partial}{\partial t} + i(\omega_E + \omega_{Da}) + v_\parallel \mathbf{n}\cdot\boldsymbol{\nabla}'\right]\hat{I}^-_{abm} \\ &\quad - \overline{e^{-iL_a} \sum_{a'} C^L_{aa'} [\hat{I}^+_{abm} e^{iL_a} \hat{I}^+_{a'bm} e^{iL_{a'}}]} = \delta_{ab} f_{bM} \hat{Z}_{bm}, \\ &\left[\frac{\partial}{\partial t} + i(\omega_E + \omega_{Da}) + v_\parallel \mathbf{n}\cdot\boldsymbol{\nabla}'\right]\hat{I}^+_{abm} \\ &\quad - \overline{e^{-iL_a} \sum_{a'} C^L_{aa'} [\hat{I}^-_{abm} e^{iL_a}, \hat{I}^-_{a'bm} e^{iL_{a'}}]} = 0, \end{aligned} \tag{C12}$$

where the superscripts + and − represent even and odd parts of the functions:

$$\begin{aligned} &\hat{H}^+_{abm}(t, \mathbf{B}) = \hat{H}^+_{abm}(-t, -\mathbf{B}), \\ &\hat{H}^-_{abm}(t, \mathbf{B}) = -\hat{H}^-_{abm}(-t, -\mathbf{B}), \\ &\hat{I}^+_{abm}(t, \mathbf{B}) = \hat{I}^+_{abm}(-t, -\mathbf{B}), \\ &\hat{I}^-_{abm}(t, \mathbf{B}) = -\hat{I}^-_{abm}(-t, -\mathbf{B}). \end{aligned} \tag{C13}$$

From Eqs. (C12) and the self-adjointness of the linearized collision operator C^L_{ab}, we can derive the following equations:

$$T_a M^{ab}_{mn} \equiv T_a \left\langle \int d^3v \sum_{\mathbf{k}_\perp} \langle (\hat{Y}_{am} + \hat{Z}_{am})(\hat{H}^{-*}_{abn} + \hat{I}^{+*}_{abn}) \rangle_{\rm ens} \right\rangle$$

$$= - \sum_{a',b'} T_{a'} \left\langle \int d^3v \, \frac{1}{f_{a'M}} \sum_{\mathbf{k}_\perp} \{ \langle (\hat{H}^{+*}_{a'am} + \hat{I}^{-*}_{a'am}) e^{-iL_{a'}} C^L_{a'b'} [(\hat{H}^{+}_{a'bn} + \hat{I}^{-}_{a'bn}) e^{iL_{a'}}, (\hat{H}^{+}_{b'bn} + \hat{I}^{-}_{b'bn}) e^{iL_{b'}}] \rangle_{\rm ens} \right.$$

$$\left. + \langle (\hat{H}^{-*}_{a'bn} + \hat{I}^{+*}_{a'bn}) e^{-iL_{a'}} C^L_{a'b'} [(\hat{H}^{-}_{a'am} + \hat{I}^{+}_{a'am}) e^{iL_{a'}}, (\hat{H}^{-}_{b'am} + \hat{I}^{+}_{b'am}) e^{iL_{b'}}] \rangle_{\rm ens} \} \right\rangle \tag{C14}$$

$$T_a N^{ab}_{mn} \equiv T_a \left\langle \int d^3v \sum_{\mathbf{k}_\perp} \langle (\hat{Y}_{am} + \hat{Z}_{am})(\hat{H}^{+*}_{abn} + \hat{I}^{-*}_{abn}) \rangle_{\rm ens} \right\rangle$$

$$= \sum_{a'} T_{a'} \left\langle \int d^3v \, \frac{1}{f_{a'M}} \sum_{\mathbf{k}_\perp} \left[\left\langle (\hat{H}^{+*}_{a'bn} + \hat{I}^{-*}_{a'bn}) \right.\right.\right.$$

$$\times \left(\frac{\partial}{\partial t} + i(\omega_E + \omega_{Da}) + v_\parallel \mathbf{n} \cdot \boldsymbol{\nabla}' \right)$$

$$\left. \times (\hat{H}^{+}_{a'am} + \hat{I}^{-}_{a'am}) \right\rangle_{\rm ens} - \left\langle (\hat{H}^{-*}_{a'am} + \hat{I}^{+*}_{a'am}) \left(\frac{\partial}{\partial t} \right.\right.$$

$$\left.\left.\left. + i(\omega_E + \omega_{Da}) + v_\parallel \mathbf{n} \cdot \boldsymbol{\nabla}' \right) (\hat{H}^{-}_{a'bn} + \hat{I}^{+}_{a'bn}) \right\rangle_{\rm ens} \right\rangle .$$

Using the self-adjointness of C^L_{ab} again, we find

$$T_a M^{ab}_{mn} = T_b M^{ba}_{nm}, \quad T_a N^{ab}_{mn} = - T_b N^{ba}_{nm}. \tag{C15}$$

From the symmetry properties given by Eqs. (C9), (C11), and (C13), it is shown that

$$\begin{aligned} M^{ab}_{mn}[\mathbf{B}, \{\hat{\boldsymbol{\phi}}(t)\}] &= M^{ab}_{mn}[-\mathbf{B}, \{\hat{\boldsymbol{\phi}}(-t)\}], \\ N^{ab}_{mn}[\mathbf{B}, \{\hat{\boldsymbol{\phi}}(t)\}] &= - N^{ab}_{mn}[-\mathbf{B}, \{\hat{\boldsymbol{\phi}}(-t)\}]. \end{aligned} \tag{C16}$$

Finally, using Eqs. (C15) and (C16), and noting that $\langle (L^A)^{ab}_{mn} \rangle \equiv M^{ab}_{mn} + N^{ab}_{mn}$, we obtain the Onsager symmetry of the quasilinear anomalous transport matrix,

$$T_a \langle (L^A)^{ab}_{mn}[\mathbf{B}, \{\hat{\boldsymbol{\phi}}(t)\}] \rangle = T_b \langle (L^A)^{ba}_{nm}[-\mathbf{B}, \{\hat{\boldsymbol{\phi}}(-t)\}] \rangle, \tag{C17}$$

which is the same as given by Eq. (48).

When the parallel correlation length for the fluctuations is much shorter than the equilibrium scale length, we can use the Fourier transform for spatial variation of the fluctuations along the magnetic field lines and a procedure similar to the above shows that, in this case, the Onsager symmetry is valid for the local quasilinear anomalous transport coefficients (without taking the magnetic-surface-average),

$$T_a (L^A)^{ab}_{mn}[\mathbf{B}, \{\hat{\boldsymbol{\phi}}(t)\}] = T_b (L^A)^{ba}_{nm}[-\mathbf{B}, \{\hat{\boldsymbol{\phi}}(-t)\}]. \tag{C18}$$

Furthermore, if we use the simple Krook collision model $C^L_a(\hat{h}_a e^{iL_a}) = - \nu_a \hat{h}_a e^{iL_a}$, we find that the quasilinear transport coefficients $(L^A)^{ab}_{mn} = \delta_{ab} (L^A)^a_{mn}$ are given by Eqs. (50) and that their antisymmetric parts that correspond to N^{ab}_{mn} vanish.

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