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Quasiclassical theory of dielectronic recombination in plasmas

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We consider the effects of plasmas on dielectronic-recombination (DR) rates. Effects of plasmas electric fields on DR rates are analyzed in detail in the space of parabolic quantum numbers. A quasiclassical approach is used to obtain general analytical expressions for DR rates in the parabolic basis for arbitrary types of ions having transitions without change of core principal quantum numbers ($\Delta n=0$ transitions) responsible for the main contribution to DR rates. The approach makes it possible to investigate scaling laws for dependences of both total and differential DR rates on atomic parameters. Effects of electron collisions and ionization are taken into account with the help of cutoff procedures. Numerical data are presented for Li- and Na-like ions under typical plasma conditions. A comparison with numerical calculations for specific ions is presented.

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I. INTRODUCTION

Effects of electric fields on dielectronic recombination (DR) rates are under broad investigation in last two decades, see Refs [1–5] and references there. These effects are connected with an evolution of highly excited atomic states populated in the DR capture of an electron colliding with ions having complex cores. The main contribution to the process comes from transitions without change of cores principle quantum numbers ($\Delta n=0$ transitions). The electric field and plasma effects can be separated into three groups: (1) An enhance of the phase-space volume contributing to the recombination due to a transformation of the ion atomic energy states from the spherical quantization to the parabolic one under the action of the field; (2) a decrease of a quantity of excited atomic states responsible for DR because of ionization and energy-mixing effects in the electric field; and (3) kinetic effects due to electron collisions after (or during) the core stabilization.

The effects mentioned are usually taken into account with the help of specific numerical calculations for a particular ion. At the same time the ion energy states responsible for DR are of an universal Rydberg type so DR effects must follow general scaling laws. It is a goal of the present paper to investigate general properties of plasmas electric-field effects on DR rates. Note that we are interested here in the action of a plasma microfield, which is much more strong as compared with laboratory electric fields. So the atomic states mixing effects can be considered to be full, an account of a partial mixing being described by a simple cutoff receipt. The first group mentioned above is taken into account in the frame of a quasiclassical approach.

To make clear the reasons for the application of quasiclassical methods, one should note that the plasma electrons responsible for strong dielectronic capture are classical ones. Really let us consider a multicharged ion with an ion charge Z and a complex atomic core having transitions without change of its principle quantum number n ($\Delta n=0$ transitions). The energy of these transitions is of the order of Z atomic units (a.u.) so the energy $E=mv^2/2$ (v is the electron velocity) of the captured electron must be smaller than Z. At the same time the ionization potential of the ion is of the order of Z^2 that is much larger than the electron energy E. These conditions are simply the conditions of a classical electron motion in the field of the multicharged ion

$$E<Z\ll Z^2 \quad \text{or} \quad Ze^2/\hbar v\gg 1. \tag{1.1}$$

Consequently, the electron captured in the field of the multicharged ion, which undergoes no change in principal quantum number, can be considered on the basis of pure classical mechanics. To do so let us consider matrix elements of the electron-electron interaction e^2/r_{12} [r_{12} is the distance between atomic (1) and colliding (2) electrons] in the dipole approximation $\mathbf{r}_1\mathbf{r}_2/r_2^3$ presenting the wave function of the system as a product of the core wave function and the excited electron wave function. Then the matrix element is equal to the product of a core matrix element $(r_1)_{if}$ between the initial (i) and finite (f) core states and the matrix element of the electric field $\mathbf{r}_2/r_2^3$ produced by the electron colliding with the nucleus. The last one can be expressed in terms of the electron acceleration $d^2\mathbf{r}_2/dt^2$ according to the electron motion equation in the ion field,

$$\mathbf{r}^2/r_2^3=Z^{-1}d^2\mathbf{r}_2/dt^2. \tag{1.2}$$

According to the correspondence principle [6], the matrix elements make a transition to the corresponding Fourier coefficients. This means that the matrix element from Eq. (1.2) is expressed in terms of the Fourier coefficients of its acceleration in the ion Coulomb field, as is well known in classical electrodynamics. The square of these Fourier coefficients determines the intensities of the classical electron radiation emission in a Coulomb field, see [7]. Note that the classical consideration is applicable even for strong inelastic electron transitions when the change of the electron energy is large as compared with its initial energy. It is due to the strong elec-

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tron acceleration in the attractive Coulomb potential being a basis of the so-called Kramers electrodynamics approach, see Refs. [8,9].

Electric-field ionization effects can be also taken into account by a cutoff of the sum over principal quantum numbers at its value corresponding to the critical value F_c of the electric-field strength (see [9])

$$F_c = Z_i^3 F_A / n^4, \tag{1.3}$$

where Z_i is the ion charge, $F_A = 6.8 \times 10^8$ V/cm is the typical value of the atomic-field strength responsible for the ionization.

Contributions of kinetic effects due to electron collisions are taken into account in the present paper as well with the help of a cutoff procedure.

II. QUASICLASSICAL THEORY FOR AUTOIONIZATION DECAY RATES

The general formula for DR recombination rates takes the form [1–5,10]

$$Q_{DR}(n) = \left(\frac{4\pi Ry}{T} \right)^{3/2} \frac{g_f}{g_i} (a_0)^3 W_R \times \exp\left(-\frac{\omega}{T} + \frac{Z^2}{2n^2 T} \right) \sum_{k,m} \frac{W_A(n,k,m)}{W_R + W_A(n,k,m)}, \tag{2.1}$$

where T is an electron temperature, $g_{i,f}$ are statistical weights on an initial i and a finite f atomic core states, W_R is a radiative transition probability inside an ion core, W_A is an autoionization decay rate of an excited atomic energy level with a principal quantum number n and parabolic electrical k and magnetic m quantum numbers, ω is a frequency for a transition with $\Delta n = 0$ inside the core, a_0 is the Bohr radius. Atomic units (a.u.) will be used below.

The radiative decay rate is expressed simply in terms of an oscillator strength f_{if} for the transition in the core (c is the speed of light)

$$W_R = 2\omega^2 f_{if} / c^3. \tag{2.2}$$

The autoionization decay rate $W_A(n,l)$ is calculated usually in terms of spherical quantum numbers and the transition to parabolic ones is performed numerically with the help of Clebsh-Gordan coefficients, see Refs. [1–5]. To obtain a general expression for DR rates we will use a quasiclassical representation both for DR rates in the spherical coordinates and for the Clebsh-Gordan coefficients.

A quasiclassical expression for an autoionization decay rate W_A may be obtained by different ways, which result in the same formulas. The first way is a direct transition to the classical limit in a general formula for the matrix elements of the radius vector taken with Coulomb wave functions. Note that in the case of Rydberg states ($n \gg 1$) there is no difference as to which types of electron transitions (free-bound, bound-bound, or free-free) are considered.

The first results were obtained by Sommerfeld [11] for free-free radiative transitions in a Coulomb field. He also performed a transition to the classical limit and obtained quasiclassical formulas for matrix elements as a function of the scattering angle. Taking into account the relationship between the scattering angle and the electron orbital momentum l, one quickly reproduces Kramers classical formulas from Sommerfeld results. The same results were obtained in Ref. [12] by the direct calculation of free-bound matrix elements with further transition to the classical limit.

The second way is connected with the relationship between the rate W_A and a partial electron excitation cross section $\sigma_{exc}(l)$ near threshold [10],

$$(2l+1) g_f W_A(nl) = Z^2 n^{-3} \omega g_i \sigma_{exc}(l) / \pi^2 a_0^2, \tag{2.3}$$

where $g_{i,f}, \omega$ are statistical weights and a transition frequency equal to the difference between the initial and final energies of the core energy levels.

The electron-excitation cross section for $\Delta n = 0$ transitions can be calculated in the frame of pure classical mechanics. To do it let us use a classical version of the equivalent photons method proposed by Fermi [14].

According to the Fermi conception [14] of equivalent photons (EP) the electromagnetic field produced by an external particle (e.g., an electron) at a multicharged-ion (MCI) location may be interpreted as a flux of equivalent photons incident on the MCI. It may be shown that such a description is applicable provided the dipole approximation for the interaction between the bound electron of the MCI and the incident electron of the plasma. The latter approximation universally treats all the processes of an energy loss by the incident electron (either due to the radiation emission during a collision with an ion or due to an inelastic nonradiative collision with an ion) as the processes of the emission of (real or equivalent, respectively) photons. The probability of both processes is determined by the dipole matrix element for the corresponding inelastic (radiative or nonradiative) transition of the incident electron.

The spectral intensity distribution of the EP may be described on the basis of the classical radiation theory (for a detailed discussion of the applicability of the classical approach for real photons, see [8]). In this case the intensity of the EP flux is simply determined by the Fourier transformation of the electron coordinates determined in their turn by the classical trajectory. Such an approach makes it possible to treat collisional processes as radiative ones, in particular, the excitation of an ion by an electron impact as an absorption of the EP by the ion.

An essential advantage of this method comes from the application of available results for purely radiative processes to the description of nonradiative processes both collisional and radiative-collisional ones. The processes discussed are of a resonant nature with respect to the absorption of the EP by the ion. In order that the Fermi method be applicable to the processes involving MCI, the effective distances $\mathbf{r}_{eff}$, which are responsible for the main contribution to the cross section of the inelastic collision of an incident electron with MCI, should be much larger than the size of the bound electron

orbit in MCI. This requirement is fulfilled especially well for MCI. Let us illustrate this point for the process of excitation of $\Delta n=0$ transitions. The electron-orbit size is of the order of $1/Z$ (in atomic units), the transition energy ΔE for $\Delta n=0$ transitions in MCI is typically of the order of Z. The values of $\mathbf{r}_{eff}$ for the corresponding cross section can be estimated from the condition that the electron angular rotation velocity $\omega_R \sim (Z/r_{eff}^3)^{1/2}$ near its turning point on the trajectory is equal to the transition energy ΔE, see [8,9],

$$\mathbf{r}_{eff} \sim \mathbf{r}_{\omega} \sim (Z/\Delta E^2)^{1/3} \sim Z^{-1/3} \gg Z^{-1}. \tag{2.4}$$

This inequality justifies the application of the dipole approximation for the potential of the interaction between the bound and incident (with space coordinate vectors $\mathbf{r}_i$ and $\mathbf{r}_e$, respectively) electrons, $V=e^2\mathbf{r}_i\cdot\mathbf{r}_e/r_e^3$. In this framework the static Coulomb interaction between the bound and incident electrons transforms to processes of the emission and the absorption of the EP by electrons, and corresponding probabilities are determined by conventional dipole matrix elements. The electric field produced by the incident electron at the location $r_i=0$ of the ion is equal to

$$\mathbf{F}(0,t) = -\mathbf{r}_e(t)/r_e^3(t), \tag{2.5}$$

where the dependence $\mathbf{r}_e(t)$ describes the classical trajectory of the incident electron. Using the equation of the motion of the incident electron in the field of the MCI $m\ddot{\mathbf{r}}_e = -Ze^2\mathbf{r}_e/r_e^3$, it is convenient to transform Eq. (2.5) into the form

$$\mathbf{F}(t) = \ddot{\mathbf{r}}_e(t)/Z. \tag{2.6}$$

The spectral distribution of the EP flux I_ω produced by the incident-electron electric field, can be expressed in terms of Fourier transforms of the field

$$I_\omega = \frac{c}{8\pi^2}\frac{1}{\omega}\{|F_{x,\omega}|^2+|F_{y,\omega}|^2\} = \frac{c\omega^3}{8\pi^2 Z^2}\{|x_\omega|^2+|y_\omega|^2\}, \tag{2.7}$$

where x and y are coordinates of the incident electron in the plane of its motion. The Fourier transformations of the electron-space coordinates in the Coulomb field are well known [7,15]. Thus we obtain

$$I_\omega = \frac{c\omega^2}{8\upsilon^4}\left\{[H_{i\nu}^{(1)'}(i\nu\varepsilon)]^2 - \frac{\varepsilon^2-1}{\varepsilon^2}[H_{i\nu}^{(1)}(i\nu\varepsilon)]^2\right\}, \tag{2.8}$$

where υ is the electron-initial velocity, $H_{i\nu}^{(1)}$ is Hankel function, ε is the eccentricity

$$\varepsilon = 1+2EM^2/Z^2; \quad \nu=\omega Z/\upsilon^3; \quad E=\upsilon^2/2; \tag{2.9}$$

E and M are the energy and the angular momentum of the incident electron, respectively.

In the limit of low EP frequencies, $\nu \ll 1$, the main contribution to the spectral distribution of the EP flux integrated over the electron-impact parameters ϱ is due to trajectories distant from the field center, ($\varrho \gg a \equiv Z/2E$), which are nearly rectilinear, with eccentricity $\varepsilon \gg 1$. In this case, Eq. (2.8) is transformed to

$$I_\omega = (c\omega/2\pi^2\upsilon_0^4)\{K_0^2(\omega\rho/\upsilon)+K_1^2(\omega\rho/\upsilon_0)\}, \tag{2.10}$$

where $K_0(x)$ and $K_1(x)$ are McDonald functions. Fermi used just Eq. (2.10) for the description of atomic states ionization by a rectilinearly moving particle [14].

For the description of the processes resulting in a loss of a considerable part of the incident-electron energy, it is necessary to consider the EP with high frequencies, namely, $\nu \gg 1$. The main contribution to the emission of such EP comes from the strongly curved electron trajectories, $\varepsilon - 1 \ll 1$, which are close to the field center, $\varrho \ll a$. In this Kramers domain we arrive at the result (see Refs. [7–9]).

$$I_\omega = \pi^{-2}Z^{-2}cMG(\omega M^3/3Z^2), \tag{2.11}$$

where $M=m\upsilon\varrho$ is the electron-orbital momentum and

$$G = u(K_{1/3}^2(u)+K_{2/3}^2(u)), \tag{2.12}$$

where $K_{1/3,2/3}$ are McDonald functions.

The equivalent photons method makes it possible to obtain a simple analytical description of collisional processes and treat them as pure radiative ones. Within this framework the excitation of MCI by an electron impact may be clearly considered as an absorption of the EP with a resonant frequency $\omega_0=\Delta E_{if}/\hbar$. The relationship between the collisional cross section σ_{exc} and the cross section σ_{abs} of the absorption of the EP can be obtained equating the number of excitation events, during the time interval dt, caused by the collisions of the MCI with the electron flux with a space density n_e and a particle velocity υ_e

$$dN_{exc} = n_e\upsilon_e\sigma_{exc}dt$$

to the corresponding number of transitions caused by the absorption of the EP produced by a single electron. This last result is multiplied by the total number of electrons in the volume dV corresponding to the time interval dt, $dV = 2\pi\varrho d\varrho\upsilon_e dt$,

$$dN_{abs} = \int 2\pi\varrho d\varrho n_e\upsilon_e dt \int d\omega(cF_\omega^2/4\pi^2\hbar\omega)\sigma_{abs}(\omega), \tag{2.13}$$

where the expression in parentheses corresponds to the spectral distribution of the EP flux (2.7) produced by a single electron with a fixed value of the impact parameter ϱ. Assuming the following relation between the total and the partial (with respect to the quantum orbital number l) cross sections

$$\sigma_{exc} = \int \sigma_{exc}^l dl,$$

we arrive at the result

$$\sigma^{l}_{exc}=2\pi(\hbar/mv_e)^2(l+1/2)\int \sigma_{abs}(\omega)(cF^2_\omega/4\pi^2\hbar\omega)d\omega. \tag{2.14}$$

Furthermore, the expression for the EP flux can be taken out of the integral at the frequency ω_0 of the radiative transition in the MCI core under consideration because of its slow frequency dependence in comparison with that of the absorption cross section. The resulting integral over ω gives the well-known expression

$$\int \sigma_{abs}(\omega)d\omega=\pi^2(c/\omega)^2 g_f 4\omega_0^2|d_{if}|^2/3\hbar c^3,$$

where d_{if} is the dipole moment matrix element of the transition considered and g_f is the statistical weight of the upper level.

Substituting the spectral distribution (2.8) of the EP flux, produced by the electron in the Coulomb field of the MCI into Eq. (2.14), we finally obtain

$$\sigma^{l}_{exc}=\frac{8\pi^3}{3}(\hbar/mv)^2\omega_0^2|d_{if}|^2 g_f v^{-4}(l+1/2) \times\{[H^{(1)\prime}_{i\nu}(i\nu\varepsilon)]^2-(\varepsilon^2-1)\varepsilon^{-2}[H^{(1)}_{i\nu}(i\nu\varepsilon)]^2\}. \tag{2.15}$$

The transition to the Kramers electrodynamics (KED) domain ($\nu\gg 1$) in Eq. (2.15) corresponds to the transition from Eq. (2.8) to Eq. (2.11).

Thus, we obtain the result in the KED domain

$$\sigma^{l}_{exc}=(8\pi/3)(\hbar/mv_e)^2(g_f/g_i)f_{if}Z^{-2} \times(l+1/2)^2 G(\omega(l+1/2)^3/3Z^2), \tag{2.16}$$

where f_{if} is the oscillator strength for the transition considered, and g_i is the statistical weight of the lower level, the function $G(u)$ is equal to Eq. (2.12).

The total excitation cross section is obtained by summing of the partial cross section (2.14) over l, yielding the expression in terms of the well-known spectral distribution for the Coulomb bremsstrahlung Gaunt factor $g(\nu)$ [13,12]

$$\sigma^{if}_{exc}=\frac{8\pi^2}{\sqrt{3}}|d_{if}|^2 g_i^{-1}v^{-2}g(Z\omega_0/(2E)^{3/2}). \tag{2.17}$$

Remember that a simple analytic approximation [8] is possible for the function $g(\nu)$.

The result (2.17) was derived earlier [13] in a somewhat different way. It should be noted that Eq. (2.17) is valid up to the excitation threshold where Kramers EP spectrum (2.16) does not depend on the incident electron energy at all. In the opposite limit of a fast incident particle, the cross section (2.17) exhibits a logarithmic (Born-type) structure. It is just the same result that was derived by Fermi for the atomic excitation and ionization by fast particles. Equation (2.17) is in a good agreement with quantum-numerical calculations as well as with experimental data [13].

The classical result (2.15) for the excitation cross section must be used in Eq. (2.3) for the determination of the autoionization decay rate.

The most interesting case corresponds to the large value of the parameter $\nu\gg 1$. For large values of ν, one finally obtains

$$W_A=\frac{f_{if}}{\pi n^3}lG\left(\frac{\omega_0 M^3}{3Z^2}\right), \tag{2.18}$$

where f_{if} is the oscillator strength for the core transition, $M=mv\varrho$ is the electron-orbital momentum.

The result (2.15) presents the autoionization decay rate $W_A(n,l)$ in the classical approximation. It coincides with the limiting case of the quantum-mechanical consideration [12] after the standard substitution $l\to l+1/2$. One can see the sharp decrease in the autoionization decay rate with the increase of the electron-orbital momentum l described by the function G. Taking into account that the essential values of the argument of the G function is never close to zero it is possible, for practical use, to change the function G by its asymptotic expansion

$$G(u)\approx\pi\exp(-2u). \tag{2.19}$$

To obtain the total autoionization decay rate, it is necessary to multiply Eq. (2.18) by $(2l+1)$ and to sum (or integrate) over l. It is more convenient to use the relationships (2.3), (2.15), and (2.17) and to express the total autoionization decay rate in terms of the total excitation cross section

$$W_A(n)=4Z^2 f_{if}g(Z\omega/(2E)^{3/2})3^{-1/2}n^{-3}v^{-2}, \tag{2.20}$$

where $g(\nu)$ is the classical Gaunt factor for bremsstruhlung (H are standard Hankel functions)

$$g(\nu)=\pi\frac{\sqrt{3}}{4}i\nu H^{(1)}_{i\nu}(i\nu\varepsilon)H^{(1)\prime}_{i\nu}(i\nu\varepsilon). \tag{2.21}$$

Equations (2.20) and (2.21) can be used for calculations of total autoionization decay rates of the atomic state with principal quantum number n. The precision of the result (2.21) is the same as the precision of the general relationship (2.3), the precision of quasiclassical cross sections being very high up to the threshold, see [13].

To obtain the final result for the total autoionization decay rate, one must substitute the value of the electron energy near the threshold $E=mv^2/2=\omega$ into Eq. (2.20) that gives

$$W_A(n)=4Z^2|d_{if}|^2 g((Z^2/8\omega)^{1/2})3^{-1/2}n^{-3}. \tag{2.22}$$

The dependence of W_A on Z is practically absent if one takes into account that $d^2_{ij}\propto Z^{-2}$, the argument of the Gaunt factor is large if one scales $\omega\propto Z$, which means that the value of g is close to 1 (practically, however, the argument is not so large).

III. TRANSFORMATION TO A PARABOLIC BASIS IN QUASICLASSICAL THEORY

The transformation to a parabolic basis may also be performed in two ways: (1) a calculation of the probabilities of an appearance of a specific value of the parabolic quantum number in the anglular-momenta distribution in a Coulomb field, and (2) an application of the asymptotic representation of Clebsh-Gordan coefficients [16]. Both methods use the specific four-dimensional symmetry properties of the Coulomb field. These properties are connected, as it is well known, with the additional integral of motion in a Coulomb field, namely, the Runge-Lentz vector $\mathbf{A}=-2e^2\langle\mathbf{r}\rangle/3a$ $(a=e^2/2E)$ where $\langle\ \rangle$ means average over the electron motion. The properties of the vector are as follows:

$$\mathbf{L}\cdot\mathbf{A}=0, \quad A^2+2EL^2=1, \quad \text{or } n^2\varepsilon^2+l^2=n^2, \tag{3.1}$$

where ε is the eccentricity of the electron orbit and we have substituted quantum numbers l,n instead of the orbital momentum and the energy.

The electron motion can be described in terms of two independent orbital momenta $\mathbf{J}_1,\mathbf{J}_2$

$$\mathbf{J}_{1,2}=\frac{1}{2}\left[\mathbf{L}\pm\left(\frac{m}{a}\right)^{1/2}\mathbf{A}\right]. \tag{3.2}$$

The main properties of the momenta are as follows:

$$\mathbf{J}_1^2=\mathbf{J}_2^2=j(j+1),$$

$$(J_1)_z=m_1=\frac{m+n_2-n_1}{2},\ (J_2)_z=m_2=\frac{m+n_1-n_2}{2},$$

$$L_z=m,\ \left(\frac{m}{a}\right)^{1/2}A_z=n_2-n_1. \tag{3.3}$$

The last equation together with Eq. (3.1) describes the connection between projections of vectors $(J_1)_Z,(J_2)_Z$ and parabolic $(n_1,n_2,k=n\varepsilon=n_1-n_2)$ and spherical (l,m) quantum numbers. The following conditions are fulfilled in our case of strongly curved classical trajectories corresponding to the formula (2.18)

$$m<l\ll n\propto j. \tag{3.4}$$

Equation (3.3) expresses some limitations on the distribution of projections of these vectors in space. Under conditions (3.4) one can consider the simplest model when $m=0$. Putting the z axis along the vector $\mathbf{A}$ and the x axis along the vector $\mathbf{L}$, we can consider a two-dimensional model for the evolution of projections J_z and J_x of both vectors $\mathbf{J}_1,\mathbf{J}_2$.

From a general point of view, the joint probability $P(l,k,n)$ of the appearance of spherical and parabolic quantum numbers l,k is equal to the ratio of the phase-space volume where conditions (3.3) are satisfied to the total phase-space volume (n^2). To determine $P(l,k,n)$ let us perform an integration over all phase space of vectors $\mathbf{J}_1,\mathbf{J}_2$ accounting for limitations (3.3). The integrals take the form

$$\begin{aligned}P(l,k,n)=&\int\int\int\int_{-j,+j}dJ_{1x}dJ_{1z}dJ_{2x}dJ_{2z}\\&\times\delta(J_{1z}+J_{2z})\,\delta(J_{1z}-J_{2z}-k)\\&\times\delta((J_{1x}^2+J_{1z}^2)^{1/2}-j)\,\delta(l-2(J^2-J_{1z}^2)^{1/2}).\end{aligned} \tag{3.5}$$

The integration is performed with the help of the properties of δ functions: $\delta(f(x))=[df(x)/dx]^{-1}\delta(x-x_0)$ $[f(x_0)=0]$ and proper limitations on the domains of variable change.

When one takes into account normalization conditions the result takes the simple form

$$P(l,k,n,m=0)=2[(n-1)^2-k^2]^{-1/2}/\pi, \tag{3.6}$$

that is the probability of the appearance of the parabolic quantum number k does not depend on l at all under conditions of small $l\ll n$. The probability distribution (3.5) coincides with the probability of the oscillating variable J_1-J_2 to take a definite value k. It corresponds to a picture of electron motion resulting in the oscillation of orbital momenta $J_{1,2}$ along the z axis with small (neglecting) projections on the x axis.

A close result for the case $m\neq0$ can be obtained from a consideration of quasiclassical limit of Clebsh-Gordan coefficients. Really the parabolic and spherical quantum numbers are connected by the sum with Clebsh-Gordan coefficients $C[j_1,j_2,j;m_1,m_2,m]$

$$\begin{aligned}&|\langle n_1n_2m|nlm'\rangle|^2\\&\quad=C^2\left[\frac{n-1}{2},\frac{n-1}{2},l;\frac{m+n_2-n_1}{2},\frac{m+n_1-n_2}{2},m'\right].\end{aligned} \tag{3.7}$$

The square of the coefficients may be considered as a joint probability $P(n,l,k,m)$ of the presence of specific quantum numbers. Making a transition to large values of all quantum numbers [16] and using the conditions (3.4) determining the domain of present interest, one arrives at the following approximation:

$$\begin{aligned}&P(n,l,k,m)\\&\quad=C^2[(n-1)/2,(n-1)/2,l;(m-k)/2,(m+k)/2,m]\\&\quad=2l[(l^2-l_{\min}^2)(l_{\max}^2-l^2)]^{-1/2}/\pi,\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}l_{\min}^2=&[(n-1)^2+m^2-k^2]/2-\{[(n-1)^2+m^2-k^2]^2\\&-4(n-1)^2m^2\}^{1/2}/2,\end{aligned} \tag{3.9}$$

$$\begin{aligned}l_{\max}^2=&[(n-1)^2+m^2-k^2]/2+\{[(n-1)^2+m^2-k^2]^2\\&-4(n-1)^2m^2\}^{1/2}/2,\end{aligned}$$

or, when $m\ll n$

$$l^2_{\min}\approx(n-1)^2m^2[(n-1)^2+m^2-k^2]^{-1},$$
$$l^2_{\max}\approx[(n-1)^2+m^2-k^2]=(n-1)^2m^2/l^2_{\min}. \quad (3.10)$$

One can see that when $m=0$ the Eq. (3.8) reduces to the classical Eq. (3.6). The difference between both probability distributions is inside a small domain $[(n-1)-k]\approx m\ll n$.

The normalization of $P(n,l,k,m)$ Eq. (3.8) is equal to 1. Really, the integration (3.8) over l^2 gives the expression

$$-\pi\arcsin\frac{-2l^2+l^2_{\max}+l^2_{\min}}{l^2_{\max}+l^2_{\min}}\Bigg|^{l^2_{\max}}_{l^2_{\min}}=1.$$

The parabolic representation of an autoionization decay rate is obtained by the multiplication of the rate in the spherical basis (2.18) by the probability (3.8) and integrating (summing) over l,

$$W_A(n,k,m)=\int_{l_{\min}}^{l_{\max}}dlP(n,l,k,m)W_A(n,l), \quad (3.11)$$

where $l_{\min},l_{\max}$ are defined by Eq. (3.9) or Eq. (3.10).

Substituting the expressions (2.18) and (3.8) for functions $W_A(n,l)$ and $P(n,l,k,m)$ and making a transformation to dimensionless variables $t=l/l_{ef}, l_{ef}=(3Z^2/\omega)^{1/3}$, it is possible to obtain

$$W_A(n,k,m)=\pi^{-1}n^{-3}f_{if}I(n,k,m), \quad (3.12)$$

where the universal function $I(n,k,m)$ is

$$I(n,k,m)=I(t_{\min},t_{\max})$$
$$=2l_{ef}/\pi\int_{t_{\min}}^{t_{\max}}dtt^2G(t^3)(t^2-t^2_{\min})^{-1/2}$$
$$\times(t^2_{\max}-t^2)^{-1/2}, \quad (3.13)$$

where G was defined by Eq. (2.12), $t_{\min}\propto(n-1)m((n-1)^2+m^2-k^2)^{-1/2}, t_{\max}=(n-1)m/t_{\min}\propto n$.

Below we will use the approximation (2.19) for our particular calculations.

One can see that for the case $t_{\max}\approx n\gg1$ Eq. (3.13) may be transformed into

$$I(t_{\min},t_{\max})\approx2l_{ef}/\pi t_{\max}\int_{t_{\min}}^{t_{\max}}dtt^2G(t^3)(t^2-t^2_{\min})^{-1/2}$$
$$\approx2l_{ef}/t_{\max}\int_{t_{\min}}^{t_{\max}}dtt^2\exp(-2t^3)(t^2-t^2_{\min})^{-1/2}$$
$$=I(t_{\min})2l_{ef}/t_{\max}. \quad (3.14)$$

The universal function $I(x)$ is presented in Fig. 1.

Limiting cases of the function $I(x)$ are as follows:

$$I(x)\approx\begin{cases}\Gamma(2/3)3^{-1}2^{-2/3}\approx0.284 & \text{for } x\ll1,\\ (\pi/12)^{1/2}x^{1/2}\exp(-2x^3) & \text{for } x\gg1.\end{cases} \quad (3.15)$$

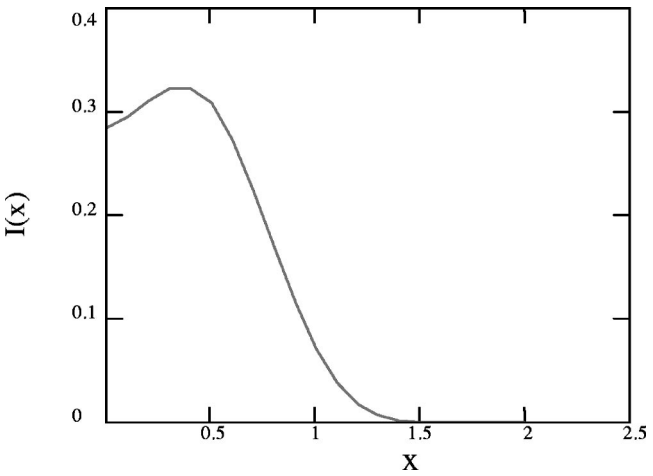

FIG. 1. Universal function $I(x)$ describing the dependence of autoionization decay rates on parabolic quantum numbers, see Eqs. (3.12)–(3.14).

The function $I(x)$ can be approximated with a good precision by the simple exponent

$$I(x)=0.284\exp(-2x^3). \quad (3.16)$$

We will use below the asymptotic expression for the function $I(x)$ for particular calculations.

The dependences of dimensionless autoionization decay rate $I(n,k,m)(\pi)^{-1}$ from Eq. (3.13) on the ''electric'' quantum number k for different values of magnetic quantum number m are presented in Fig. 2 for the Li-like ion Zn XXVIII ($Z=30$). One can see that the greatest contribution to the k phase space comes from small m values.

IV. DIFFERENTIAL DR RATES

To obtain differential (as regard to principal quantum numbers n) and total DR decay rates it is necessary to per-

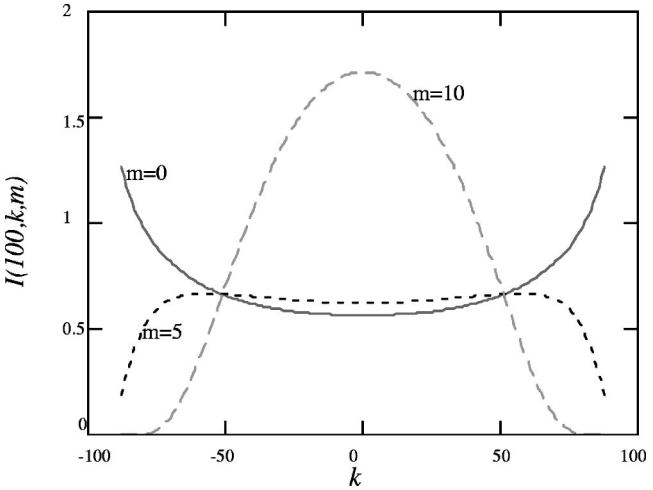

FIG. 2. Distribution of autoionization decay rates over ''electric'' quantum numbers k at different values of magnetic quantum numbers m for the Li-like ion Zn XXVIII at the principal quantum number $n=100$. Here the scale for $I(100,k,10)$ equals the scale for $I(100,k,0)$ multiplied by 10^1.

form two (or three) summations in the Eq. (2.1). It is more convenient to deal with a reduced DR rate, $q^*(n)$ equals to the term with the sum in the Eq. (2.1), that is,

$$q^*(n,Z)=Q_{DR}(n)/B(Z,T,\omega,n)=\sum_{k,m}\frac{W_A(n,k,m)}{W_R+W_A(n,k,m)}, \tag{4.1}$$

where

$$B(Z,T,\omega,n)=\left(\frac{4\pi Ry}{T}\right)^{3/2}\frac{g_f}{g_i}(a_0)^3 W_R\exp\left(-\frac{\omega}{T}+\frac{Z^2}{2n^2T}\right).$$

The sum in Eq. (4.1) accounts for a conversion from the spherical basis to parabolic one. The detailed consideration of the atomic basis transformation under the action of perturbations is presented in Ref. [17]. Our treatment deals with a particular case of perturbations constant in time. So it is sufficient for our consideration to restrict ourselves by the transformation coefficients presented by Eqs. (3.8) and (3.11).

The quantity $q^*(n,Z)$ is equal to a number of atomic states with a given energy (n) contributing to DR rates. When $W_R=0$, the number is equal to the statistical weight of the energy level, that is, equal to n^2. If $W_R\neq 0$ the number depends on the type of the atomic quantization being spherical or parabolic.

Let us take into account that the autoionization decay rates in the Eq. (4.1) are expressed in terms of the universal function (3.16) with the argument depending on parameters $l^2_{\min}$ and $l^2_{\max}$ in accordance with Eq. (3.11). To do this let us change the sum in Eq. (4.1) by the integral over corresponding quantum numbers and then make a transformation to the variables $l^2_{\min}, l^2_{\max}$. The Jacobian J of the transformation is obtained by a direct calculation

$$J(n,l^2_{\min},l^2_{\max})=\frac{l^2_{\min}-l^2_{\max}}{4l_{\min}l_{\max}(n-1)^2}\times\left[1+\frac{l^2_{\min}l^2_{\max}}{(n-1)^4}-\frac{l^2_{\min}+l^2_{\max}}{(n-1)^2}\right]^{-1/2}. \tag{4.2}$$

Extracting in the sum (4.1) the dependence on the principal quantum number n, we arrive at the expression

$$q^*(n,Z)=2\int_0^{(n-1)^2}dl^2_{\max}\int_0^{l^2_{\max}}dl^2_{\min}\frac{J(n,l^2_{\min},l^2_{\max})}{1+[n/n^*(l_{\max},l_{\min})]^3}, \tag{4.3}$$

where the effective value of the principle quantum number n^* is introduced with the account of Eqs. (2.2), (3.12), (3.14), and (3.16),

$$n^{*3}=0.284\frac{c^3l^2_{ef}}{\pi\omega^2 l_{\max}}\exp(-2l^3_{\min}/l^3_{ef})=L(n-1)^{-1}z^{-1/2}\exp(-A_n y^{3/2}),$$

$$L=0.284c^3l^2_{ef}\pi^{-1}\omega^{-2},\qquad A_n=2(n-1)^3l^{-3}_{ef}\gg 1. \tag{4.4}$$

Here the dimensionless variables $y=l^2_{\min}/(n-1)^2$ and $z=l^2_{\max}/(n-1)^2$ and parameters L, A_n are also introduced.

Further making the transformation to the variables y and z in Eq. (4.3), we obtain

$$q^*(n,Z)=\frac{1}{2}(n-1)^2\int_0^1\frac{dz}{\sqrt{z}}\int_0^z\frac{dy(z-y)}{[y(1+zy-z-y)]^{1/2}[1+n^3(n-1)\sqrt{z}\exp(A_n y^{3/2})/L]}. \tag{4.5}$$

One can check that for W_R (that is, $L\to\infty$) the quantity of $q^*(n,Z)$ is equal to $(n-1)^2$, which is just the statistical weight (with small corrections) of the energy level n.

The Eq. (4.5) solves the problem of the distribution of DR rates over principal quantum numbers n.

To make clear the dependence, let us take into account that under condition $A_n\gg 1$ effective values of the variable y are small as compared with z being of the order of unity. Neglecting, where it is possible, the magnitude of y as compared with z, expanding the integration over y to infinity, and changing variables, one arrives at the expression

$$q^*(n,Z)=\frac{3.14l_{ef}}{2^{4/3}3}n\int_0^\infty\frac{du}{u^{2/3}}(1+n^4e^u/L)^{-1}. \tag{4.6}$$

One can see that the dependence of DR rates on n is described by the universal function $J(\alpha)$ such that

$$Q_{DR}(n)=B(Z,T,\omega,n)q^*(n,Z), \tag{4.7}$$

where

$$q^*(n,Z)=1.25l_{ef}nJ(n^4\pi\omega^2/(0.284c^3l^2_{ef})) \tag{4.8}$$

and

$$J(\alpha)=\alpha\int_1^\infty dx\ln^{1/3}(x)/(1+\alpha x)^2\approx\begin{cases}\ln^{1/3}(1/\alpha) & \text{if } \alpha\ll 1,\\ 0.89\alpha^{-1} & \text{if } \alpha\gg 1.\end{cases} \tag{4.9}$$

The function $J(\alpha)$ is presented in Fig. 3.

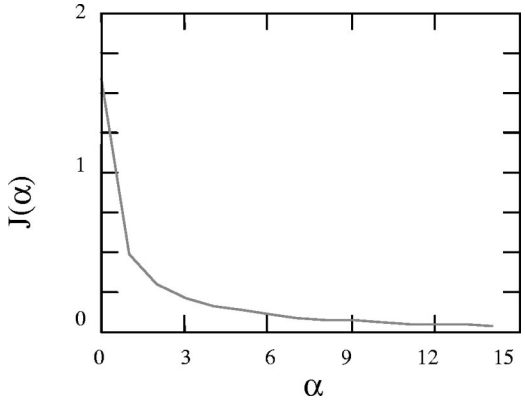

FIG. 3. Universal function $J(\alpha)$ describing the distribution of DR rates over principle quantum numbers n, see Eqs. (4.8) and (4.9).

Let us apply general results (4.7)–(4.9) to the case of the Li-like carbon ion C^{3+} and make a comparison with the particular calculations [2,3]. Substituting all numerical constants into Eqs. (4.7)–(4.9) one obtains the distribution of DR rates (in units 10^{-12} cm^3/s for the electron temperature $T_e=10^5$ K) over principle quantum numbers presented in Fig. 4 (solid curve). The specific data [3] are represented by the dotted line. Our data, at their maximum in Fig. 4, are closer to data [2] and near two times lager than the data [3] (see comments in [3]). The difference may be due to the proper account of the normalization conditions in Eq. (4.5) [see comments below Eq. (4.5)].

It follows from the quasiclassical consideration that the conditions for the C^{+3} ion are very close to the conditions for the Mg^{+1} presented in [1]. Really one can check that both ions mentioned above have the same radiation decay rates and the arguments of the J function in Eq. (4.8) are also very close for them. The only difference is the value of the parameter l_{ef} equal to 4.6 for C^{+3} and 2.7 for Mg^{+1}. The corresponding results are presented in Fig. 5. The close correspondence of both data in Figs. 4 and 5 is a confirmation of the classical scaling low following from the general result (4.8).

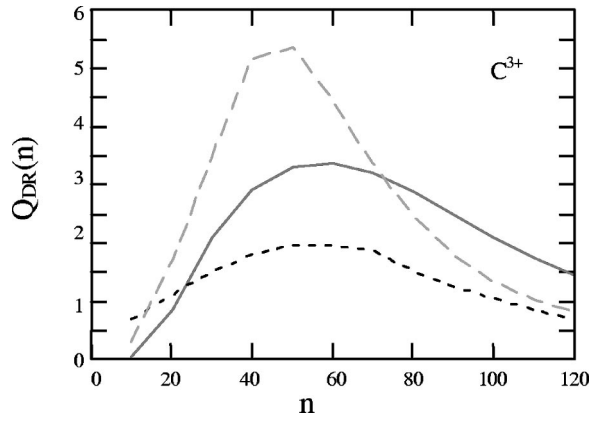

FIG. 4. Distribution of DR rates (in units 10^{-12} cm^3/s) over n for the C^{3+} ion at the electron temperature $T_e=10^5$ K: solid curve, universal quasiclassical formula (4.7)–(4.9); dotted line, calculation [3]; long dashed line, calculation [2].

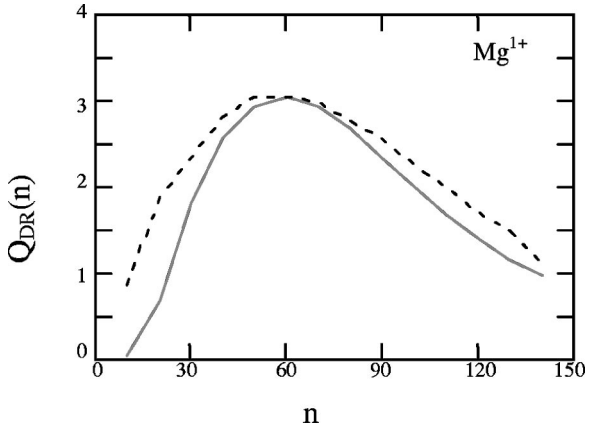

FIG. 5. The same as in Fig. 4 but for the Mg^{1+} ion: solid curve, quasiclassical formula (4.7)–(4.9); dotted line, calculation [1] (multiplied by the factor 2).

V. TOTAL DR RATES

To obtain total recombination rates, in the parabolic basis one must integrate a differential distribution over all values of principle quantum numbers from a small value (put to be zero below) up to a particular value n_{max} depending on the cutoff conditions in plasmas. Corresponding results can be also expressed in term of a universal function

$$Q_{DR}^{tot}=B(Z,T,\omega,n)q^{*tot}, \tag{5.1}$$

where

$$q^{*tot}=0.08c^{3/2}\frac{l_{ef}^2}{\omega}P(n_{\max}^2\omega\sqrt{\pi}/(0.533c^{3/2}l_{ef})) \tag{5.2}$$

and

$$P(x)=\int_0^1\frac{dz}{\ln^{2/3}(1/z)}\arctan\frac{x}{z}\approx\begin{cases}\dfrac{\pi}{2}\Gamma(1/3)\approx 4.2 & \text{if } x\gg 1\\ 3x\ln^{1/3}(1/x) & \text{if } x\ll 1.\end{cases} \tag{5.3}$$

The universal function $P(x)$ is presented in Fig. 6.

Using the data (5.1)–(5.3) it is possible to investigate a dependence of DR rates on electric-field strengths expressing the value of $n_{\max}$ as a function of an electric-field strength F from Eq. (1.3). Below we will pick up the value of $n_{\max}$ in accordance with [3] to make clearer the comparison with numerical data, that is,

$$n_{\max}=(6.8\times 10^8 Z_i^3/F)^{1/4}. \tag{5.4}$$

Figure 7 presents the dependence of DR rates on the field strength (V/cm). The data [2] are also presented in the Fig. 7. These data demonstrate the typical maximum in the increase of the DR rates, which corresponds to an essential increase of the phase space due to the action of the field. The field ionization results in a decrease of DR rates with the increase of field strengths. Our model corresponds to relatively large values of electric-field strengths, that is, to the

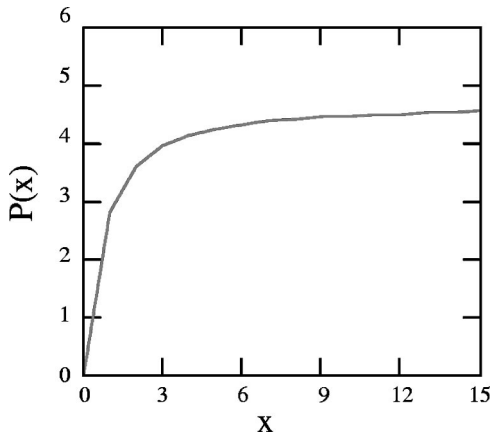

FIG. 6. Universal function $P(x)$ (5.3) describing the dependence of total DR rates on cutoff parameters.

full mixing of atomic states, so the only decrease due to ionization is seen in the Fig. 7.

The simple way to take into account plasma density effects is to connect the value of $n_{\max}$ with the electron plasma density. It can be done easily by the substitution into Eq. (5.4) of the Holsmark field strength given by the formula

$$F(\mathrm{V/cm}) = 1.3\times 10^{-6} Z_i N_i^{2/3} \ \mathrm{cm}^{-3},$$

$$n_{\max} = (5.23\times 10^{14} Z_i^2 / N_i^{2/3})^{1/4}, \qquad (5.5)$$

where one can take into account the relationship $N_e = Z_i N_i$ for one-component plasmas or substitute an independent value N_i for many-component plasmas (for example, $N_e = N_{\mathrm{H}^+}$ for hydrogen plasmas considered in [2,3]) The comparison between our data and data of Ref. [3] (multiplied by the factor 2, see comments at the end of Sec. IV) is presented in Fig. 8. One can see a good correspondence between universal formulas (5.1)–(5.3) and the specific calculation.

The kinetic effects due to electron collisions can be also taken into account in the following approximate manner. The main effect of electron collisions is a decrease of recombination effects due to the secondary ionization of the captured electron by other plasma electrons. These effects are described by an attenuation factor $j(n, N_e, T_e)$ equal to a probability for the electron captured to reach a ground atomic state. The probability can be estimated as the ratio of the radiative decay rate A_n from the captured atomic state and the ionization frequency $N_e\langle v\sigma_i\rangle$ from the state due to electron collisions. Both these rates are estimated for Rydberg atomic states as follows [18]:

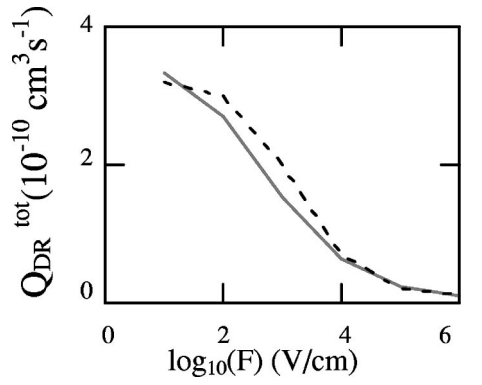

FIG. 7. Dependence of total DR rates (5.1) for the C^{3+} ion on the electric-field strength: solid line, quasiclassical formulas (5.1) and (5.2); dotted line, calculations [2].

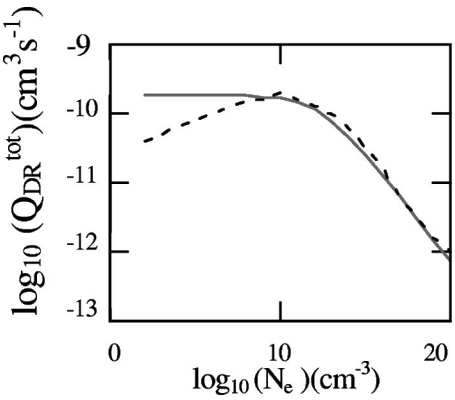

FIG. 8. Effect of plasma electron density N_e on the DR rate for the C^{3+} ion for the electron temperatures $T_e = 10^5$ K (solid line, quasiclassical formula (5.1); dotted line, calculations [3]).

$$A_n(\mathrm{s}^{-1}) = 2.4\times 10^{10} Z_i^4 / n^5,$$

$$\langle v\sigma_i\rangle = 10^{-7} n^2 Z_i^{-2} (1 \ \mathrm{Ry}/T_e)^{1/2}. \qquad (5.6)$$

Using these approximations one obtains for the attenuation factor

$$j(n, N_e, T_e) = [1 + (n/n_0)^7]^{-1},$$

$$n_0 = Z_i [2.4\times 10^{17} (T_e/\mathrm{Ry})^{1/2} / Z_i N_e(\mathrm{cm}^{-3})]^{1/7}. \qquad (5.7)$$

It is seen that there is a sharp dependence of the factor on the value of the principle quantum number that means that the electron secondary ionization effects can be also taken into account with the help of a cutoff procedure. So the value n_0 can be used as the cutoff parameter together with the $n_{\max}$ accounting for ion field ionization effects described by Eq. (5.5) because both effects result in the ionization of the captured electron. So the specific value of $n_{\max}$ in Eq. (5.2) must be put to be the minimum of two magnitudes (5.5) and (5.7).

The effects of both plasma ion microfields and electron collisions on DR rates for the C^{3+} ions are presented in Fig. 9 as a function of electron densities for the electron temperature $T_e = 10^5$ K. One can see that the ionization by electrons produces larger effect on DR rates than does the ionization by the electric field. The results of the present simplified consideration are also in a reasonable correspondence with numerical data [3].

Note that Refs. [2] and [3] deal with plasma effects by solving a set of rate equations, whereas the present paper uses the cutoff procedure. The procedure takes into account diagonal elements of the relaxation matrix only. This is the reason for the discrepancy between data in Fig. 9. It is necessary to note, however, that the discrepancy is not so large

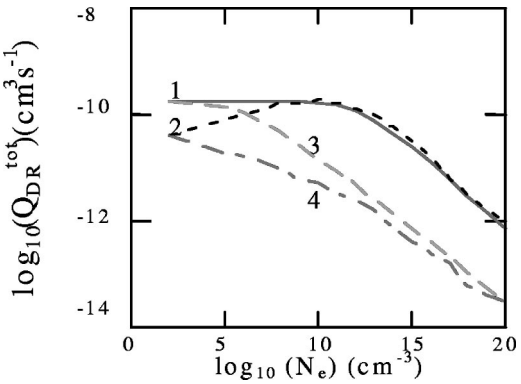

FIG. 9. Effects of ion electric field (upper curves: 1, quasiclassical formula (5.1); 2, calculation [3]) and electron collisions (lower curves: 3, quasiclassical formula (5.1); 4, calculation [3]) on DR rates for C^{3+} in a hydrogen plasma with $T_e=10^5$ K.

as to be of a great importance because the matrix elements of the relaxation matrix can be also modified by the plasma microfield.

VI. CONCLUSION

The above consideration results in the general quasiclassical formulas (3.12), (4.8)–(5.3) for differential and total DR rates as functions of the atomic and cutoff parameters. It describes effects of relatively strong electric fields (full mixing of atomic states) on DR rates accompanied by transitions in atomic cores without change in their principle quantum numbers ($\Delta n=0$ transitions). These transitions are responsible for the most of the contribution to atomic processes in collisions of plasma electrons with complex ions. The present results make it possible to calculate DR rates in a simple way for every ion having $\Delta n=0$ transitions in the core.

The above results can be generalized to the case of partial l mixing in the following way. Let us write down the condition of the total mixing comparing the energy shift due to a quantum defect δ_l and the matrix elements of the atom electric-field interaction

$$3n(n^2-l^2)^{1/2}F>Z^3\delta_l/n^3. \tag{6.1}$$

The condition (6.1) determines a minimum value of the orbital momentum l consistent with full mixing conditions. The value must be substituted instead of the parameter $l_{\min}$ in Eq. (3.11). A contribution of atomic states with $l<l_{\min}$ from the Eq. (6.1) is taken into account in Eq. (2.1) in the usual manner as in the conventional spherical basis.

Let us estimate an enhanced factor for DR rates. To do it, one can calculate the DR rate in a spherical basis with the help of Eq. (2.18). The general expression for the effective DR rate $q^*_{sph}(n,l)$ into particular values n,l in a spherical basis takes the form [10,11]

$$q^*_{sph}(n,l)=(2l+1)[(n/n^*)^3+1]^{-1}, \tag{6.2}$$

where

$$n^*(l)=\{3c^3(2l+1)G(\omega(l+1/2)^3/3Z^2)/(2\pi\omega f_{if})\}^{1/3}. \tag{6.3}$$

Using the asymptotic expression for the function $G(u)$, one can present Eq. (6.3) in the form [see the value l_{ef} in Eqs. (3.11)–(3.13)]

$$n^*(l)=n^*(0)\exp[-(l/l_{ef})^3], \tag{6.4}$$

$$n^*(0)=c[3l_{ef}/(2\omega^2)]^{1/3}\gg 1. \tag{6.5}$$

When integrating Eq. (6.2) over l, it is natural to change variables to $n^*(l)$ according Eqs. (6.3) and (6.4). One can obtain the following simple estimation for the total effective DR rate $q^*_{sph}(n)$:

$$q^*_{sph}(n)\sim l^2_{ef}ln^{2/3}[n^*(0)/n], \tag{6.6}$$

that is, it practically does not depend on values of n. This value must be compared with the one in the parabolic basis given by Eq. (4.8). Neglecting slow logarithmic dependences, we arrive at the simple estimation

$$q^*_{par}(n)/q^*_{sph}(n)\sim n^*(0)/l_{ef}, \tag{6.7}$$

where we have introduced the designation ''par'' for the result (4.8) obtained above in the parabolic basis. The same estimation is approximately true for the total DR recombination rates.

It is seen from the estimations that the enhanced factor for DR in an electric field is approximately equal to the ratio of the effective volume in the n space to the effective volume in the l space for the spherical basis. This ratio depends on a specific atomic structure of a recombining ion determining effective values of parameters in Eq. (6.7). Practically for most ions the value of l_{ef} changes from 3 up to 10 whereas $n^*(0)$ is of order of 10^2 that means the enhanced factor is near 20–30, which is in a reasonable correspondence with numerical calculations [1–5].

ACKNOWLEDGMENTS

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