Neoclassical electron and ion transport in toroidally rotating plasmas

H. Sugama
National Institute for Fusion Science, Nagoya 464-01, Japan

W. Horton
Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712

(Received 10 January 1997; accepted 28 February 1997)

Neoclassical transport processes of electrons and ions are investigated in detail for toroidally rotating axisymmetric plasmas with large flow velocities on the order of the ion thermal speed. The Onsager relations for the flow-dependent neoclassical transport coefficients are derived from the symmetry properties of the drift kinetic equation with the self-adjoint collision operator. The complete neoclassical transport matrix with the Onsager symmetry is obtained for the rotating plasma consisting of electrons and single-species ions in the Pfirsch–Schlüter and banana regimes. It is found that the inward banana fluxes of particles and toroidal momentum are driven by the parallel electric field, which are phenomena coupled through the Onsager symmetric off-diagonal coefficients to the parallel currents caused by the radial thermodynamic forces conjugate to the inward fluxes, respectively. © *1997 American Institute of Physics.* [S1070-664X(97)00806-9]

I. INTRODUCTION

Improved confinement modes of tokamak plasmas such as high-confinement modes (H-modes)[1] and reversed shear configurations[2] are attracting considerable attention as promising means for achieving controlled fusion. Such a reduction of the transport level is generally considered as caused by the large radial electric field shear (or sheared flow). In the Japan Atomic Energy Research Institute Tokamak-60 Upgrade (JT-60U),[3] the internal transport barrier (ITB) with the steep ion temperature gradient is formed in the region where the gradient of the toroidal flow is steepest.[4] In rotating plasmas with the large flow velocities on the order of the ion thermal speed v_{Ti}, the toroidal flow shear influences the transport of particles, heat, and momentum as an additional thermodynamic force, although, in conventional neoclassical theories,[5–7] the flow velocities are assumed to be on the order of δv_{Ti} and the direct effects of the flow shear on the transport do not appear in the lowest order. Here $\delta \equiv \rho_i / L$ is the drift ordering parameter, ρ_i the ion thermal gyroradius, and L the equilibrium scale length. It is important to derive the transport equations including the flow shear effects at the same order as particle and thermal transport for understanding the ITB physics. Neoclassical ion transport equations for rotating plasmas were obtained by Hinton and Wong[8] and by Catto *et al.*[9] However, neoclassical electron fluxes are also required for a comprehensive description of transport processes. For example, the neoclassical parallel (bootstrap) current is associated with the parallel electron viscosity and is necessary for determining the equilibrium configuration self-consistently. In the present work, we derive full transport equations for neoclassical electron and ion fluxes in the rotating plasma with the toroidal flow velocity on the order of the ion thermal speed.

Hereafter we consider only axisymmetric systems, for which the magnetic field is given by

$$\mathbf{B} = I(\Psi)\nabla\zeta + \nabla\zeta \times \nabla\Psi, \tag{1}$$

where ζ is the toroidal angle, Ψ represents the poloidal flux, and $I(\Psi) = RB_T$. In the axisymmetric systems, the poloidal flow decays in a few transit or collision times and the lowest-order flow velocity $\mathbf{V}_0$ is in the toroidal direction:[8]

$$\mathbf{V}_0 = V_0 \hat{\boldsymbol{\zeta}}, \quad V_0 = RV^{\zeta} = -Rc\Phi_0'(\Psi), \tag{2}$$

where $\Phi_0(\Psi)$ denotes the lowest-order electrostatic potential in δ (which corresponds to Φ_{-1} in the paper by Hinton and Wong[8]) and $\mathbf{E}_0 \equiv -\nabla\Phi_0 \equiv -\Phi_0'\nabla\Psi$. The toroidal angular velocity $V^{\zeta} = -c\Phi_0'$ is directly given by the radial electric field and is a flux-surface quantity.

For particle species a with the mass m_a and the charge e_a, the phase space variables $(\mathbf{x}', \varepsilon, \mu, \xi)$ are defined in terms of the spatial coordinates $\mathbf{x}$ in the laboratory frame and the velocity $\mathbf{v}' \equiv \mathbf{v} - \mathbf{V}_0$ in the moving frame as[8,10]

$$\mathbf{x}' = \mathbf{x}, \quad \epsilon = \frac{1}{2} m_a (v')^2 + \Xi_a, \quad \mu = \frac{m_a (v_\perp')^2}{2B},$$
$$\frac{\mathbf{v}_\perp'}{v_\perp'} = \mathbf{e}_1 \cos\xi + \mathbf{e}_2 \sin\xi. \tag{3}$$

Here $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{b} \equiv \mathbf{B}/B)$ are unit vectors which form a right-handed orthogonal system at each point, $\mathbf{v}' = v_\parallel' \mathbf{b} + \mathbf{v}_\perp'$, $v_\parallel' = \mathbf{v}' \cdot \mathbf{b}$, and

$$\Xi_a \equiv e_a \tilde{\Phi}_1 - \frac{1}{2} m_a V_0^2, \tag{4}$$

where $\tilde{\Phi}_1 \equiv \Phi_1 - \langle \Phi_1 \rangle [= \mathscr{O}(\delta)]$ is the poloidal-angle-dependent part of the electrostatic potential. The magnetic flux surface average is denoted by $\langle \cdot \rangle$. The lowest-order distribution function is the Maxwellian which is written as

$$f_{a0} = n_a \left(\frac{m_a}{2\pi T_a} \right)^{3/2} \exp\left(-\frac{m_a (v')^2}{2T_a} \right)$$
$$= N_a \left(\frac{m_a}{2\pi T_a} \right)^{3/2} \exp\left(-\frac{\varepsilon}{T_a} \right), \tag{5}$$

where the temperature $T_a = T_a(\Psi)$ and $N_a = N_a(\Psi)$ are flux-surface functions although generally the density n_a depends on the poloidal angle θ through Ξ_a and is given by[8]

$$n_a = N_a \exp\left(-\frac{\Xi_a}{T_a} \right). \tag{6}$$

This dependence of the density n_a on the poloidal angle θ is one of the causes which complicates the derivation of the classical and neoclassical transport coefficients for the rotating plasma. For plasmas consisting of electrons and single-species ions with charge $e_i \equiv Z_i e$, we have[8]

$$\begin{aligned} \frac{e}{T_e}\tilde{\Phi}_1 &= \frac{m_i (V^\zeta)^2 (R^2 - \langle R^2 \rangle)}{2(Z_i T_e + T_i)}, \\ Z_i N_i(\Psi) &= N_e(\Psi) \exp\left(-\frac{m_i (V^\zeta)^2 \langle R^2 \rangle}{2 T_i} \right), \end{aligned} \tag{7}$$

where the charge neutrality condition $\Sigma_a e_a n_a = 0$ is used and $m_e/m_i (\ll 1)$ is neglected.

In toroidally rotating axisymmetric systems, the linearized drift-kinetic equation is written as[8–10]

$$\begin{aligned} v_\parallel' \mathbf{b} \cdot \nabla \overline{g_a} - C_a^L(\overline{g_a}) = \frac{1}{T_a} f_{a0} (W_{a1} X_{a1} + W_{a2} X_{a2} \\ + W_{aV} X_V + W_{aE} X_E), \end{aligned} \tag{8}$$

where C_a^L denotes the linearized collision operator [see Eq. (8) in Ref. 11] and $\overline{g_a}$ is defined in terms of the first-order gyrophase-averaged distribution function $\overline{f_{a1}}$ as

$$\overline{g_a} \equiv \overline{f_{a1}} - f_{a0} \frac{e_a}{T_a} \int^l \frac{dl}{B} \left(B E_\parallel^{(2)} - \frac{B^2}{\langle B^2 \rangle} \langle B E_\parallel^{(2)} \rangle \right). \tag{9}$$

Here $\int^l dl$ denotes the integral along the magnetic field line, and $E_\parallel^{(2)} \equiv \mathbf{b} \cdot (-\nabla \Phi^{(2)} - c^{-1} \partial \mathbf{A} / \partial t)$ is the second-order parallel electric field. The thermodynamic forces $(X_{a1}, X_{a2}, X_V, X_E)$ are flux-surface quantities defined by

$$\begin{aligned} X_{a1} &\equiv -\frac{1}{N_a} \frac{\partial (N_a T_a)}{\partial \Psi} - e_a \frac{\partial \langle \Phi_1 \rangle}{\partial \Psi}, \quad X_{a2} \equiv -\frac{\partial T_a}{\partial \Psi}, \\ X_V &\equiv -\frac{\partial V^\zeta}{\partial \Psi} = c \frac{\partial^2 \Phi_0}{\partial \Psi^2}, \quad X_E \equiv \frac{\langle B E_\parallel^{(A)} \rangle}{\langle B^2 \rangle^{1/2}}. \end{aligned} \tag{10}$$

The functions $(W_{a1}, W_{a2}, W_{aV}, W_{aE})$ are defined by

$$\begin{aligned} W_{a1} &\equiv \frac{m_a c}{e_a} v_\parallel' \mathbf{b} \cdot \nabla \left(R^2 V^\zeta + \frac{I}{B} v_\parallel' \right) \equiv v_\parallel' \mathbf{b} \cdot \nabla U_{a1}, \\ W_{a2} &\equiv W_{a1} \left(\frac{\varepsilon}{T_a} - \frac{5}{2} \right) \equiv v_\parallel' \mathbf{b} \cdot \nabla U_{a2}, \\ W_{aV} &\equiv \frac{m_a c}{2 e_a} v_\parallel' \mathbf{b} \cdot \nabla \left[m_a \left(R^2 V^\zeta + \frac{I}{B} v_\parallel' \right)^2 + \mu \frac{R^2 B_P^2}{B} \right] \\ &\equiv v_\parallel' \mathbf{b} \cdot \nabla U_{aV}, \\ W_{aE} &\equiv \frac{e_a v_\parallel' B}{\langle B^2 \rangle^{1/2}}. \end{aligned} \tag{11}$$

The neoclassical entropy production[10,11] is kinetically defined in terms of $\overline{f_{a1}}$ and C_a^L and is rewritten in the thermodynamic form by using Eq. (8). The surface-averaged total neoclassical entropy production is given by

$$\begin{aligned} \sum_a T_a \langle \sigma_a^{\rm ncl} \rangle &\equiv -\sum_a T_a \left\langle \int d^3 v \frac{\overline{f_{a1}}}{f_{a0}} C_a^L(\overline{f_{a1}}) \right\rangle \\ &= \sum_a \left(\Gamma_a^{\rm ncl} X_{a1} + \frac{1}{T_a} q_a^{\rm ncl} X_{a2} + \Pi_a^{\rm ncl} X_V \right) \\ &\quad + J_E X_E, \end{aligned} \tag{12}$$

where the neoclassical transport fluxes $(\Gamma_a^{\rm ncl}, q_a^{\rm ncl}/T_a, \Pi_a^{\rm ncl}, J_E)$ conjugate to the forces $(X_{a1}, X_{a2}, X_V, X_E)$ are defined by

$$\begin{aligned} \Gamma_a^{\rm ncl} &\equiv \left\langle \int d^3 v \overline{g_a} W_{a1} \right\rangle, \quad \frac{1}{T_a} q_a^{\rm ncl} \equiv \left\langle \int d^3 v \overline{g_a} W_{a2} \right\rangle, \\ \Pi_a^{\rm ncl} &\equiv \left\langle \int d^3 v \overline{g_a} W_{aV} \right\rangle, \\ J_E &\equiv \frac{\langle B J_\parallel \rangle}{\langle B^2 \rangle^{1/2}} \equiv \sum_a \left\langle \int d^3 v \overline{g_a} W_{aE} \right\rangle. \end{aligned} \tag{13}$$

Here $\Gamma_a^{\rm ncl}$, $q_a^{\rm ncl}$, and $\Pi_a^{\rm ncl}$ denote the surface-averaged radial fluxes of particles, heat, and toroidal (angular) momentum, respectively, and J_E represents the surface-averaged parallel current. The neoclassical transport equations connecting the conjugate pairs of the fluxes and forces are written as

$$\begin{aligned} \Gamma_a^{\rm ncl} &= \sum_b (L_{11}^{ab} X_{b1} + L_{12}^{ab} X_{b2}) + L_{1V}^a X_V + L_{1E}^a X_E, \\ \frac{1}{T_a} q_a^{\rm ncl} &= \sum_b (L_{21}^{ab} X_{b1} + L_{22}^{ab} X_{b2}) + L_{2V}^a X_V + L_{2E}^a X_E, \\ \sum_a \Pi_a^{\rm ncl} &= \sum_b (L_{V1}^b X_{b1} + L_{V2}^b X_{b2}) + L_{VV} X_V + L_{VE} X_E, \\ J_E &= \sum_b (L_{E1}^b X_{b1} + L_{E2}^b X_{b2}) + L_{EV} X_V + L_{EE} X_E, \end{aligned} \tag{14}$$

where the transport coefficients are dependent on the radial electric field through the toroidal angular velocity $V^\zeta = -c \Phi_0'$.

The remaining parts of this work are organized as follows. In Sec. II, using the formal solution of the linearized drift kinetic equation (8), we prove the Onsager symmetry of the neoclassical transport matrix for the rotating plasma consisting of electrons and multi-species ions with arbitrary collision frequencies. In Sec. III, we describe the transport fluxes other than the neoclassical fluxes to give the total transport of the particles, heat, and toroidal momentum. In the cases of single-species ions, the explicit forms of the neoclassical transport matrices for the Pfirsch–Schlüter and banana regimes are given in Sec. IV and Sec. V, respectively. Conclusions and discussion are given in Sec. VII. Appendix A shows the classical transport coefficients for the rotating plasma. In Appendix B, the first-order parallel flows and the parallel momentum equations, which are useful to derive the neoclassical transport equations, are obtained from the drift kinetic equation. The parallel viscosity coefficients for the plateau regime, from which all the plateau transport coefficients except for L_{VV} can be derived, are shown in Appendix C.

II. ONSAGER SYMMETRY OF NEOCLASSICAL TRANSPORT EQUATIONS FOR ROTATING PLASMAS

In order to prove the Onsager symmetry of the neoclassical transport equations, it is useful to note that the solution of the linearized drift kinetic equation (8) is written as

$$\overline{g_a} = \sum_b \, (G_{ab1}X_{b1} + G_{ab2}X_{b2}) + G_{aV}X_V + G_{aE}X_E, \quad (15)$$

where G_{abm} $(m=1,2)$ and G_{aM} $(M=V,E)$ are defined as the solutions of the following equations:

$$v'_\parallel \mathbf{b}\cdot\nabla G_{abm} - \sum_{a'} C^L_{aa'}(G_{abm}, G_{a'bm}) = \delta_{ab}\frac{1}{T_b} f_{b0} W_{bm} \quad (m=1,2),$$
$$v'_\parallel \mathbf{b}\cdot\nabla G_{aM} - \sum_{a'} C^L_{aa'}(G_{aM}, G_{a'M}) = \frac{1}{T_a} f_{a0} W_{aM} \quad (M=V,E). \quad (16)$$

Substituting Eq. (15) into Eq. (13), and comparing it with Eq. (14), we find that the neoclassical transport coefficients are given by

$$L^{ab}_{mn} = \left\langle \int d^3v \, W_{am} G_{abn} \right\rangle \quad (m,n=1,2),$$
$$L^{a}_{mM} = \left\langle \int d^3v \, W_{am} G_{aM} \right\rangle \quad (m=1,2; M=V,E),$$
$$L^{b}_{Mm} = \sum_a \left\langle \int d^3v \, W_{aM} G_{abm} \right\rangle \quad (m=1,2; M=V,E),$$
$$L_{MN} = \sum_a \left\langle \int d^3v \, W_{aM} G_{aN} \right\rangle \quad (M,N=V,E). \quad (17)$$

Here, let us consider separately two types of variable transformations, i.e., $v'_\parallel \to -v'_\parallel$ and $V^\zeta \to -V^\zeta$. For an arbitrary function F of $v'_\parallel$ and V^ζ, we define F^{++}, F^{+-}, F^{-+}, and F^{--} as parts of F which are even–even, even–odd, odd–even, and odd–odd with respect to the transformations $v'_\parallel \to -v'_\parallel$ and $V^\zeta \to -V^\zeta$, respectively. From Eq. (11), we find that

$$W_{am} = W^{++}_{am} + W^{--}_{am}, \quad W^{+-}_{am} = W^{-+}_{am} = 0 \quad (m=1,2),$$
$$W_{aM} = W^{+-}_{aM} + W^{-+}_{aM}, \quad W^{++}_{aM} = W^{--}_{aM} = 0 \quad (M=V,E). \quad (18)$$

Then, the first equation in Eqs. (16) is divided into the $++$, $+-$, $-+$, and $--$ parts as

$$v'_\parallel \mathbf{b}\cdot\nabla G^{-+}_{abm} - \sum_{a'} C^L_{aa'}(G^{++}_{abm}, G^{++}_{a'bm}) = \delta_{ab}\frac{1}{T_b} f_{b0} W^{++}_{bm},$$
$$v'_\parallel \mathbf{b}\cdot\nabla G^{--}_{abm} - \sum_{a'} C^L_{aa'}(G^{+-}_{abm}, G^{+-}_{a'bm}) = 0,$$
$$v'_\parallel \mathbf{b}\cdot\nabla G^{++}_{abm} - \sum_{a'} C^L_{aa'}(G^{-+}_{abm}, G^{-+}_{a'bm}) = 0,$$
$$v'_\parallel \mathbf{b}\cdot\nabla G^{+-}_{abm} - \sum_{a'} C^L_{aa'}(G^{--}_{abm}, G^{--}_{a'bm}) = \delta_{ab}\frac{1}{T_b} f_{b0} W^{--}_{bm} \quad (m=1,2). \quad (19)$$

Similarly, the second equation in Eqs. (16) is divided into parts with these symmetries, which is straightforward and not shown here. Next, we regard the transport coefficients as functions of V^ζ and write them as the sum of even and odd parts with respect to V^ζ: for example, L^{ab}_{mn} $(m,n=1,2)$ given in Eq. (17) are written as

$$L^{ab}_{mn} = L^{ab+}_{mn} + L^{ab-}_{mn} \quad (m,n=1,2), \quad (20)$$

where

$$L^{ab+}_{mn}(V^\zeta) = L^{ab+}_{mn}(-V^\zeta) = \left\langle \int d^3v (W^{++}_{am} G^{++}_{abn} + W^{--}_{am} G^{--}_{abn}) \right\rangle,$$
$$L^{ab-}_{mn}(V^\zeta) = -L^{ab-}_{mn}(-V^\zeta) = \left\langle \int d^3v (W^{++}_{am} G^{+-}_{abn} + W^{--}_{am} G^{-+}_{abn}) \right\rangle \quad (m,n=1,2). \quad (21)$$

From Eqs. (19) and (21), we obtain

$$L^{ab+}_{mn} = -\sum_{a',b'} T_{a'} \left\langle \int d^3v \frac{1}{f_{a'0}} \times [G^{++}_{a'bn} C^L_{a'b'}(G^{++}_{a'am}, G^{++}_{b'am}) + G^{--}_{a'bn} C^L_{a'b'}(G^{--}_{a'am}, G^{--}_{b'am}) + G^{+-}_{a'am} C^L_{a'b'}(G^{+-}_{a'bn}, G^{+-}_{b'bn}) + G^{-+}_{a'am} C^L_{a'b'}(G^{-+}_{a'bn}, G^{-+}_{b'bn})] \right\rangle = L^{ba+}_{nm},$$
$$L^{ab-}_{mn} = \sum_{a'} T_{a'} \left\langle \int d^3v \frac{1}{f_{a'0}} [G^{+-}_{a'bn} v'_\parallel \mathbf{b}\cdot\nabla G^{-+}_{a'am} + G^{-+}_{a'bn} v'_\parallel \mathbf{b}\cdot\nabla G^{+-}_{a'am} - G^{++}_{a'am} v'_\parallel \mathbf{b}\cdot\nabla G^{--}_{a'bn} - G^{--}_{a'am} v'_\parallel \mathbf{b}\cdot\nabla G^{++}_{a'bn}] \right\rangle = -L^{ba-}_{nm} \quad (m,n=1,2), \quad (22)$$

where we have used the self-adjointness of the linearized collision operator [see Eq. (9) in Ref. 11] and the antisymmetry relation

$$\left\langle \int d^3v \, \chi v'_\parallel \mathbf{b}\cdot\nabla \psi \right\rangle = -\left\langle \int d^3v \, \psi v'_\parallel \mathbf{b}\cdot\nabla \chi \right\rangle \quad (\chi, \psi\text{: arbitrary functions}). \quad (23)$$

Equation (22) gives the Onsager symmetry for the coefficients L_{mn}^{ab} $(m,n=1,2)$ which is rewritten as $L_{mn}^{ab}(V^\zeta)=L_{nm}^{ba}(-V^\zeta)$. In the same way as above, we can derive the Onsager symmetry for the other coefficients. To summarize, the Onsager relations for all the neoclassical transport coefficients are given by

$$\begin{aligned} &L_{mn}^{ab}(V^\zeta)=L_{nm}^{ba}(-V^\zeta) \quad (m,n=1,2),\\ &L_{MN}(V^\zeta)=L_{NM}(-V^\zeta) \quad (M,N=V,E),\\ &L_{mM}^{a}(V^\zeta)=-L_{Mm}^{a}(-V^\zeta) \quad (m=1,2;M=V,E). \end{aligned} \tag{24}$$

We can see from the derivation here that the Onsager relations in Eq. (24) are robustly valid even for the cases of multi-species ions and arbitrary collision frequencies.

Here, let us consider the case in which the system has up–down symmetry $B(\theta)=B(-\theta)$ (θ: a poloidal angle defined such that $\theta=0$ on the plane of reflection symmetry). In this case, it is convenient to use the transformation $\mathscr{T}:(v_\parallel',\theta,V^\zeta)\to(-v_\parallel',-\theta,-V^\zeta)$. We note that, under the transformation $\mathscr{T}$, W_{aV} is invariant while $W_{am}(m=1,2)$ and W_{aE} change their signs. Also, the operators $v_\parallel'\mathbf{b}\cdot\nabla$ and C_a^L commute with $\mathscr{T}$. Therefore, we see that G_{aV} is symmetric and G_{abm} $(m=1,2)$ and G_{aE} are antisymmetric with respect to $\mathscr{T}$. Then, it is found from Eq. (17) that L_{mn}^{ab}, L_{mE}^{a}, L_{VV}, and L_{EE} are even while L_{mV}^{a} and L_{VE} are odd in V_ζ. Thus, for the system with up–down symmetry, we obtain the restricted forms of the Onsager relations,

$$\begin{aligned} &L_{mn}^{ab}(V^\zeta)=L_{mn}^{ab}(-V^\zeta)=L_{nm}^{ba}(V^\zeta) \quad (m,n=1,2),\\ &L_{mV}^{a}(V^\zeta)=-L_{mV}^{a}(-V^\zeta)=L_{Vm}^{a}(V^\zeta) \quad (m=1,2),\\ &L_{mE}^{a}(V^\zeta)=L_{mE}^{a}(-V^\zeta)=-L_{Em}^{a}(V^\zeta) \quad (m=1,2),\\ &L_{VE}(V^\zeta)=-L_{VE}(-V^\zeta)=-L_{EV}(V^\zeta),\\ &L_{VV}(V^\zeta)=L_{VV}(-V^\zeta),\\ &L_{EE}(V^\zeta)=L_{EE}(-V^\zeta). \end{aligned} \tag{25}$$

III. TRANSPORT FLUXES OTHER THAN NEOCLASSICAL FLUXES

The particle, heat, and toroidal momentum fluxes for species a consist of the neoclassical and other transport parts, and are written as[10]

$$\begin{aligned} \Gamma_a &\equiv \left\langle \int d^3v f_a \mathbf{v}\cdot\nabla\Psi \right\rangle\\ &=\Gamma_a^{\rm cl}+\Gamma_a^{\rm ncl}+\Gamma_a^{H}+\Gamma_a^{(E)}+\Gamma_a^{\rm anom},\\ q_a &\equiv T_a\left\langle \int d^3v f_a\left(\frac{\varepsilon}{T_a}-\frac{5}{2}\right)\mathbf{v}\cdot\nabla\Psi \right\rangle\\ &=q_a^{\rm cl}+q_a^{\rm ncl}+q_a^{H}+q_a^{(E)}+q_a^{\rm anom},\\ \Pi_a &\equiv \left\langle \int d^3v f_a m_a v_\zeta \mathbf{v}\cdot\nabla\Psi \right\rangle\\ &=\Pi_a^{\rm cl}+\Pi_a^{\rm ncl}+\Pi_a^{H}+\Pi_a^{(E)}+\Pi_a^{\rm anom}. \end{aligned} \tag{26}$$

Here the classical fluxes $\Gamma_a^{\rm cl}$, $q_a^{\rm cl}$, and $\Pi_a^{\rm cl}$ are caused by particles' gyromotion with collisions. Their definitions and the classical transport coefficients for the rotating plasma are shown in Ref. 10. The momentum conservation in collisions assures the intrinsic ambipolarity of the classical particle fluxes, which implies that $\Sigma_a e_a\Gamma_a^{\rm cl}=0$ is valid for arbitrary values of the radial electric field. In Appendix A, the classical transport coefficients for the case of single-species ions are given. The fluxes $\Gamma_a^{(E)}$, $q_a^{(E)}$, and $\Pi_a^{(E)}$ [see Eq. (20) in Ref. 10 for their definitions] are given from the inductive electric field $\mathbf{E}^{(A)}\equiv -c^{-1}\partial\mathbf{A}/\partial t$ and do not contribute to the entropy production. The anomalous transport fluxes $\Gamma_a^{\rm anom}$, $q_a^{\rm anom}$, and $\Pi_a^{\rm anom}$ are driven by turbulent fluctuations and are defined in terms of the fluctuation–particle interaction operator [see Eq. (26) in Ref. 10]. Then, the intrinsic ambipolarity of the particle fluxes $\Gamma_a^{\rm anom}$ and $\Gamma_a^{(E)}$ are separately derived from the charge neutrality: $\Sigma_a e_a\Gamma_a^{\rm anom}=\Sigma_a e_a\Gamma_a^{(E)}=0$. The fluxes Γ_a^{H}, q_a^{H}, and Π_a^{H} are defined in Ref. 10, and are related to the thermodynamic forces X_{a1}, X_{a2}, and X_V through the *nondissipative antisymmetric* transport coefficients which satisfy the Onsager symmetry Eq. (24) [see Eqs. (23) and (24) in Ref. 10] and vanish if the system has up–down symmetry.

It can be shown that the sum of the neoclassical fluxes $(\Gamma_a^{\rm ncl},q_a^{\rm ncl},\Pi_a^{\rm ncl})$ and $(\Gamma_a^{H},q_a^{H},\Pi_a^{H})$ gives

$$\begin{aligned} \Gamma_a^{\rm ncl}+\Gamma_a^{H}&=-cI\langle n_a\rangle\frac{\langle BE_\parallel^{(A)}\rangle}{\langle B^2\rangle}-\frac{cI}{e_a}\left\langle\frac{F_{\parallel a1}}{B}\right\rangle,\\ \frac{1}{T_a}(q_a^{\rm ncl}+q_a^{H})&=-cI\left\langle n_a\frac{\Xi_a}{T_a}\right\rangle\frac{\langle BE_\parallel^{(A)}\rangle}{\langle B^2\rangle}\\ &\quad-\frac{c}{e_a}\left\langle\int d^3v\, m_a\left(\frac{Iv_\parallel'}{B}+R^2V^\zeta\right)\right.\\ &\quad\left.\times\left(\frac{\varepsilon}{T_a}-\frac{5}{2}\right)C_a^L(\overline{g_a})\right\rangle,\\ \Pi_a^{\rm ncl}+\Pi_a^{H}&=-m_acIV^\zeta\langle n_aR^2\rangle\frac{\langle BE_\parallel^{(A)}\rangle}{\langle B^2\rangle}\\ &\quad-\frac{m_ac}{2e_a}\left\langle\int d^3v\left[m_a\left(R^2V^\zeta+\frac{I}{B}v_\parallel'\right)^2\right.\right.\\ &\quad\left.\left.+\mu\frac{R^2B_P^2}{B}\right]C_a^L(\overline{g_a})\right\rangle, \end{aligned} \tag{27}$$

where $F_{\parallel a1}$ is the parallel component of the friction force $\mathbf{F}_{a1}\equiv\int d^3v\, m_a\mathbf{v}'C_a(f_a)$. Those fluxes in Eq. (27) that include $(\Gamma_a^{H},q_a^{H},\Pi_a^{H})$ are referred to as the neoclassical fluxes by Hinton and Wong and by Catto *et al.* In Ref. 10 and in the present work, $(\Gamma_a^{H},q_a^{H},\Pi_a^{H})$ are considered separately from the neoclassical fluxes since the former result from the collisionless particles' gyromotion and are related to the nondissipative parallel gyroviscosity. Now that the Onsager symmetry is shown to be satisfied by the transport coefficients for both $(\Gamma_a^{\rm ncl},q_a^{\rm ncl},\Pi_a^{\rm ncl})$ and $(\Gamma_a^{H},q_a^{H},\Pi_a^{H})$, the symmetry is also valid for the transport coefficients for their total fluxes in Eq. (27). From the charge neutrality and the momentum conservation in collisions with Eq. (27), we find that the particle fluxes $(\Gamma_a^{\rm ncl}+\Gamma_a^{H})$ are intrinsically ambipolar:

$$\sum_a e_a(\Gamma_a^{\rm ncl}+\Gamma_a^H)=0. \tag{28}$$

Using the ambipolarity condition in Eq. (28), the number of the pairs of the particle fluxes $(\Gamma_a^{\rm ncl}+\Gamma_a^H)$ and the conjugate thermodynamic forces appearing in the transport equations can be reduced by one without breaking the Onsager symmetry of the transport matrix.[11] In the following sections, we consider the case of single-species ions (i) with charge $e_i=Z_ie$ per particle, and derive the transport equations combining the five transport fluxes $[(\Gamma_e^{\rm ncl}+\Gamma_e^H),(q_e^{\rm ncl}+q_e^H)/T_e, (q_i^{\rm ncl}+q_i^H)/T_i,(\Pi_i^{\rm ncl}+\Pi_i^H),J_E]$ with the five thermodynamic forces $(X_{e1}^*,X_{e2},X_{i2},X_V,X_E)$:

$$\begin{bmatrix} \Gamma_e^{\rm ncl}+\Gamma_e^H \\ \dfrac{1}{T_e}(q_e^{\rm ncl}+q_e^H) \\ \dfrac{1}{T_i}(q_i^{\rm ncl}+q_i^H) \\ \Pi_i^{\rm ncl}+\Pi_i^H \\ J_E \end{bmatrix} = \begin{bmatrix} L_{11}^{ee} & L_{12}^{ee} & L_{12}^{ei} & L_{1V}^{e} & L_{1E}^{e} \\ L_{21}^{ee} & L_{22}^{ee} & L_{22}^{ei} & L_{2V}^{e} & L_{2E}^{e} \\ L_{21}^{ie} & L_{22}^{ie} & L_{22}^{ii} & L_{2V}^{i} & L_{2E}^{i} \\ L_{V1}^{e} & L_{V2}^{e} & L_{V2}^{i} & L_{VV} & L_{VE} \\ L_{E1}^{e} & L_{E2}^{e} & L_{E2}^{i} & L_{EV} & L_{EE} \end{bmatrix} \times \begin{bmatrix} X_{e1}^* \\ X_{e2} \\ X_{i2} \\ X_V \\ X_E \end{bmatrix}, \tag{29}$$

where the Onsager symmetry for the 5×5 transport matrix is already guaranteed. Here, we have neglected $\Pi_e^{\rm ncl}+\Pi_e^H$ which is $\mathcal{O}(m_e/m_i)$ smaller than $\Pi_i^{\rm ncl}+\Pi_i^H$. The first thermodynamic force X_{e1}^* is defined by

$$X_{e1}^*\equiv X_{e1}+\frac{X_{i1}}{Z_i}=-\frac{1}{N_e}\frac{\partial(N_eT_e)}{\partial\Psi}-\frac{1}{Z_iN_i}\frac{\partial(N_iT_i)}{\partial\Psi}. \tag{30}$$

With this reduction, we see that the $\mathcal{O}(\delta)$ radial electric field $-\partial\langle\Phi_1\rangle/\partial\Psi$ disappears from the thermodynamic forces. Thus, in the axisymmetric toroidally rotating system, the $\mathcal{O}(\delta)$ radial electric field neither affects the transport nor is determined by the ambipolar condition. Recall that the transport coefficients in Eq. (29) depend on the radial electric field of $\mathcal{O}(\delta^0)$ [not $\mathcal{O}(\delta)$] through the toroidal angular velocity $V^\zeta=-c\,\partial\Phi_0/\partial\Psi$ and that the thermodynamic force X_V is proportional to the radial gradient of the $\mathcal{O}(\delta^0)$ radial electric field [see Eq. (10)].

IV. PFIRSCH–SCHLÜTER REGIME

In the Pfirsch–Schlüter regime, the ratio of the particles' mean-free path to the equilibrium scale length $\lambda\equiv v_{Ta}\tau_{aa}/L$ is used as a small ordering parameter, where $v_{Ta}\equiv\sqrt{2T_a/m_a}$ is the thermal velocity and τ_{aa} the collision time defined in Ref. 6. In this section, we derive the full neoclassical transport equations for the rotating plasma consisting of electrons and single-species ions in the Pfirsch–Schlüter regime. Here we retain terms up to $\mathcal{O}[(m_e/m_i)^{1/2}]$ in order to obtain the electron transport coefficients, which are not considered in Refs. 8 and 9. In Ref. 8, the rotation speed divided by the ion thermal velocity V_0/v_{Ti} is assumed to be as small as $\lambda^{1/2}$ although, without this assumption, our results here are valid even for $V_0\sim v_{Ti}$.

According to the conventional analytical technique for the Pfirsch–Schlüter transport, let us expand the $\mathcal{O}(\delta)$ distribution function $\overline{g_a}$ in terms of the expansion parameter λ as

$$\overline{g_a}=\overline{g_a^{(-1)}}+\overline{g_a^{(0)}}+\overline{g_a^{(1)}}+\cdots. \tag{31}$$

In the lowest-order with respect to λ, the linearized drift kinetic equation (8) reduces to $C_a^L(\overline{g_a^{(-1)}})=0$, from which we have

$$\overline{g_a^{(-1)}}=\left(\alpha_a^{(-1)}+\frac{m_a}{T_a}u_{\parallel 1}^{(-1)}v_\parallel'+\frac{T_{a1}^{(-1)}}{T_a}\frac{m_a(v')^2}{2T_a}\right)\times f_{a0}(\varepsilon,\Psi). \tag{32}$$

Here, the $\mathcal{O}(\delta\lambda^{-1})$ quantities $\alpha_a^{(-1)}$, $u_{\parallel 1}^{(-1)}$, and $T_{a1}^{(-1)}$ are independent of the velocity variables (ε,μ). In Appendix B, some relations on the parallel flows and the parallel momentum balance equations are derived from the drift kinetic equation. The quantity $u_{\parallel 1}^{(-1)}$ represents the $\mathcal{O}(\delta\lambda^{-1})$ average parallel flow velocity, and is related by Eq. (B4) in Appendix B to $\mathcal{O}(\delta\lambda^{-1})$ surface quantities $\Gamma_{a\theta}^{(-1)}(\Psi)$ and $q_{a\theta}^{(-1)}(\Psi)$ as $n_au_{\parallel 1}^{(-1)}=B\Gamma_{a\theta}^{(-1)}$ and $q_{\parallel a1}^{(-1)}=\Xi_a n_a u_{\parallel 1}^{(-1)} = Bq_{a\theta}^{(-1)}$. Then, we have $\Xi_a\Gamma_{a\theta}^{(-1)}=q_{a\theta}^{(-1)}$, and therefore obtain $u_{\parallel 1}^{(-1)}=\Gamma_{a\theta}^{(-1)}=q_{a\theta}^{(-1)}=0$ by noting from Eqs. (4) and (7) that Ξ_a is dependent on the poloidal angle θ. In the next order with respect to λ, the linearized drift kinetic equation (8) is written as

$$C_a^L(\overline{g_a^{(0)}})=v_\parallel'\mathbf{b}\cdot\nabla\overline{g_a^{(-1)}}-\frac{1}{T_a}f_{a0}W_{aE}X_E, \tag{33}$$

where $X_E\equiv\langle BE_\parallel^{(A)}\rangle/\langle B^2\rangle^{1/2}$ is considered to be on the order of $\lambda^{(-1)}$ since it is balanced with the parallel current multiplied by the resistivity ($\propto\lambda^{(-1)}$) although the other thermodynamic forces X_{a1}, X_{a2}, and X_V are regarded as $\mathcal{O}(\lambda^0)$ quantities.

In the derivation of the transport equations for the Pfirsch–Schlüter regime, only the leading order terms of $\mathcal{O}(\delta^2\lambda^{-1})$ are retained in the radial fluxes of the particles, heat, and toroidal momentum, which are written from Eq. (27) as

$$\Gamma_e^{\rm ncl}+\Gamma_e^H=-cI\langle n_e\rangle\frac{\langle BE_\parallel^{(A)}\rangle}{\langle B^2\rangle}+\frac{cI}{e}\left\langle\frac{F_{\parallel e1}}{B}\right\rangle,$$

$$\frac{1}{T_e}(q_e^{\rm ncl}+q_e^H)=-cI\langle n_e\Delta_e\rangle\frac{\langle BE_\parallel^{(A)}\rangle}{\langle B^2\rangle}+\frac{cI}{e}\left\langle\frac{F_{\parallel e1}\Delta_e}{B}\right\rangle+\frac{cI}{e}\left\langle\frac{F_{\parallel e2}}{B}\right\rangle,$$

$$\frac{1}{T_i}(q_i^{\rm ncl}+q_i^H)=-\frac{cI}{Z_i}\langle n_e\Delta_i\rangle\frac{\langle BE_\parallel^{(A)}\rangle}{\langle B^2\rangle}+\frac{cI}{Z_ie}\left\langle\frac{F_{\parallel e1}\Delta_i}{B}\right\rangle-\frac{cI}{Z_ie}\left\langle\frac{F_{\parallel i2}}{B}\right\rangle,$$

$$\Pi_i^{\rm ncl}+\Pi_i^H=-\frac{cI}{Z_i}m_i V^\zeta \langle n_e R^2\rangle \frac{\langle BE_\parallel^{(A)}\rangle}{\langle B^2\rangle} + \frac{cI}{Z_i e} m_i V^\zeta \left\langle \frac{F_{\parallel e1}}{B} R^2 \right\rangle, \tag{34}$$

where we have used the definitions $\Delta_a \equiv \Xi_a/T_a$, $\mathbf{F}_{a2}\equiv \int d^3v\, m_a \mathbf{v}'[m_a(v')^2/2T_a - 5/2]C_a(f_a)$ $(a=e,i)$, and the momentum balance $\mathbf{F}_{e1}+\mathbf{F}_{i1}=0$ in collisions.

Multiplying Eq. (33) by $m_a v_\parallel'(-1)^j L_j^{(3/2)}(x^2)$ $(j=0,1,2,\ldots)$ $[L_0^{(3/2)}(x^2)=1,\ L_1^{(3/2)}(x^2)=5/2-x^2,\ldots$: the Laguerre polynomials; $x^2\equiv m_a(v')^2/2T_a]$ and integrating them in the velocity space give the $\mathscr{O}(\delta\lambda^{(-1)})$ parallel momentum balance equations. Then, we obtain $F_{\parallel aj}\equiv \int d^3v\, m_a \mathbf{v}_\parallel'(-1)^{j-1}L_{j-1}^{(3/2)}(x^2)C_a(f_a)=0$ for $j\geqslant 3$ and

$$\begin{bmatrix} T_e \mathbf{B}\cdot\nabla\alpha_e^{(-1)} + \frac{5}{2}\mathbf{B}\cdot\nabla T_e^{(-1)} - T_e^{(-1)}\mathbf{B}\cdot\nabla\Delta_e + eB^2\langle BE_\parallel^{(A)}\rangle/\langle B^2\rangle \\ \frac{5}{2}\mathbf{B}\cdot\nabla T_e^{(-1)} \end{bmatrix}$$
$$=\begin{bmatrix} BF_{\parallel e1}/n_e \\ BF_{\parallel e2}/n_e \end{bmatrix} = -\frac{m_e}{n_e\tau_{ee}} \begin{bmatrix} \hat\lambda_{11}^e & -\hat\lambda_{12}^e \\ -\hat\lambda_{12}^e & \hat\lambda_{22}^e \end{bmatrix} \begin{bmatrix} n_e(u_{\parallel e1}^{(0)} - u_{\parallel i1}^{(0)})B \\ \frac{2}{5}(q_{\parallel e1}^{(0)}/T_e - \Delta_e n_e u_{\parallel e1}^{(0)})B \end{bmatrix}, \tag{35}$$

$$\frac{5}{2}\mathbf{B}\cdot\nabla T_i^{(-1)} = B\frac{F_{\parallel i2}}{n_i} = -\frac{m_i}{n_i\tau_{ii}}\hat\lambda_{22}^i \frac{2}{5}\left(\frac{q_{\parallel i1}^{(0)}}{T_i} - \Delta_i n_i u_{\parallel i1}^{(0)}\right)B, \tag{36}$$

which correspond to the leading-order $\mathscr{O}(\delta\lambda^{-1})$ parts of Eqs. (B5). Here, the dimensionless friction coefficients $\hat\lambda_{11}^e$, $\hat\lambda_{12}^e$, $\hat\lambda_{22}^e$, and $\hat\lambda_{22}^i$ are related to the dimensionless coefficients $\tilde\sigma_\parallel$, $\tilde\alpha_\parallel$, $\tilde\kappa_\parallel^e$, and $\tilde\kappa_\parallel^i$ given in Ref. 7 as

$$\mathbf{\Lambda}_e \equiv \begin{bmatrix} \hat\lambda_{11}^e & -\hat\lambda_{12}^e \\ -\hat\lambda_{12}^e & \hat\lambda_{22}^e \end{bmatrix} = \begin{bmatrix} \tilde\sigma_\parallel & -\sqrt{\frac{2}{5}}\tilde\alpha_\parallel \\ -\sqrt{\frac{2}{5}}\tilde\alpha_\parallel & \frac{2}{5}\tilde\kappa_\parallel^e \end{bmatrix}^{-1}, \quad \hat\lambda_{22}^i = \left(\frac{2}{5}\tilde\kappa_\parallel^i\right)^{-1}. \tag{37}$$

Using the results of the 29-moment (29M) approximation in Ref. 7, we have $\hat\lambda_{11}^e=0.672$, $\hat\lambda_{12}^e=0.558$, $\hat\lambda_{22}^e=1.945$, and $\hat\lambda_{22}^i=1.110$ for the case of $Z_i=1$.

Using the magnetic surface average operations $\langle\cdot\rangle$ and $\langle\Delta_a\cdot\rangle$ $(\Delta_a\equiv\Xi_a/T_a; a=e,i)$ in the parallel momentum balance equations (35) and (36), we obtain the following equations:

$$\begin{bmatrix} \langle\Delta_e\mathbf{B}\cdot\nabla T_e^{(-1)}\rangle + e\langle BE_\parallel^{(A)}\rangle \\ 0 \end{bmatrix} = -\frac{m_e}{n_e\tau_{ee}} \begin{bmatrix} \hat\lambda_{11}^e & -\hat\lambda_{12}^e \\ -\hat\lambda_{12}^e & \hat\lambda_{22}^e \end{bmatrix} \left(\begin{bmatrix} \langle B^2\rangle(\Gamma_{e\theta}^{(0)} - Z_i\Gamma_{i\theta}^{(0)}) \\ \frac{2}{5}\langle B^2\rangle q_{e\theta}^{(0)}/T_e - \frac{2}{5}\langle B^2\Delta_e\rangle\Gamma_{e\theta}^{(0)} \end{bmatrix} - \frac{cI}{e}\begin{bmatrix} \langle n_e\rangle X_{e1}^* + \langle n_e\Delta_e\rangle X_{e2} + \langle n_e\Delta_i\rangle X_{i2}/Z_i + m_i V^\zeta\langle n_e R^2\rangle X_V/Z_i \\ \langle n_e\rangle X_{e2} \end{bmatrix} \right), \tag{38}$$

$$\frac{5}{2}\langle\Delta_e\mathbf{B}\cdot\nabla T_e^{(-1)}\rangle = -\frac{m_e}{n_e\tau_{ee}}[-\hat\lambda_{12}^e \quad \hat\lambda_{22}^e]\left(\begin{bmatrix} \langle B^2\Delta_e\rangle(\Gamma_{e\theta}^{(0)} - Z_i\Gamma_{i\theta}^{(0)}) \\ \frac{2}{5}\langle B^2\Delta_e\rangle q_{e\theta}^{(0)}/T_e - \frac{2}{5}\langle B^2\Delta_e^2\rangle\Gamma_{e\theta}^{(0)} \end{bmatrix} - \frac{cI}{e}\begin{bmatrix} \langle n_e\Delta_e\rangle X_{e1}^* + \langle n_e\Delta_e^2\rangle X_{e2} \\ \langle n_e\Delta_e\rangle X_{e2} \end{bmatrix} - \frac{cI}{e}\begin{bmatrix} \langle n_e\Delta_e\Delta_i\rangle X_{i2}/Z_i + m_i V^\zeta\langle n_e R^2\Delta_e\rangle X_V/Z_i \\ 0 \end{bmatrix} \right), \tag{39}$$

$$\frac{2}{5}\left(\langle B^2\rangle\frac{q_{i\theta}^{(0)}}{T_i} - \langle B^2\Delta_i\rangle\Gamma_{i\theta}^{(0)}\right) + \frac{cI}{Z_i e}\langle n_i\rangle X_{i2} = 0, \tag{40}$$

$$\frac{2}{5}\left\{\langle B^2\Delta_i\rangle\frac{q_{i\theta}^{(0)}}{T_i} - \langle B^2\Delta_i^2\rangle\Gamma_{i\theta}^{(0)}\right\} + \frac{cI}{Z_i e}\langle n_i\Delta_i\rangle X_{i2} = -\frac{5}{2}\frac{n_i\tau_{ii}}{m_i\hat\lambda_{22}^i}\langle\Delta_i\mathbf{B}\cdot\nabla T_i^{(-1)}\rangle, \tag{41}$$

where it should be noted that $n_a\tau_{aa}$ $(a=e,i)$ are independent of the poloidal angle θ. We also have from Eq. (B6),

$$Z_e\langle\Delta_e\mathbf{B}\cdot\nabla T_e^{(-1)}\rangle + \langle\Delta_i\mathbf{B}\cdot\nabla T_i^{(-1)}\rangle = 0. \tag{42}$$

Now, using Eqs. (38)–(42) and (B4), we can express the $\mathscr{O}(\delta\lambda^{(0)})$ poloidal flows $(\Gamma_{a\theta}^{(0)}, q_{a\theta}^{(0)})$, the $\mathscr{O}(\delta\lambda^{(0)})$ parallel flows $(u_{\|a1}^{(0)}, q_{\|a1}^{(0)})$, and the parallel friction forces $(F_{\|a1}, F_{\|a2})$ in the linear forms of the thermodynamic forces $(X_{e1}^{*}, X_{e2}, X_{i2}, X_V, X_E)$. Then, we can calculate the transport fluxes in Eq. (34) and obtain the transport coefficients in Eq. (29). The resultant transport coefficients for the Pfirsch–Schlüter regime are given by

$$\begin{bmatrix} L_{11}^{ee} & L_{12}^{ee} \\ L_{12}^{ee} & L_{22}^{ee} \end{bmatrix} = \frac{m_e}{n_e\tau_{ee}} \frac{c^2I^2}{e^2}\left(\left\langle \frac{n_e^2}{B^2}\mathbf{U}_e^T\mathbf{\Lambda}_e\mathbf{U}_e \right\rangle - \langle n_e\mathbf{U}_e^T\mathbf{\Lambda}_e\mathbf{V}_e^T\rangle\langle B^2\mathbf{V}_e\mathbf{\Lambda}_e\mathbf{V}_e^T\rangle^{-1}\langle n_e\mathbf{V}_e\mathbf{\Lambda}_e\mathbf{U}_e\rangle \right), \tag{43}$$

$$\begin{bmatrix} L_{12}^{ei} \\ L_{22}^{ei} \end{bmatrix} = \begin{bmatrix} L_{21}^{ie} \\ L_{22}^{ie} \end{bmatrix} = \frac{m_e}{n_e\tau_{ee}} \frac{c^2I^2}{e^2}\Bigg\{ \frac{5}{2}\frac{\{\langle B^2\rangle\langle n_e\Delta_i\rangle - \langle B^2\Delta_i\rangle\langle n_e\rangle\}}{\{\langle B^2\rangle\langle B^2\Delta_i^2\rangle - \langle B^2\Delta_i\rangle^2\}}\langle n_e\mathbf{U}_e^T\rangle + \left\langle \frac{n_e^2}{B^2}\Delta_i\mathbf{U}_e^T\right\rangle - \langle n_e\mathbf{U}_e^T\mathbf{\Lambda}_e\mathbf{V}_e^T\rangle$$

$$\times\langle B^2\mathbf{V}_e\mathbf{\Lambda}_e\mathbf{V}_e^T\rangle^{-1}\left(\frac{5}{2}\frac{\{\langle B^2\rangle\langle n_e\Delta_i\rangle - \langle B^2\Delta_i\rangle\langle n_e\rangle\}}{\{\langle B^2\rangle\langle B^2\Delta_i^2\rangle - \langle B^2\Delta_i\rangle^2\}}\langle B^2\mathbf{V}_e\rangle + \langle n_e\Delta_i\mathbf{V}_e\rangle\right)\Bigg\}\begin{bmatrix} \hat{\lambda}_{11}^e \\ -\hat{\lambda}_{12}^e \end{bmatrix}, \tag{44}$$

$$\begin{bmatrix} L_{1V}^{e} \\ L_{2V}^{e} \end{bmatrix} = \begin{bmatrix} L_{V1}^{e} \\ L_{V2}^{e} \end{bmatrix} = \frac{m_e}{n_e\tau_{ee}} \frac{c^2I^2}{e^2}\frac{m_iV^\zeta}{Z_i}\left(\left\langle \frac{n_e^2R^2}{B^2}\mathbf{U}_e^T\right\rangle - \langle n_e\mathbf{U}_e^T\mathbf{\Lambda}_e\mathbf{V}_e^T\rangle\langle B^2\mathbf{V}_e\mathbf{\Lambda}_e\mathbf{V}_e^T\rangle^{-1}\langle n_eR^2\mathbf{V}_e\rangle\right)\begin{bmatrix} \hat{\lambda}_{11}^e \\ -\hat{\lambda}_{12}^e \end{bmatrix}, \tag{45}$$

$$\begin{bmatrix} L_{1E}^{e} \\ L_{2E}^{e} \end{bmatrix} = -\begin{bmatrix} L_{E1}^{e} \\ L_{E2}^{e} \end{bmatrix} = cI\left(\langle n_e\mathbf{U}_e^T\mathbf{\Lambda}_e\mathbf{V}_e^T\rangle\langle B^2\mathbf{V}_e\mathbf{\Lambda}_e\mathbf{V}_e^T\rangle^{-1}\begin{bmatrix} \langle B^2\rangle^{1/2} \\ 0 \end{bmatrix} - \frac{1}{\langle B^2\rangle^{1/2}}\begin{bmatrix} \langle n_e\rangle \\ \langle n_e\Delta_e\rangle \end{bmatrix}\right), \tag{46}$$

$$L_{22}^{ii} = \frac{m_i\hat{\lambda}_{22}^i}{n_i\tau_{ii}}\frac{c^2I^2}{Z_i^2e^2}\left(\left\langle \frac{n_i^2}{B^2}\right\rangle + [\langle n_i\rangle \quad \langle n_i\Delta_i\rangle]\begin{bmatrix} \langle B^2\rangle & \langle B^2\Delta_i\rangle \\ \langle B^2\Delta_i\rangle & \langle B^2\Delta_i^2\rangle \end{bmatrix}^{-1}\begin{bmatrix} \langle n_i\rangle \\ \langle n_i\Delta_i\rangle \end{bmatrix}\right) + \frac{m_e}{n_e\tau_{ee}}\frac{c^2I^2}{Z_i^2e^2}\Bigg\{ \hat{\lambda}_{11}^e\Bigg(\left\langle \frac{n_e^2\Delta_i^2}{B^2}\right\rangle$$

$$+ 5\langle n_e\Delta_i\rangle\frac{\{\langle B^2\rangle\langle n_e\Delta_i\rangle - \langle B^2\Delta_i\rangle\langle n_e\rangle\}}{\{\langle B^2\rangle\langle B^2\Delta_i^2\rangle - \langle B^2\Delta_i\rangle^2\}} + \frac{25}{4}\langle B^2\rangle\frac{\{\langle B^2\rangle\langle n_e\Delta_i\rangle - \langle B^2\Delta_i\rangle\langle n_e\rangle\}^2}{\{\langle B^2\rangle\langle B^2\Delta_i^2\rangle - \langle B^2\Delta_i\rangle^2\}^2}\Bigg) - [\hat{\lambda}_{11}^e \quad -\hat{\lambda}_{12}^e]$$

$$\times\left(\frac{5}{2}\frac{\{\langle B^2\rangle\langle n_e\Delta_i\rangle - \langle B^2\Delta_i\rangle\langle n_e\rangle\}}{\{\langle B^2\rangle\langle B^2\Delta_i^2\rangle - \langle B^2\Delta_i\rangle^2\}}\langle B^2\mathbf{V}_e^T\rangle + \langle n_e\Delta_i\mathbf{V}_e^T\rangle\right)\langle B^2\mathbf{V}_e\mathbf{\Lambda}_e\mathbf{V}_e^T\rangle^{-1}\Bigg(\frac{5}{2}\frac{\{\langle B^2\rangle\langle n_e\Delta_i\rangle - \langle B^2\Delta_i\rangle\langle n_e\rangle\}}{\{\langle B^2\rangle\langle B^2\Delta_i^2\rangle - \langle B^2\Delta_i\rangle^2\}}\langle B^2\mathbf{V}_e\rangle$$

$$+ \langle n_e\Delta_i\mathbf{V}_e\rangle\Bigg)\begin{bmatrix} \hat{\lambda}_{11}^e \\ -\hat{\lambda}_{12}^e \end{bmatrix}\Bigg\}, \tag{47}$$

$$L_{2V}^{i} = L_{V2}^{i} = m_iV^\zeta\frac{m_e}{n_e\tau_{ee}}\frac{c^2I^2}{Z_i^2e^2}\Bigg\{ \hat{\lambda}_{11}^e\left(\left\langle \frac{n_e^2R^2\Delta_i}{B^2}\right\rangle + \frac{5}{2}\langle n_eR^2\rangle\frac{\{\langle B^2\rangle\langle n_e\Delta_i\rangle - \langle B^2\Delta_i\rangle\langle n_e\rangle\}}{\{\langle B^2\rangle\langle B^2\Delta_i^2\rangle - \langle B^2\Delta_i\rangle^2\}}\right) - [\hat{\lambda}_{11}^e \quad -\hat{\lambda}_{12}^e]$$

$$\times\left(\frac{5}{2}\frac{\{\langle B^2\rangle\langle n_e\Delta_i\rangle - \langle B^2\Delta_i\rangle\langle n_e\rangle\}}{\{\langle B^2\rangle\langle B^2\Delta_i^2\rangle - \langle B^2\Delta_i\rangle^2\}}\langle B^2\mathbf{V}_e^T\rangle + \langle n_e\Delta_i\mathbf{V}_e^T\rangle\right)\langle B^2\mathbf{V}_e\mathbf{\Lambda}_e\mathbf{V}_e^T\rangle^{-1}\langle n_eR^2\mathbf{V}_e\rangle\begin{bmatrix} \hat{\lambda}_{11}^e \\ -\hat{\lambda}_{12}^e \end{bmatrix}\Bigg\}, \tag{48}$$

$$L_{2E}^{i} = -L_{E2}^{i} = -\frac{cI}{Z_i}\Bigg\{\left(\frac{\langle n_e\Delta_i\rangle}{\langle B^2\rangle^{1/2}} + \frac{5}{2}\langle B^2\rangle^{1/2}\frac{\{\langle B^2\rangle\langle n_e\Delta_i\rangle - \langle B^2\Delta_i\rangle\langle n_e\rangle\}}{\{\langle B^2\rangle\langle B^2\Delta_i^2\rangle - \langle B^2\Delta_i\rangle^2\}}\right) - [\hat{\lambda}_{11}^e \quad -\hat{\lambda}_{12}^e]$$

$$\times\left(\frac{5}{2}\frac{\{\langle B^2\rangle\langle n_e\Delta_i\rangle - \langle B^2\Delta_i\rangle\langle n_e\rangle\}}{\{\langle B^2\rangle\langle B^2\Delta_i^2\rangle - \langle B^2\Delta_i\rangle^2\}}\langle B^2\mathbf{V}_e^T\rangle + \langle n_e\Delta_i\mathbf{V}_e^T\rangle\right)\langle B^2\mathbf{V}_e\mathbf{\Lambda}_e\mathbf{V}_e^T\rangle^{-1}\begin{bmatrix} \langle B^2\rangle^{1/2} \\ 0 \end{bmatrix}\Bigg\}, \tag{49}$$

$$L_{VV} = m_i^2(V^\zeta)^2\frac{m_e}{n_e\tau_{ee}}\frac{c^2I^2}{Z_i^2e^2}\left(\hat{\lambda}_{11}^e\left\langle \frac{n_e^2R^4}{B^2}\right\rangle - [\hat{\lambda}_{11}^e \quad -\hat{\lambda}_{12}^e]\langle n_eR^2\mathbf{V}_e^T\rangle\langle B^2\mathbf{V}_e\mathbf{\Lambda}_e\mathbf{V}_e^T\rangle^{-1}\langle n_eR^2\mathbf{V}_e\rangle\begin{bmatrix} \hat{\lambda}_{11}^e \\ -\hat{\lambda}_{12}^e \end{bmatrix}\right), \tag{50}$$

$$L_{VE} = -L_{EV} = -m_iV^\zeta\frac{cI}{Z_i}\left(\frac{\langle n_eR^2\rangle}{\langle B^2\rangle^{1/2}} - [\hat{\lambda}_{11}^e \quad -\hat{\lambda}_{12}^e]\langle n_eR^2\mathbf{V}_e^T\rangle\langle B^2\mathbf{V}_e\mathbf{\Lambda}_e\mathbf{V}_e^T\rangle^{-1}\begin{bmatrix} \langle B^2\rangle^{1/2} \\ 0 \end{bmatrix}\right), \tag{51}$$

$$L_{EE} = \frac{n_e\tau_{ee}e^2}{m_e}[\langle B^2\rangle^{1/2} \quad 0]\langle B^2\mathbf{V}_e\mathbf{\Lambda}_e\mathbf{V}_e^T\rangle^{-1}\begin{bmatrix} \langle B^2\rangle^{1/2} \\ 0 \end{bmatrix}, \tag{52}$$

where terms up to $\mathcal{O}[(m_e/m_i)^{1/2}]$ are retained, and the 2×2 matrices $\mathbf{\Lambda}_e$, $\mathbf{U}_e$, and $\mathbf{V}_e$ are defined by Eq. (37) and

$$\mathbf{U}_e \equiv \begin{bmatrix} 1 & \Delta_e \\ 0 & 1 \end{bmatrix}, \quad \mathbf{V}_e \equiv \begin{bmatrix} 1 & -\frac{2}{5}\Delta_e \\ 0 & 1 \end{bmatrix}. \tag{53}$$

In Eqs. (43)–(52), $\mathbf{M}^T$ denotes the transpose of $\mathbf{M}$ and $\langle \mathbf{M} \rangle$ is defined by $\langle \mathbf{M} \rangle \equiv [\langle M_{jk} \rangle]$ for an arbitrary matrix $\mathbf{M} = [M_{jk}]$.

It is found that, even if there is no up–down symmetry, the Pfirsch–Schlüter transport coefficients in Eqs. (43)–(52) satisfy the restricted version of the Onsager symmetry given in Eq. (25) since the inhomogeneity of the magnetic field is ignored within the mean free path in the Pfirsch–Schlüter regime [see also Eq. (A3) in Appendix A showing that the restricted version of the Onsager symmetry is also valid for the classical transport coefficients].

V. BANANA REGIME

In order to analytically obtain the neoclassical transport coefficients for the banana regime, we hereafter consider the large aspect ratio toroidal system and use the toroidal coordinates (r,θ,ζ) where the minor radius r is a label for magnetic surfaces. The major radius is given by $R = R_0 + r\cos\theta$ (R_0: the distance between the major axis and the magnetic axis) and $r/R_0 \ll 1$ is assumed. The banana regime is represented by $\omega_{Ta}\tau_{aa} \gg (R_0/r)^{3/2}$ where $\omega_{Ta} \equiv v_{Ta}/(qR_0)$ is the transit frequency and $q \equiv rB_T/(R_0 B_P)$ is the safety factor. When $\omega_{Ta}\tau_{aa} \gg 1$, the dominant parts of the radial transport fluxes in Eq. (27) are given by

$$\begin{aligned}
\Gamma_e^{\text{ncl}} + \Gamma_e^H &\simeq \frac{cI}{eB_0^2}\left\langle \int d^3v\, m_e (v_\parallel')^2 \mathbf{B}\cdot\nabla \overline{h_e} \right\rangle, \\
\frac{1}{T_e}(q_e^{\text{ncl}} + q_e^H) &\simeq \frac{cI}{eB_0^2}\left\langle \int d^3v\, m_e (v_\parallel')^2 \left(\frac{m_e (v')^2}{2T_e} - \frac{5}{2} \right) \mathbf{B}\cdot\nabla \overline{h_e} \right\rangle, \\
\frac{1}{T_i}(q_i^{\text{ncl}} + q_i^H) &\simeq -\frac{cI}{Z_i e B_0^2}\left\langle \int d^3v\, m_i (v_\parallel')^2 \left(\frac{m_i (v')^2}{2T_e} - \frac{5}{2} \right) \mathbf{B}\cdot\nabla \overline{h_i} \right\rangle \\
&\quad - \frac{cI}{Z_i e B_0^2}\langle \Delta_i \rangle \left\langle \int d^3v\, m_i (v_\parallel')^2 \mathbf{B}\cdot\nabla \overline{h_i} \right\rangle, \\
\Pi_i^{\text{ncl}} + \Pi_i^H &\simeq -\frac{m_i c}{2 Z_i e}\left\langle \int d^3v \left(m_i \frac{I^2}{B^2}(v_\parallel')^2 + \mu \frac{R^2 B_P^2}{B} \right) C_i^L(\overline{g_i}) \right\rangle \\
&\quad - m_i R_0^2 V^\zeta \frac{cI}{Z_i e B_0^2}\left\langle \int d^3v\, m_i (v_\parallel')^2 \mathbf{B}\cdot\nabla \overline{h_i} \right\rangle,
\end{aligned} \tag{54}$$

where the distribution functions $\overline{h_a}$ $(a = e,i)$ are defined by Eq. (B2).

It is shown from Eqs. (4), (6), and (7) that, in the lowest-order with respect to the inverse aspect ratio $r/R_0 \ll 1$, the electron and ion densities are also regarded as surface functions: $n_e = Z_i n_i \simeq N_e$ where $\mathcal{O}(m_e/m_i)$ terms are neglected. Then, the surface-averaged parallel momentum balance equations for $\omega_{Ta}\tau_{aa} \gg 1$ are obtained from Eqs. (B4), (B5), and (B6) as

$$\begin{aligned}
&\begin{bmatrix} \left\langle \int d^3v\, m_e (v_\parallel')^2 \mathbf{B}\cdot\nabla \overline{h_e} \right\rangle + n_e e \langle B E_\parallel^{(A)} \rangle \\ \left\langle \int d^3v\, m_e (v_\parallel')^2 \left(\frac{m_e (v')^2}{2T_e} - \frac{5}{2} \right) \mathbf{B}\cdot\nabla \overline{h_e} \right\rangle \end{bmatrix} \\
&= -\frac{m_e}{\tau_{ee}} \begin{bmatrix} \hat{l}_{11}^e & -\hat{l}_{12}^e \\ -\hat{l}_{12}^e & \hat{l}_{22}^e \end{bmatrix} \left(B_0^2 \begin{bmatrix} (\Gamma_{e\theta} - Z_i \Gamma_{i\theta}) \\ \frac{2}{5}\overline{q_{e\theta}}/T_e \end{bmatrix} \right. \\
&\quad \left. - \frac{n_e c I}{e} \begin{bmatrix} X_{e1}^* + \langle \Delta_i \rangle X_{i2}/Z_i + m_i R_0^2 V^\zeta X_V / Z_i \\ X_{e2} \end{bmatrix} \right),
\end{aligned} \tag{55}$$

$$\begin{aligned}
&\left\langle \int d^3v\, m_i (v_\parallel')^2 \left(\frac{m_i (v')^2}{2T_e} - \frac{5}{2} \right) \mathbf{B}\cdot\nabla \overline{h_i} \right\rangle \\
&= -\hat{l}_{22}^i \frac{m_i}{\tau_{ii}} \left(\frac{2}{5} B_0^2 \frac{\overline{q_{i\theta}}}{T_i} + \frac{n_i c I}{Z_i e} X_{i2} \right),
\end{aligned} \tag{56}$$

$$\left\langle \int d^3v\, m_e (v_\parallel')^2 \mathbf{B}\cdot\nabla \overline{h_e} \right\rangle + \left\langle \int d^3v\, m_i (v_\parallel')^2 \mathbf{B}\cdot\nabla \overline{h_i} \right\rangle = 0, \tag{57}$$

where the dimensionless friction coefficients are given by

$$\hat{l}_{11}^e = Z_i, \quad \hat{l}_{12}^e = \frac{3}{2} Z_i, \quad \hat{l}_{22}^e = \sqrt{2} + \frac{13}{4} Z_i, \quad \hat{l}_{22}^i = \sqrt{2}. \tag{58}$$

In Eqs. (55) and (56), we have used the notation $\overline{q_{a\theta}} \equiv q_{a\theta} - \langle \Xi_a \rangle \Gamma_{a\theta}$ $(a = e,i)$ and the 13-moment (13M) approximation to express the friction forces in terms of the flows.

Now, let us use the banana regime parameter $(R_0/r)^{3/2}(\omega_{Ta}\tau_{aa})^{-1} \ll 1$ to expand the distribution functions as

$$\begin{aligned}
\overline{g_a} &= \overline{g_a^{(0)}} + \overline{g_a^{(1)}} + \cdots, \\
\overline{h_a} &= \overline{h_a^{(0)}} + \overline{h_a^{(1)}} + \cdots.
\end{aligned} \tag{59}$$

The lowest order of the linearized drift kinetic equation is written as $v_\parallel' \mathbf{b}\cdot\nabla \overline{h_a^{(0)}} = 0$ which shows that $\overline{h_a^{(0)}}$ is independent of the poloidal angle θ:

$$\overline{h_a^{(0)}} = \overline{h_a^{(0)}}(\varepsilon, \mu; \Psi). \tag{60}$$

Thus $\overline{h_a^{(0)}}$ $(a = e,i)$ make no direct contribution to the neoclassical fluxes as shown by substituting Eq. (60) into Eq. (54). In the next order, the linearized drift kinetic equation gives

$$v_\parallel' \mathbf{b}\cdot\nabla \overline{h_a^{(1)}} = C_a^L(\overline{h_a^{(0)}}) + \frac{1}{T_a} C_a^L [f_{a0}(U_{a1}X_{a1} + U_{a2}X_{a2} + U_{aV}X_V)] + \frac{1}{T_a} f_{a0} W_{aE} X_E . \qquad (61)$$

Then, we have the solvability conditions[6] for Eq. (61) as

$$\oint \frac{dl}{v_\parallel'} C_a^L(\overline{h_a^{(+)}}) = -\oint \frac{dl}{v_\parallel'} C_a^L \left[f_{a0} \frac{m_a c}{e_a T_a} \left\{ \frac{m_a (v')^2}{2T_a} R^2 V^\zeta X_{a2} + \frac{1}{2}\left(m_a \frac{I^2}{B^2}(v_\parallel')^2 + \mu \frac{R^2 B_P^2}{B}\right) X_V \right\} \right], \qquad (62)$$

$$\oint \frac{dl}{v_\parallel'} C_a^L(\overline{h_a^{(-)}}) = -\oint \frac{dl}{v_\parallel'} \left(C_a^L \left[f_{a0} v_\parallel' \frac{m_a c I}{e_a T_a B} \left\{ X_{a1} + \left(\frac{\varepsilon}{T_a} - \frac{5}{2}\right) X_{a2} + m_a R^2 V^\zeta X_V \right\} \right] + \frac{e_a}{T_a} \frac{B}{\langle B^2\rangle^{1/2}} v_\parallel' X_E \right), \qquad (63)$$

where $\overline{h_a^{(0)}}$ is divided into the even $(+)$ and odd $(-)$ parts in $v_\parallel'$:

$$\overline{h_a^{(0)}} = \overline{h_a^{(+)}} + \overline{h_a^{(-)}} . \qquad (64)$$

We need to calculate the lowest-order parallel viscosities $\langle \int d^3v\, m_a (v_\parallel')^2 \mathbf{B}\cdot\nabla \overline{h_a^{(1)}}\rangle$ and $\langle \int d^3v\, m_a (v_\parallel')^2 \{[m_a (v')^2/2T_a] - \frac{5}{2}\}\ \mathbf{B}\cdot\nabla \overline{h_a^{(1)}}\rangle$ $(a=e,i)$ in order to obtain the radial fluxes in Eq. (54). It is found from Eq. (61) that only the odd part $\overline{h_a^{(-)}}$ is necessary for calculation of those lowest-order parallel viscosities. Following the standard procedure by Hirshman and Sigmar,[6] we can obtain the solution $\hat{h}_a^{(-)}$ of Eq. (63) and derive the parallel viscosities, which are written in the linear forms of the poloidal flows:

$$\begin{bmatrix} \left\langle \int d^3v\, m_a (v_\parallel')^2 \mathbf{B}\cdot\nabla \overline{h_a}\right\rangle \\ \left\langle \int d^3v\, m_a (v_\parallel')^2 (m_a (v')^2/2T_a - \frac{5}{2}) \mathbf{B}\cdot\nabla \overline{h_a}\right\rangle \end{bmatrix} = 1.469 \left(\frac{r}{R_0}\right)^{1/2} \frac{m_a}{\tau_{aa}} B_0^2 \begin{bmatrix} \hat{\mu}_{a1} & \hat{\mu}_{a2} \\ \hat{\mu}_{a2} & \hat{\mu}_{a3} \end{bmatrix} \begin{bmatrix} \Gamma_{a\theta} \\ \frac{2}{5}\overline{q_{a\theta}}/T_a \end{bmatrix}. \qquad (65)$$

The dimensionless coefficients $\hat{\mu}_{aj}$ $(a=e,i; j=1,2,3)$ are defined by

$$\hat{\mu}_{aj} = \frac{8}{3\sqrt{\pi}} \int_0^\infty dx\, x^4 e^{-x^2} \left(x^2 - \frac{5}{2}\right)^{j-1} \times \left(1 + \frac{\Upsilon}{x^2}\right)^{1/2} \tau_{aa} \nu_D^a(x) \quad (a=e,i; j=1,2,3), \qquad (66)$$

where the velocity-dependent collision frequency $\nu_D^a = \nu_D^a(x)$ $(x \equiv v'/v_{Ta})$ is defined in Ref. 6, and Υ denotes the square of the toroidal flow velocity normalized by the sound wave velocity: $\Upsilon \equiv m_i V_0^2/(Z_i T_e + T_i)$. For $0 \leqslant \Upsilon \leqslant 1$, the banana regime dimensionless coefficients $\hat{\mu}_{aj}$ $(a=e,i; j=1,2,3)$ defined by Eq. (66) are fitted as

$$\begin{aligned} \hat{\mu}_{e1} &= 0.533(1+0.923\Upsilon - 0.501\Upsilon^2 + 0.199\Upsilon^3) + (1+2.064\Upsilon - 1.690\Upsilon^2 + 0.765\Upsilon^3) Z_i , \\ \hat{\mu}_{e2} &= -0.625(1+1.551\Upsilon - 0.960\Upsilon^2 + 0.392\Upsilon^3) - 1.500(1+3.124\Upsilon - 2.709\Upsilon^2 + 1.239\Upsilon^3) Z_i , \\ \hat{\mu}_{e3} &= 1.386(1+1.533\Upsilon - 0.998\Upsilon^2 + 0.414\Upsilon^3) + 3.250(1+3.392\Upsilon - 3.029\Upsilon^2 + 1.395\Upsilon^3) Z_i , \\ \hat{\mu}_{i1} &= 0.533(1+0.923\Upsilon - 0.501\Upsilon^2 + 0.199\Upsilon^3), \\ \hat{\mu}_{i2} &= -0.625(1+1.551\Upsilon - 0.960\Upsilon^2 + 0.392\Upsilon^3), \\ \hat{\mu}_{i3} &= 1.386(1+1.533\Upsilon - 0.998\Upsilon^2 + 0.414\Upsilon^3), \end{aligned} \qquad (67)$$

where the ion–electron collision contributions of $\mathcal{O}(m_e/m_i)$ are neglected. Appendix C shows the parallel viscosity coefficients for the plateau regime, from which all the plateau transport coefficients except for L_{VV} can be derived.

Now, by using Eqs. (55)–(57), (65), and (B4), we can express the parallel viscosities $\langle \int d^3v\, m_a (v_\parallel')^2 \mathbf{B}\cdot\nabla \overline{h_a}\rangle$, $\langle \int d^3v\, m_a (v_\parallel')^2 \{[m_a (v')^2/2T_a] - \frac{5}{2}\} \mathbf{B}\cdot\nabla \overline{h_a}\rangle$ $(a=e,i)$ and the parallel current $J_E = e\langle n_e (u_{\parallel i} - u_{\parallel e})\rangle / \langle B^2\rangle^{1/2}$ in the linear forms of the thermodynamic forces $(X_{e1}^*, X_{e2}, X_{i2}, X_V, X_E)$. We find that the effects of the toroidal flow velocity (not its shear) on the electron and ion parallel viscosities for the banana regime are included only through Υ in Eq. (66). As in Ref. 8, we have used here the approximate expression for the parallel velocity $v_\parallel'$ of the trapped particle in the toroidally rotating plasma with the large aspect ratio

$$\frac{m_a (v_\parallel')^2}{2T_a} \simeq x^2 \left[1 - \frac{\mu B_0}{T_a x^2} + \frac{r}{R_0}\left(1 + \frac{\Upsilon}{x^2}\right) \cos\theta \right] \quad (a=e,i). \qquad (68)$$

In the right-hand side of Eq. (68), the term proportional to Υ is derived from the poloidal dependent part of the potential function Ξ_a which consists of the electrostatic potential and the effective gravity potential due to the centrifugal force [see Eqs. (2), (4), and (7)]. The sum of this poloidal variation $\widetilde{\Xi}_a$ and the poloidal magnetic variation forms the well for trapped particles, which is expressed by $(r/R_0) \times (1+\Upsilon/x^2)\cos\theta$ in Eq. (68). Thus, the toroidal rotation increases the trapped particles' population and accordingly the parallel viscosity coefficients as shown by the enhancement factor $(1+\Upsilon/x^2)^{1/2}$ in the right-hand side of Eq. (66).

In order to obtain the full transport equations, we need to also derive the linear thermodynamic expression of $\langle \int d^3v [m_i I^2 (v_\parallel')^2/B^2 + \mu R^2 B_P^2/B] C_i^L(\overline{g_i})\rangle$ which is necessary for the radial flux of the toroidal momentum in Eq. (54). This requires the solution $\hat{h}_i^{(+)}$ of Eq. (62), which can be given as in Ref. 8 by minimizing the positive definite functional:

$$\left\langle \int d^3v \frac{1}{f_{i0}} \left[\hat{h}_i^{(+)} + \frac{m_i c X_V}{2 Z_i e T_i} f_{i0} \left(m_i \frac{I^2}{B^2} (v_\parallel')^2 + \mu \frac{R^2 B_P^2}{B} \right) \right] \times C_{ii}^L \left[\hat{h}_i^{(+)} + \frac{m_i c X_V}{2 Z_i e T_i} f_{i0} \left(m_i \frac{I^2}{B^2} (v_\parallel')^2 + \mu \frac{R^2 B_P^2}{B} \right) \right] \right\rangle, \tag{69}$$

where C_{ii}^L denotes the linearized ion–ion collision operator with the ion–electron collisions neglected. Then we have the approximate solution

$$\hat{h}_i^{(+)} = \frac{m_i c I^2 X_V}{Z_i e T_i} \frac{\langle n_i^2/B^3 \rangle}{\langle n_i^2/B^2 \rangle} \mu f_{i0}, \tag{70}$$

from which $\langle \int d^3v [m_i I^2 (v_\parallel')^2/B^2 + \mu R^2 B_P^2/B] C_i^L(\overline{g_i}) \rangle$ is given in the linear form of X_V.

Thus, the final banana transport formulas form a 5×5 system of the coefficients as follows:

$$\begin{bmatrix} L_{11}^{ee} & L_{12}^{ee} \\ L_{12}^{ee} & L_{22}^{ee} \end{bmatrix} = 1.469 \left(\frac{r}{R_0} \right)^{1/2} \frac{n_e m_e c^2 |\nabla\Psi|^2}{e^2 B_P^2 \tau_{ee}} \begin{bmatrix} \hat{\mu}_{e1} & \hat{\mu}_{e2} \\ \hat{\mu}_{e2} & \hat{\mu}_{e3} \end{bmatrix}, \tag{71}$$

$$\begin{bmatrix} L_{12}^{ei} \\ L_{22}^{ei} \end{bmatrix} = \begin{bmatrix} L_{21}^{ie} \\ L_{22}^{ie} \end{bmatrix} = \frac{1}{Z_i} \left(\frac{\hat{\mu}_{i2}}{\hat{\mu}_{i1}} - \frac{m_i V_0^2}{2 T_i} \right) \begin{bmatrix} L_{11}^{ee} \\ L_{12}^{ee} \end{bmatrix}, \tag{72}$$

$$\begin{bmatrix} L_{1V}^{e} \\ L_{2V}^{e} \end{bmatrix} = \begin{bmatrix} L_{V1}^{e} \\ L_{V2}^{e} \end{bmatrix} = \frac{m_i R_0 V_0}{Z_i} \begin{bmatrix} L_{11}^{ee} \\ L_{12}^{ee} \end{bmatrix}, \tag{73}$$

$$\begin{bmatrix} L_{1E}^{e} \\ L_{2E}^{e} \end{bmatrix} = - \begin{bmatrix} L_{E1}^{e} \\ L_{E2}^{e} \end{bmatrix} = -1.469 \left(\frac{r}{R_0} \right)^{1/2} \frac{n_e c}{B_P} \frac{|\nabla\Psi|}{[\hat{l}_{11}^e \hat{l}_{22}^e - (\hat{l}_{12}^e)^2]} \times \begin{bmatrix} \hat{\mu}_{e1} & \hat{\mu}_{e2} \\ \hat{\mu}_{e2} & \hat{\mu}_{e3} \end{bmatrix} \begin{bmatrix} \hat{l}_{22}^e \\ \hat{l}_{12}^e \end{bmatrix}, \tag{74}$$

$$L_{22}^{ii} = 1.469 \left(\frac{r}{R_0} \right)^{1/2} \frac{n_i m_i c^2 |\nabla\Psi|^2}{Z_i^2 e^2 B_P^2 \tau_{ii}} \left(\hat{\mu}_{i3} - \frac{(\hat{\mu}_{i2})^2}{\hat{\mu}_{i1}} \right) + \frac{1}{Z_i^2} \left(\frac{\hat{\mu}_{i2}}{\hat{\mu}_{i1}} - \frac{m_i V_0^2}{2 T_i} \right)^2 L_{11}^{ee}, \tag{75}$$

$$L_{2V}^{i} = L_{V2}^{i} = \frac{m_i R_0 V_0}{Z_i^2} \left(\frac{\hat{\mu}_{i2}}{\hat{\mu}_{i1}} - \frac{m_i V_0^2}{2 T_i} \right) L_{11}^{ee}, \tag{76}$$

$$L_{2E}^{i} = -L_{E2}^{i} = \frac{1}{Z_i} \left(\frac{\hat{\mu}_{i2}}{\hat{\mu}_{i1}} - \frac{m_i V_0^2}{2 T_i} \right) L_{1E}^{e}, \tag{77}$$

$$L_{VV} = \frac{\sqrt{2}}{10} \left(\frac{r}{R_0} \right)^2 \frac{n_i T_i m_i^2 c^2 R_0^2 |\nabla\Psi|^2}{Z_i^2 e^2 B_P^2 \tau_{ii}} + \frac{m_i^2 R_0^2 V_0^2}{Z_i^2} L_{11}^{ee}, \tag{78}$$

$$L_{VE} = -L_{EV} = \frac{m_i R_0 V_0}{Z_i} L_{1E}^{e}, \tag{79}$$

$$L_{EE} = \sigma_S \left(1 - \frac{1.469 (r/R_0)^{1/2}}{\hat{l}_{22}^e [\hat{l}_{11}^e \hat{l}_{22}^e - (\hat{l}_{12}^e)^2]} [\hat{l}_{22}^e \quad \hat{l}_{12}^e] \begin{bmatrix} \hat{\mu}_{e1} & \hat{\mu}_{e2} \\ \hat{\mu}_{e2} & \hat{\mu}_{e3} \end{bmatrix} \times \begin{bmatrix} \hat{l}_{22}^e \\ \hat{l}_{12}^e \end{bmatrix} \right), \tag{80}$$

where $\sigma_S \equiv (e^2 n_e \tau_{ee}/m_e) \hat{l}_{22}^e / [\hat{l}_{11}^e \hat{l}_{22}^e - (\hat{l}_{12}^e)^2]$ denotes the Spitzer resistivity and $B_P \equiv |\nabla\Psi|/R$ represents the poloidal magnetic field. The dimensionless friction coefficients $\hat{l}_{jk}^e$ $(j,k=1,2)$ are written in Eq. (58) and the dimensionless viscosity coefficients $\hat{\mu}_{aj}$ $(a=e,i; j=1,2,3)$ are given by Eqs. (66) and (67). Recall that Ψ is used to define the radial transport fluxes and the radial thermodynamic forces in such a way that $q_a \equiv \langle \mathbf{q}_a \cdot \nabla\Psi \rangle$ and $X_{a2} \equiv -\partial T_a/\partial\Psi$. When we use r instead of Ψ to define the radial fluxes and forces, the resultant transport coefficients are immediately given by replacing $|\nabla\Psi|$ in Eqs. (71)–(80) with the unity.

We find that, in the large aspect ratio system, the banana transport coefficients in Eqs. (71)–(80) are much larger than the classical transport coefficients in Eq. (A2) by a factor of $\mathscr{O}[q^2(R_0/r)^{3/2}]$ except for the diagonal banana coefficient for the toroidal momentum transport $L_{VV} \simeq (1/10\sqrt{2}) \times q^2(\rho_i^2/\tau_{ii}) n_i m_i R_0^2 |\nabla\Psi|^2$ which is comparable to the classical one $L_{VV}^{\rm cl} \simeq (3/5\sqrt{2})(\rho_i^2/\tau_{ii}) n_i m_i R_0^2 |\nabla\Psi|^2$. All the coefficients in Eqs. (71)–(80) are functions of V_0 as seen from the explicit appearance of V_0 and from the flow-dependent viscosity coefficients [see Eq. (66)]. From Eq. (75) with the small electron mass terms neglected, the toroidal flow dependence of the ion thermal diffusivity L_{22}^{ii} appears through $[\hat{\mu}_{i3} - (\hat{\mu}_{i2})^2/\hat{\mu}_{i1}] \equiv 0.653 F(\Upsilon)$ where the enhancement factor $F(\Upsilon)$ for the ion thermal diffusivity is fitted for $0 \leqslant \Upsilon \leqslant 1$ as

$$F(\Upsilon) = 1 + 0.765\Upsilon - 0.631\Upsilon^2 + 0.280\Upsilon^3. \tag{81}$$

This enhancement factor is in good agreement with that given by Catto *et al.*,[9] $F(\Upsilon) = 1 + 0.75\Upsilon - 0.60\Upsilon^2 + 0.26\Upsilon^3$ [see Eq. (98) in Ref. 9 and note that Υ is written as X in their notation], in spite of the difference between the solution methods: our calculation is based on the moment expansion method with the 13M approximation while they use the variational technique. We find from Eqs. (67), (71), and (81) that the banana particle diffusivity and the banana electron and ion thermal diffusivities are monotonically increasing functions of Υ. This is because the potential well due to the toroidal rotation increases the number of the trapped particles as mentioned after Eq. (68).

The transport coefficients in Eqs. (71)–(80) satisfy the restricted version of the Onsager symmetry given in Eq. (25) since we have used the large aspect ratio approximation where the magnetic surfaces ($r=$ const) have circular cross sections. A well-known pair of Onsager symmetric neoclassical transport coefficients is that of L_{1E}^e and $L_{E1}^e (= -L_{1E}^e)$ [see Eq. (74)]. The off-diagonal coefficient $L_{1E}^e < 0$ indicates that the parallel electric field X_E gives the inward particle flux $\Gamma_e = L_{1E}^e X_E < 0$ due to trapped particles, which is known as the Ware pinch effect.[12] The counterpart $L_{E1}^e (= -L_{1E}^e)$ represents that the thermodynamic force X_{e1}^* produces the parallel current (the bootstrap current) $J_E = L_{E1}^e X_{e1}^*$. Since the toroidal momentum transport Π_i and the flow shear X_V enter the transport equations for the toroidally rotating plasma as a new conjugate flux-force pair, there appears a new physically important pair of Onsager symmetric neoclassical transport coefficients L_{VE} and $L_{EV}(=-L_{VE})$. The coefficient $L_{VE}=(m_i R_0 V_0/Z_i)L_{1E}^e$ [see Eq. (79)] shows that the parallel electric field X_E gives the inward toroidal momentum flux $\Pi_i=L_{VE}X_E$ (which has the opposite sign to V_0) due to the pinched trapped ions with the mean toroidal velocity V_0. From its partner $L_{EV}(=-L_{VE})$, we find that the flow shear X_V drives the parallel current $J_E=L_{EV}X_V$.

It is shown from Eqs. (4), (6), (30), and (79) that the sum of the currents driven by X_{e1}^* and X_V is rewritten as $L_{E1}^e[X_{e1}^*+(m_i R V_0/Z_i)X_V]=(L_{E1}^e/n_e|\nabla\Psi|)[-(\partial P/\partial r)+n_i m_i(V_0^2/R)(\partial R/\partial r)-(n_i m_i V_0^2/2T_i)(\partial T_i/\partial r)]$ with the flow shear term canceled. Here the total pressure gradient term $\partial P/\partial r\equiv\partial(n_e T_e+n_i T_i)/\partial r$ and the centrifugal force term $n_i m_i(V_0^2/R)(\partial R/\partial r)=-n_i m_i \mathbf{V}_0\cdot\nabla\mathbf{V}_0\cdot\nabla R$ appear since they give the perpendicular current $J_\perp$ through the equilibrium equation $\mathbf{J}\times\mathbf{B}/c=\nabla P-n_i m_i\mathbf{V}_0\cdot\nabla\mathbf{V}_0$ and accordingly drive the electron poloidal flow, from which the electron parallel viscosity [see Eq. (65)] and therefore the neoclassical parallel current are produced.

Using the current coefficients in Eqs. (74), (77), (79), and (80) with the viscosity coefficients (67), the full expression of the parallel current for the toroidally rotating plasma is written as

$$J_E=-\left(\frac{r}{R_0}\right)^{1/2}\frac{c}{B_P}\left[2.411F_{E1}^e(\Upsilon)\frac{dP}{dr}-1.800F_{E2}^e(\Upsilon)n_e\frac{dT_e}{dr}-2.828F_{E2}^i(\Upsilon)n_i\frac{dT_i}{dr}\right]+\sigma_S\left[1-1.832\left(\frac{r}{R_0}\right)^{1/2}F_{EE}(\Upsilon)\right]X_E, \tag{82}$$

with the enhancement factors

$$\begin{aligned}F_{E1}^e(\Upsilon)&=1+0.868\Upsilon-0.539\Upsilon^2+0.229\Upsilon^3,\\ F_{E2}^e(\Upsilon)&=1+2.248\Upsilon-1.661\Upsilon^2+0.727\Upsilon^3,\\ F_{E2}^i(\Upsilon)&=1+1.494\Upsilon-1.022\Upsilon^2+0.434\Upsilon^3,\\ F_{EE}(\Upsilon)&=1+0.431\Upsilon-0.184\Upsilon^2+0.072\Upsilon^3,\end{aligned} \tag{83}$$

where $Z_i=1$ and $0\leqslant\Upsilon\leqslant 1$ are assumed, and the centrifugal force term of $\mathcal{O}[\Upsilon(r/R_0)(dP/dr)]$ is neglected.

VI. CONCLUSIONS AND DISCUSSION

In this work, we have studied neoclassical transport for the axisymmetric system with the large toroidal flow velocity $(\sim v_{Ti})$. In the toroidally rotating plasma, the transport equations involve a new pair of the transport flux and the thermodynamic force: the radial flux of the toroidal momentum and the toroidal flow shear which is proportional to the radial electric field shear. For general rotating plasmas consisting of multi-species particles in arbitrary collisional regimes, the Onsager symmetry of the neoclassical transport matrix is proved by using the formal solution of the linearized drift kinetic equation with the self-adjoint collision operator, and its restricted form for the system with up–down symmetry is also shown. The complete neoclassical electron and ion transport equations are derived for the Pfirsch–Schlüter and banana regimes in the case of single-species ions and the Onsager symmetry is directly confirmed by them.

We have found that the toroidal rotation causes the centrifugal force and the poloidal variation of the electrostatic potential, which result in the increase of the trapped particles and therefore the enhancement of the parallel viscosities, the particle and thermal diffusivities, and the transport coefficients concerned with the pressure gradient driven (bootstrap) current and the Ware pinch. It is also shown that the parallel inductive electric field drives the inward banana flux of the toroidal momentum in addition to the Ware pinch of the particles. These inward particle and momentum fluxes driven by the parallel electric field are related through two pairs of the Onsager symmetric off-diagonal coefficients to the parallel currents driven by the radial thermodynamic forces conjugate to the inward fluxes, respectively. For large toroidal flows such as observed in JT-60U,[4,13] toroidal flow effects on the parallel current coefficients [see Eqs. (82) and (83)] are roughly estimated to reach the order of ~10% of those without taking account of the flow effects, and thus should not be neglected for accurate calculation of the current profile and the magnetic configuration.

Nagashima *et al.*[13] obtained from the JT-60U experiment the toroidal momentum diffusivity χ_ϕ and the inward velocity $v_{\rm inward}$ for the toroidal momentum transport. These are related to the transport coefficients given in the present paper by $L_{VV}=n_i m_i R_0^2\chi_\phi|\nabla\Psi|^2$ and $L_{VE}^e X_E=-n_i m_i R_0 V_0 v_{\rm inward}|\nabla\Psi|$, and are written from the results in Sec. V as $\chi_\phi=(1/\sqrt{2})(\frac{1}{10}q^2+\frac{3}{5})\rho_i^2/\tau_{ii}$ and $v_{\rm inward}=2.411Z_i^{-1}F_{E1}^e(\Upsilon)(r/R_0)^{1/2}cE_\parallel/B_P$ for the banana regime. Their experimental results give typically $\chi_\phi\sim 1\ {\rm m^2/s}$ and $v_{\rm inward}\sim 1$ m/s, which are much larger than the predictions by the above neoclassical model $\chi_\phi\sim 10^{-4}\ {\rm m^2/s}$ and $v_{\rm inward}\leqslant 0.1$ m/s. Thus, the radial transport of the toroidal momentum is considered to be dominated by the anomalous processes. In our previous paper,[10] the anomalous transport fluxes for the rotating plasma are formulated based on the gyrokinetic equations, and the simple expression for the anomalous toroidal momentum diffusivity is given for the mixing length level of the ion temperature gradient (ITG) driven turbulence [see Eq. (66) in Ref. 10 where μ_i^A corresponds to χ_ϕ]. This mixing length type estimation can give a larger momentum diffusivity on the order of the experimentally observed one. However, in order to describe the anomalous pinch of the toroidal momentum and explain the significant reduction of the transport at the ITB, a more elaborate investigation on the anomalous transport fluxes in the rotating plasma is required as a future task.

ACKNOWLEDGMENTS

The author (H. S.) thanks Professor M. Okamoto for his encouragement of this work.

This work is supported in part by the Grant-in-Aid from the Japanese Ministry of Education, Science, and Culture, and in part by the U.S. Department of Energy, Grant No. DE-FG05-80ET-53088.

APPENDIX A: CLASSICAL TRANSPORT FOR TOROIDALLY ROTATING PLASMAS CONSISTING OF ELECTRONS AND SINGLE-SPECIES IONS

The classical transport equations for the rotating plasma consisting of electrons and multi-species ions are derived in Appendix A of Ref. 10. In the case of single-species ions, they are written as

$$\begin{bmatrix} \Gamma_e^{\rm cl} \\ \frac{1}{T_e} q_e^{\rm cl} \\ \frac{1}{T_i} q_i^{\rm cl} \\ \Pi_i^{\rm cl} \end{bmatrix} = \begin{bmatrix} (L^{\rm cl})_{11}^{ee} & (L^{\rm cl})_{12}^{ee} & (L^{\rm cl})_{12}^{ei} & (L^{\rm cl})_{1V}^{e} \\ (L^{\rm cl})_{21}^{ee} & (L^{\rm cl})_{22}^{ee} & (L^{\rm cl})_{22}^{ei} & (L^{\rm cl})_{2V}^{e} \\ (L^{\rm cl})_{21}^{ie} & (L^{\rm cl})_{22}^{ie} & (L^{\rm cl})_{22}^{ii} & (L^{\rm cl})_{2V}^{i} \\ (L^{\rm cl})_{V1}^{e} & (L^{\rm cl})_{V2}^{e} & (L^{\rm cl})_{V2}^{i} & (L^{\rm cl})_{VV} \end{bmatrix} \times \begin{bmatrix} X_{e1}^* \\ X_{e2} \\ X_{i2} \\ X_V \end{bmatrix}, \tag{A1}$$

where the classical transport coefficients are given by

$$\begin{bmatrix} (L^{\rm cl})_{11}^{ee} & (L^{\rm cl})_{12}^{ee} \\ (L^{\rm cl})_{21}^{ee} & (L^{\rm cl})_{22}^{ee} \end{bmatrix} = \left\langle \frac{n_e m_e}{\tau_{ee}} \frac{c^2 R^2 B_P^2}{e^2 B^2} \begin{bmatrix} 1 & 0 \\ \Delta_e & 1 \end{bmatrix} \times \begin{bmatrix} \hat{l}_{11}^e & -\hat{l}_{12}^e \\ -\hat{l}_{12}^e & \hat{l}_{22}^e \end{bmatrix} \begin{bmatrix} 1 & \Delta_e \\ 0 & 1 \end{bmatrix} \right\rangle,$$

$$\begin{bmatrix} (L^{\rm cl})_{12}^{ei} \\ (L^{\rm cl})_{22}^{ei} \end{bmatrix} = \begin{bmatrix} (L^{\rm cl})_{21}^{ie} \\ (L^{\rm cl})_{22}^{ie} \end{bmatrix} = \left\langle \frac{n_e m_e}{\tau_{ee}} \frac{c^2 R^2 B_P^2}{Z_i e^2 B^2} \Delta_i \begin{bmatrix} 1 & 0 \\ \Delta_e & 1 \end{bmatrix} \times \begin{bmatrix} \hat{l}_{11}^e \\ -\hat{l}_{12}^e \end{bmatrix} \right\rangle,$$

$$\begin{bmatrix} (L^{\rm cl})_{1V}^{e} \\ (L^{\rm cl})_{2V}^{e} \end{bmatrix} = \begin{bmatrix} (L^{\rm cl})_{V1}^{e} \\ (L^{\rm cl})_{V2}^{e} \end{bmatrix} = \left\langle m_i R^2 V^\zeta \frac{n_e m_e}{\tau_{ee}} \frac{c^2 R^2 B_P^2}{Z_i e^2 B^2} \begin{bmatrix} 1 & 0 \\ \Delta_e & 1 \end{bmatrix} \times \begin{bmatrix} \hat{l}_{11}^e \\ -\hat{l}_{12}^e \end{bmatrix} \right\rangle, \tag{A2}$$

$$(L^{\rm cl})_{22}^{ii} = \left\langle \frac{n_i m_i}{\tau_{ii}} \frac{c^2 R^2 B_P^2}{Z_i^2 e^2 B^2} \left(\hat{l}_{22}^i + \hat{l}_{11}^e \Delta_i^2 \frac{n_e m_e \tau_{ii}}{n_i m_i \tau_{ee}} \right) \right\rangle,$$

$$(L^{\rm cl})_{2V}^{i} = (L^{\rm cl})_{V2}^{i} = \left\langle m_i R^2 V^\zeta \frac{n_e m_e}{\tau_{ee}} \frac{c^2 R^2 B_P^2}{Z_i^2 e^2 B^2} \Delta_i \hat{l}_{11}^e \right\rangle,$$

$$(L^{\rm cl})_{VV} = \frac{m_i^2 c^2}{Z_i^2 e^2} \left\langle \frac{R^2 B_P^2}{B^2} \left(\frac{(R^2 B_P^2 + 4I^2)}{4B^2} \frac{n_i T_i}{\tau_{ii}} \hat{l}_V^i + R^4 (V^\zeta)^2 \frac{n_e m_e}{\tau_{ee}} \hat{l}_{11}^e \right) \right\rangle.$$

Here, the poloidal magnetic field is given by $B_P = |\nabla \Psi|/R$ and the collision times τ_{aa} $(a=e,i)$ are defined in Ref. 6. The dimensionless friction coefficients are given by $\hat{l}_{11}^e = Z_i$, $\hat{l}_{12}^e = \frac{3}{2} Z_i$, $\hat{l}_{22}^e = \sqrt{2} + \frac{13}{4} Z_i$, $\hat{l}_{22}^i = \sqrt{2}$, and $\hat{l}_i^V = \frac{3}{5}\sqrt{2}$. The classical ion particle flux is given from the electron particle flux through the intrinsic ambipolarity condition $\Gamma_i^{\rm cl} = \Gamma_e^{\rm cl}/Z_i$. With respect to the small mass ratio m_e/m_i, we have retained the terms up to $\mathscr{O}[(m_e/m_i)^{1/2}]$ in Eqs. (A1) and (A2) although the terms of $\mathscr{O}(m_e/m_i)$ such as $\Pi_e^{\rm cl}$ have been neglected. We see from Eqs. (A2) that the classical transport coefficients satisfy the Onsager symmetry

$$\begin{aligned} &(L^{\rm cl})_{mn}^{ab}(V^\zeta) = (L^{\rm cl})_{mn}^{ab}(-V^\zeta) = (L^{\rm cl})_{nm}^{ba}(V^\zeta) \\ &\qquad (a,b=e,i; m,n=1,2), \\ &(L^{\rm cl})_{mV}^{a}(V^\zeta) = -(L^{\rm cl})_{mV}^{a}(-V^\zeta) = (L^{\rm cl})_{Vm}^{a}(V^\zeta) \\ &\qquad (a=e,i; m=1,2), \\ &(L^{\rm cl})_{VV}(V^\zeta) = (L^{\rm cl})_{VV}(-V^\zeta), \end{aligned} \tag{A3}$$

which has the same form as Eq. (25) and is valid even without up–down symmetry since the classical transport is a spatially local process.

APPENDIX B: THE FIRST-ORDER PARALLEL FLOWS AND PARALLEL MOMENTUM BALANCE EQUATIONS

From Eqs. (8) and (11), the linearized drift kinetic equation is rewritten as

$$v_\parallel' \mathbf{b} \cdot \nabla \overline{h_a} - C_a^L(\overline{g_a}) = \frac{1}{T_a} f_{a0} W_{aE} X_E, \tag{B1}$$

where

$$\overline{h_a} \equiv \overline{g_a} - \frac{1}{T_a} f_{a0} (U_{a1} X_{a1} + U_{a2} X_{a2} + U_{aV} X_V). \tag{B2}$$

Multiplying Eq. (B1) by the unity and $(\varepsilon/T_a - 5/2)$, and integrating them in the velocity space give the continuity equation and the energy balance equation of $\mathscr{O}(\delta)$ as

$$\begin{aligned} &\mathbf{B} \cdot \nabla \left(\int d^3 v \, \overline{h_a} \frac{v_\parallel'}{B} \right) = \mathbf{B} \cdot \nabla \left(n_a \frac{u_{\parallel a1}}{B} \right) + \nabla \cdot (n_a \mathbf{u}_{\perp a1}) = 0, \\ &\mathbf{B} \cdot \nabla \left[\int d^3 v \, \overline{h_a} \left(\frac{\varepsilon}{T_a} - \frac{5}{2} \right) \frac{v_\parallel'}{B} \right] = \mathbf{B} \cdot \nabla \left(\frac{q_{\parallel a1}}{B} \right) + \nabla \cdot (\mathbf{q}_{\perp a1}) = 0. \end{aligned} \tag{B3}$$

Integrating Eq. (B3) along the magnetic field line, we have the $\mathscr{O}(\delta)$ parallel flows:

$$\begin{aligned} n_a u_{\parallel a1} &\equiv \int d^3 v \, \overline{f_{a1}} v_\parallel' \\ &= B \Gamma_{a\theta}(\Psi) + \frac{n_a c I}{e_a B} (X_{a1} + \Delta_a X_{a2} + m_a R^2 V^\zeta X_V), \end{aligned} \tag{B4}$$

$$\frac{q_{\|a1}}{T_a} \equiv \int d^3 v \overline{f_{a1}} \left(\frac{\varepsilon}{T_a} - \frac{5}{2} \right) v'_{\|} = B \frac{q_{a\theta}(\Psi)}{T_a} + \frac{n_a c I}{e_a B} \left[\Delta_a X_{a1} + \left(\frac{5}{2} + \Delta_a^2 \right) X_{a2} + \Delta_a m_a R^2 V^{\zeta} X_V \right],$$

where the surface quantities $\Gamma_{a\theta}(\Psi)$ and $q_{a\theta}(\Psi)$ are obtained as integration constants.

Multiplying Eq. (B1) by $m_a v'_{\|}$ and $m_a v'_{\|}[m_a(v')^2/2T_a - 5/2]$, and integrating them in the velocity space gives the parallel momentum balance equations of $\mathcal{O}(\delta)$:

$$\begin{aligned} &\int d^3 v m_a (v'_{\|})^2 \mathbf{b} \cdot \nabla \overline{h_a} - n_a e_a B \frac{\langle B E_{\|}^{(A)} \rangle}{\langle B^2 \rangle} \\ &= \int d^3 v m_a v'_{\|} C_a^L(\overline{g_a}) \equiv F_{\|a1}, \\ &\int d^3 v m_a (v'_{\|})^2 \left(\frac{m_a (v')^2}{2T_a} - \frac{5}{2} \right) \mathbf{b} \cdot \nabla \overline{h_a} \\ &= \int d^3 v m_a v'_{\|} \left(\frac{m_a (v')^2}{2T_a} - \frac{5}{2} \right) C_a^L(\overline{g_a}) \equiv F_{\|a2}. \end{aligned} \quad \text{(B5)}$$

Using the charge neutrality $\Sigma_a n_a e_a = 0$ and the momentum conservation in collisions $\Sigma_a \mathbf{F}_{a1} = 0$ with Eq. (B5), we obtain

$$\sum_a \int d^3 v m_a (v'_{\|})^2 \mathbf{b} \cdot \nabla \overline{h_a} = 0. \quad \text{(B6)}$$

Equation (B6) expresses the balance of the total stresses in the rest frame of the plasma.

APPENDIX C: PARALLEL VISCOSITY COEFFICIENTS FOR THE PLATEAU REGIME

Here, we derive the parallel viscosity coefficients for the plateau regime for the large aspect ratio toroidal system [see Eqs. (65) and (66) for the banana regime] where $(R_0/r)^{3/2} \gg \omega_{Ta} \tau_{aa} \gg 1$ is satisfied. For that purpose, it is convenient to rewrite the $\mathcal{O}(\delta)$ distribution function $\overline{h_a}$ as

$$\overline{h_a} = \overline{h_a^{(l=1)}} + \overline{k_a}, \quad \text{(C1)}$$

where $\overline{h_a^{(l=1)}}$ is the $l=1$ component in the expansion by the Legendre polynomial $P_l(\eta)$ of $\eta \equiv v'_{\|}/v'$, which is written in the 13M approximation as

$$\overline{h_a^{(l=1)}} \equiv f_{a0} \frac{m_a v'_{\|}}{T_a} \frac{B}{n_a} \left[\Gamma_{a\theta} + \frac{2}{5} \frac{\overline{q_{a\theta}}}{T_a} \left(\frac{m_a (v')^2}{2T_a} - \frac{5}{2} \right) \right]. \quad \text{(C2)}$$

Then, let us divide $\overline{k_a}$ into the even (+) and odd (−) parts $\overline{k_a^{(+)}}$ and $\overline{k_a^{(-)}}$ with respect to the transformation $(v_{\|}, \theta) \rightarrow (-v_{\|}, -\theta)$: $\overline{k_a} = \overline{k_a^{(+)}} + \overline{k_a^{(-)}}$. It should be noted that only the odd part $\overline{k_a^{(-)}}$ contributes to the parallel viscosities $\langle \int d^3 v m_a (v'_{\|})^2 \mathbf{B} \cdot \nabla \overline{h_a} \rangle$ and $\langle \int d^3 v m_a (v'_{\|})^2 \{[m_a (v')^2/2T_a] - \frac{5}{2}\} \mathbf{B} \cdot \nabla \overline{h_a} \rangle$. From Eq. (C1), (C2), and (B1), we have the drift kinetic equation for $\overline{k_a^{(-)}}$ in the plateau regime as

$$\left(\eta \frac{\partial}{\partial \theta} - \frac{\overline{\nu_a}}{2} \frac{\partial^2}{\partial \eta^2} \right) \overline{k_a^{(-)}} = \frac{1}{2} \frac{r}{R_0} \sin\theta \left(1 + \frac{\Upsilon}{x^2} \right) f_{a0} \frac{m_a v'}{T_a} \frac{B_0}{n_a} \times \left[\Gamma_{a\theta} + \frac{2}{5} \frac{\overline{q_{a\theta}}}{T_a} \left(\frac{m_a (v')^2}{2T_a} - \frac{5}{2} \right) \right], \quad \text{(C3)}$$

which is solved to give

$$\begin{aligned} \overline{k_a^{(-)}} &= \frac{r}{R_0} \left(1 + \frac{\Upsilon}{x^2} \right) f_{a0} \frac{m_a v'}{2T_a} \frac{B_0}{n_a} \\ &\times \left[\Gamma_{a\theta} + \frac{2}{5} \frac{\overline{q_{a\theta}}}{T_a} \left(\frac{m_a (v')^2}{2T_a} - \frac{5}{2} \right) \right] \overline{\nu}_a^{-1/3} \\ &\times \int_0^\infty d\tau \sin(\theta - \overline{\nu}_a^{-1/3} \eta \tau) e^{-\tau^3/6}, \end{aligned} \quad \text{(C4)}$$

where $\overline{\nu_a} \equiv (\omega_{Ta} \tau_{aa})^{-1} \tau_{aa} \nu_D^a(x)/x$ $(x \equiv v'/v_{Ta})$. Then, by using Eq. (C4), we obtain the parallel viscosities as

$$\begin{bmatrix} \left\langle \int d^3 v m_a (v'_{\|})^2 \mathbf{B} \cdot \nabla \overline{h_a} \right\rangle \\ \left\langle \int d^3 v m_a (v'_{\|})^2 \left(\frac{m_a (v')^2}{2T_a} - \frac{5}{2} \right) \mathbf{B} \cdot \nabla \overline{h_a} \right\rangle \end{bmatrix} = \frac{\sqrt{\pi}}{2} m_a B_0^2 \left(\frac{r}{R_0} \right)^2 \frac{v_{Ta}}{q R_0} \begin{bmatrix} \hat{\mu}_{a1} & \hat{\mu}_{a2} \\ \hat{\mu}_{a2} & \hat{\mu}_{a3} \end{bmatrix} \begin{bmatrix} \Gamma_{a\theta} \\ \frac{2}{5} \overline{q_{a\theta}}/T_a \end{bmatrix}. \quad \text{(C5)}$$

Here the dimensionless coefficients $\hat{\mu}_{aj} (j=1,2,3)$ are defined by

$$\hat{\mu}_{aj} = \int_0^\infty dx\, x^5 e^{-x^2} \left(x^2 - \frac{5}{2} \right)^{j-1} \left(1 + \frac{\Upsilon}{x^2} \right)^2 \quad (a=e,i;\ j=1,2,3), \quad \text{(C6)}$$

which gives

$$\begin{aligned} \hat{\mu}_{a1} &= 1 + \Upsilon + \frac{1}{2} \Upsilon^2, \\ \hat{\mu}_{a2} &= \frac{1}{2} - \frac{1}{2} \Upsilon - \frac{3}{4} \Upsilon^2, \\ \hat{\mu}_{a3} &= \frac{13}{4} + \frac{9}{4} \Upsilon + \frac{13}{8} \Upsilon^2. \end{aligned} \quad \text{(C7)}$$

Noting that Eqs. (54)–(57) are still valid for the plateau regime and using them with Eqs. (C5)–(C7), we can express the parallel viscosities $\langle \int d^3 v m_a (v'_{\|})^2 \mathbf{B} \cdot \nabla \overline{h_a} \rangle$, $\langle \int d^3 v m_a (v'_{\|})^2 \{[m_a (v')^2/2T_a] - \frac{5}{2}\} \mathbf{B} \cdot \nabla \overline{h_a} \rangle$ $(a=e,i)$ and the parallel current $J_E = e \langle n_e (u_{\|i} - u_{\|e}) \rangle / \langle B^2 \rangle^{1/2}$ in the linear forms of the thermodynamic forces $(X_{e1}^*, X_{e2}, X_{i2}, X_V, X_E)$. Accordingly, we can obtain all the transport coefficients in Eq. (29) for the plateau regime except for L_{VV}, which are immediately given from Eqs. (71)–(77), (79), and (80) by replacing the banana parallel viscosities $1.469 m_a B_0^2 (r/R_0)^{1/2} \tau_{aa}^{-1} \hat{\mu}_{aj}$ $(a = e,i;\ j = 1,2,3)$ [see Eqs.

(65)–(67)] in them with the plateau viscosities $(\sqrt{\pi}/2) m_a B_0^2 (r/R_0)^2 (v_{Ta}/qR_0) \hat{\mu}_{aj}$ $(a=e,i;\ j=1,2,3)$ [see Eqs. (C5)–(C7)].

[1] ASDEX Team, Nucl. Fusion **29**, 1959 (1989).

[2] F. M. Levinton, M. C. Zarnstorff, S. H. Batha, M. Bell, R. E. Bell, R. V. Budny, C. Bush, Z. Chang, E. Fredrickson, A. Janos, J. Manickam, A. Ramsey, S. A. Sabbagh, G. L. Shmidt, E. J. Synakowski, and G. Taylor, Phys. Rev. Lett. **75**, 4417 (1995).

[3] H. Ninomiya and the JT-60 Team, Phys. Fluids B **4**, 2070 (1992).

[4] Y. Koide, T. Takizuka, S. Takeji, S. Ishida, M. Kikuchi, Y. Kamada, T. Ozeki, Y. Neyatani, H. Shirai, M. Mori, and S. Tsuji-Iio, Plasma Phys. Controlled Fusion **38**, 1011 (1996).

[5] F. L. Hinton and R. D. Hazeltine, Rev. Mod. Phys. **42**, 239 (1976).

[6] S. P. Hirshman and D. J. Sigmar, Nucl. Fusion **21**, 1079 (1981).

[7] R. Balescu, *Transport Processes in Plasmas* (2 vols.) (North-Holland, Amsterdam, 1988).

[8] F. L. Hinton and S. K. Wong, Phys. Fluids **28**, 3082 (1985).

[9] P. J. Catto, I. B. Bernstein, and M. Tessarotto, Phys. Fluids **30**, 2784 (1987).

[10] H. Sugama and W. Horton, Phys. Plasmas **4**, 405 (1997).

[11] H. Sugama and W. Horton, Phys. Plasmas **3**, 304 (1996).

[12] A. A. Ware, Phys. Rev. Lett. **25**, 15 (1970).

[13] K. Nagashima, Y. Koide, and H. Shirai, Nucl. Fusion **34**, 449 (1994).