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How to calculate the neoclassical viscosity, diffusion, and current coefficients in general toroidal plasmas

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A novel method to obtain the full neoclassical transport matrix for general toroidal plasmas by using the solution of the linearized drift kinetic equation with the pitch-angle-scattering collision operator is shown. In this method, the neoclassical coefficients for both poloidal and toroidal viscosities in toroidal helical systems can be obtained, and the neoclassical transport coefficients for the radial particle and heat fluxes and the bootstrap current with the nondiagonal coupling between unlike-species particles are derived from combining the viscosity-flow relations, the friction-flow relations, and the parallel momentum balance equations. Since the collisional momentum conservation is properly retained, the well-known intrinsic ambipolar condition of the neoclassical particle fluxes in symmetric systems is recovered. Thus, these resultant neoclassical diffusion and viscosity coefficients are applicable to evaluating accurately how the neoclassical transport in quasi-symmetric toroidal systems deviates from that in exactly symmetric systems. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1512917]

I. INTRODUCTION

Neoclassical transport theory[1–3] describes diffusion processes caused by binary Coulomb collisions between charged particles in magnetically confined plasmas. In most fusion plasma experiments, observed particle and heat fluxes across magnetic surfaces are dominated not by neoclassical transport but by turbulent or anomalous transport,[4] although the neoclassical transport theory is still useful for predicting transport fluxes tangential to magnetic surfaces such as poloidal and toroidal flows and bootstrap currents. Especially for nonaxisymmetric systems, neoclassical analyses are important because neoclassical transport fluxes due to particles trapped in helical ripples[5–8] are expected to be significantly large for high temperature and play a key role in determining the radial electric field under the ambipolar-diffusion condition.[9] Recently, quasi-symmetric toroidal systems such as quasi-axisymmetric systems are attracting much attention as an advanced concept of helical devices, in which the neoclassical ripple transport and the neoclassical viscosity against flows in the direction of symmetry are nearly suppressed by optimizing the helical configuration so as to make the magnetic field strength independent of a certain symmetry coordinate.[10–12] Thus, there are many demands for accurate and fast calculation of neoclassical quantities including the particle and heat diffusivities, the bootstrap-current coefficients, and the viscosity coefficients for the poloidal and toroidal flows.

The neoclassical transport coefficients are obtained from solution of the drift kinetic equation.[13,14] Because of complexity of the magnetic geometry, calculation of the neoclassical transport in helical systems often employs numerical methods.[15–23] The Drift Kinetic Equation Solver (DKES)[19,20] is one of powerful numerical codes to directly solve the drift kinetic equation. However, we should note that, even in such numerical calculations, approximated collision operators such as the pitch-angle-scattering (or Lorentz) collision model are generally used instead of the full Landau collision term.[24] By using this collision model, perturbed distribution functions of unlike species and of different kinetic energies can be solved independently, and therefore the neoclassical transport coefficients can quickly be calculated. However, since such simple collision models neglect the field particle collision part and break the collisional momentum conservation, the resultant transport coefficients neither contain the nondiagonal part connecting fluxes and forces of unlike species, nor recover the well-known intrinsic ambipolarity of the radial particle fluxes in the symmetric limit.[1–3] These errors seem to be a serious problem, especially when using the numerical results to show how the neoclassical transport in designed quasi-symmetric configurations differ from that in exactly symmetric systems. In the present work, it is shown how to obtain the neoclassical transport coefficients in general toroidal systems including the coupling effects between unlike-species particles as well as the collisional momentum conservation.

Here, we follow the basic idea of the moment method by Hirshman and Sigmar[2] that, in order to derive the neoclassical transport coefficients, the fluid momentum balance equations and the friction-flow relations, in which the collisional momentum conservation is already taken into account, are used together with the viscosity-flow relations obtained from the solution of the drift kinetic equation. Since the test particle portion of the collision operator dominates over the field

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particle portion for the $l=2$ spherical harmonic perturbations of the distribution functions,[2] it is more accurate to use the solution of the drift kinetic equation with the pitch-angle-scattering collision model for derivation of the neoclassical viscosity coefficients than for direct calculation of the neoclassical particle and thermal diffusivities, which are significantly influenced by neglect of the field particle portion. In tokamaks, the neoclassical toroidal viscosity vanishes due to the axisymmetry, and analytical expressions of the viscosity-flow relations are obtained for any collisionality in the Pfirsch–Schlüter, plateau, and banana regimes.[2,3] Analytical calculations of the the parallel viscosity coefficient in finite-aspect-ratio tokamaks are shown to be in good agreement with numerical results.[25,26] In general toroidal systems with no symmetry, we need to calculate viscosities in both poloidal and toroidal directions, and these viscosity coefficients are analytically derived for the Pfirsch–Schlüter and plateau regimes.[27,28] However, for the banana regime, analytical formulas are given only for the parallel viscosities.[5,29–31] In order to accurately calculate both poloidal and toroidal viscosity coefficients in toroidal helical systems for low-collisionality regimes, we need to make use of numerical solution of the drift kinetic solution as effectively as possible, and the present paper shows how to do that.

Taguchi also showed another method to calculate the neoclassical transport coefficients in nonaxisymmetric multispecies plasmas.[32] He ingeniously used a momentum conserving collision operator and its self-adjoint property to derive the particle and heat diffusivities and the bootstrap current coefficient. In addition to these transport coefficients, our method gives a useful recipe to obtain the neoclassical viscosity coefficients, which play an important role in determining plasma rotation profiles. Since our work follows a line of the moment method, it is more transparently connected or applicable to past theoretical studies of neoclassical transport in nonaxisymmetric systems[5,27–31,33] which are also based on the moment method. Furthermore, in the present study, the validity of our procedures is satisfactorily verified by numerical examples, in which our results are compared with analytical formulas on the parallel viscosity, the ripple transport coefficient, and the geometrical factor of the bootstrap current in various collision frequency regimes. In this sense, our work is a generalization of previous comparative studies between numerical and analytical evaluations of neoclassical coefficients of viscosities and other fluxes in tokamaks[25,26] to the case of nonaxisymmetric systems.

The rest of this work is organized as follows. In Sec. II, we derive the linearized drift kinetic equation for the distribution functions with $l=1$ spherical harmonic part subtracted, based on which two types of conjugate pairs of neoclassical fluxes and forces are specified. When we take the parallel flows and the radial gradients as thermodynamic forces, the parallel viscosities and the radial transport are regarded as fluxes conjugate to those forces. We can also consider the poloidal and toroidal viscosities to be fluxes conjugate to the poloidal and toroidal flows as forces. The inner product of these fluxes and forces represents the entropy production rate associated with the neoclassical transport processes.[33,34] In Sec. III, it is shown that the conjugate pairs of fluxes and forces defined in Sec. II are related to each other by the Onsager-symmetric matrices.[35] The poloidal and toroidal viscosity coefficients are included as elements in one of the matrices. We find how to calculate these matrices by using their relations to the monoenergetic diffusion tensor obtained as an output of commonly used numerical codes such as the DKES. Once these Onsager-symmetric matrices are derived, all neoclassical transport coefficients for the radial particle and heat fluxes and the bootstrap current are immediately obtained. In Sec. IV, numerical examples of these procedures are shown and compared with several analytical predictions. Conclusions are given in Sec. V. For readers' convenience, useful formulas and relations for the Boozer[36] and Hamada[37] coordinates, the poloidal and toroidal viscosity coefficients, and other neoclassical transport coefficients are written in Appendices A, B, and C, respectively. Also, the case of symmetric systems is described in Appendix D. Finally, Appendix E shows how to treat effects of the $\mathbf{E}\times\mathbf{B}$ drift on the neoclassical transport coefficients.

II. CONJUGATE PAIRS OF NEOCLASSICAL FLUXES AND FORCES

In general toroidal configurations, the magnetic field is written in terms of the flux coordinates (s,θ,ζ) as

$$\mathbf{B}=\psi'\nabla s\times\nabla\theta+\chi'\nabla\zeta\times\nabla s=B_s\nabla s+B_\theta\nabla\theta+B_\zeta\nabla\zeta, \tag{1}$$

where θ and ζ represent the poloidal and toroidal angles, respectively, s is an arbitrary label of a flux surface. The poloidal and toroidal fluxes are given by $2\pi\chi(s)=(2\pi)^{-1}\int_{V(s)}d^3x\mathbf{B}\cdot\nabla\theta$ and $2\pi\psi(s)=(2\pi)^{-1}\int_{V(s)}d^3x\mathbf{B}\cdot\nabla\zeta$, respectively, where $V(s)$ is the volume enclosed by the flux surface with the label s. The derivative with respect to s is denoted by $'=d/ds$ so that $\psi'=d\psi/ds$ and $\chi'=d\chi/ds$. The covariant radial, poloidal, and toroidal components of the magnetic field $\mathbf{B}$ are written as $B_s\equiv\mathbf{B}\cdot\partial\mathbf{x}/\partial s\equiv\sqrt{g}\mathbf{B}\cdot(\nabla\theta\times\nabla\zeta)$, $B_\theta\equiv\mathbf{B}\cdot\partial\mathbf{x}/\partial\theta\equiv\sqrt{g}\mathbf{B}\cdot(\nabla\zeta\times\nabla s)$, and $B_\zeta\equiv\mathbf{B}\cdot\partial\mathbf{x}/\partial\zeta\equiv\sqrt{g}\mathbf{B}\cdot(\nabla s\times\nabla\theta)$, respectively, where $\sqrt{g}\equiv[\nabla s\cdot(\nabla\theta\times\nabla\zeta)]^{-1}$ represents the Jacobian for the coordinates (s,θ,ζ). Here, we may regard (s,θ,ζ) as either Boozer,[36] Hamada[37] coordinates, or arbitrary other flux coordinates. Useful formulas for the Boozer and Hamada coordinates are written in Appendix A, where it is also shown that the symmetry condition for the magnetic field strength in the Boozer coordinates is equivalent to that in the Hamada coordinates.

The distribution function for the particle species a with the mass m_a and the charge e_a is written as

$$f_a=f_{aM}\left[1+\frac{e_a}{T_a}\int^l\frac{dl}{B}\left(BE_\parallel-\frac{B^2}{\langle B^2\rangle}\langle BE_\parallel\rangle\right)\right]+f_{a1}, \tag{2}$$

where the local Maxwellian distribution function is represented by $f_{aM}\equiv\pi^{-3/2}n_a v_{Ta}^{-3}\exp(-x_a^2)$ with the equilibrium density n_a, the temperature T_a, the thermal velocity $v_{Ta}\equiv(2T_a/m_a)^{1/2}$, and the normalized velocity $x_a\equiv v/v_{Ta}$. Here, $E_\parallel\equiv\mathbf{E}\cdot\mathbf{b}$ ($\mathbf{b}\equiv\mathbf{B}/B$: the unit vector tangential to the magnetic field) is the parallel electric field, $\int^l dl$ denotes the integral along the magnetic field line, and

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$\langle\cdot\rangle\equiv\oint d\theta\oint d\zeta\sqrt{g}\cdot/V'$ with $V'\equiv\oint d\theta\oint d\zeta\sqrt{g}$ represents the flux surface average. The neoclassical transport is caused by the deviation f_{a1} from the local Maxwellian. We should note that the drift kinetic theory is concerned with the gyrophase-averaged part of the distribution function and that f_{a1} is regarded as a gyrophase-averaged function in the present work.

The linearized drift kinetic equation is given by

$$V_\parallel f_{a1}-C_a^L(f_{a1})=-\mathbf{v}_{da}\cdot\nabla f_{aM}+\frac{e_a}{T_a}v_\parallel B\frac{\langle BE_\parallel\rangle}{\langle B^2\rangle}f_{aM}, \quad (3)$$

where the operator $V_\parallel\equiv v_\parallel\mathbf{b}\cdot\nabla$ and the guiding center drift velocity $\mathbf{v}_{da}=(c/e_aB)\mathbf{b}\times(m_av_\parallel^2\mathbf{b}\cdot\nabla\mathbf{b}+\mu\nabla B+e_a\nabla\Phi)$ are used, and f_{a1} and f_{aM} are regarded as functions of the phase-space variables $(\mathbf{x},E,\mu)$ ($\mathbf{x}$: the particle's position, $E\equiv\frac{1}{2}m_av^2+e_a\Phi$: The particle's energy; $\mu\equiv m_av_\perp 2/2B$: the magnetic moment). Here, the linearized collision operator C_a^L is defined by[2]

$$C_a^L(f_{a1})=\sum_b\left[C_{ab}(f_{a1},f_{bM})+C_{ab}(f_{aM},f_{b1})\right], \quad (4)$$

where C_{ab} represents the Landau collision operator for collisions between the species a and b.

Hereafter, we use $(\mathbf{x},v,\xi)$ $(\xi\equiv v_\parallel/v)$ as the phase-space variables instead of $(\mathbf{x},E,\mu)$. Then, the collisionless orbit operator $V_\parallel$ is represented by

$$V_\parallel=v\xi\mathbf{b}\cdot\nabla-\frac{1}{2}v(1-\xi^2)(\mathbf{b}\cdot\nabla\ln B)\frac{\partial}{\partial\xi}, \quad (5)$$

where the second term in the right-hand side is related to the magnetic mirror force. The $\mathbf{E}\times\mathbf{B}$ and magnetic drifts are not included in $V_\parallel$. A more general case including the $\mathbf{E}\times\mathbf{B}$ drift operator is treated in Appendix E. Let us consider the expansion of an arbitrary function $F(\mathbf{x},v,\xi)$ by the Legendre polynomials $P_l(\xi)$ $[P_0(\xi)=1,P_1(\xi)=\xi,P_2(\xi)=\frac{3}{2}\xi^2-\frac{1}{2},\ldots]$ as

$$F(\mathbf{x},v,\xi)=\sum_{l=0}^{\infty}F^{(l)}(\mathbf{x},v,\xi),$$

$$F^{(l)}(\mathbf{x},v,\xi)=P_l(\xi)\frac{2l+1}{2}\int_{-1}^{1}d\eta P_l(\eta)F(\mathbf{x},v,\eta). \quad (6)$$

The $l=1$ Legendre component $f_{a1}^{(l=1)}$ of the distribution function f_{a1} is associated with the parallel flows and is expanded by the Laguerre polynomials $L_j^{(3/2)}(x_a^2)$ $[L_0^{(3/2)}(x_a^2)=1,L_1^{(3/2)}(x_a^2)=\frac{5}{2}-x_a^2,\ldots]$ as

$$f_{a1}^{(l=1)}\equiv(v_\parallel/v)\frac{3}{2}\int_{-1}^{1}d(v_\parallel/v)(v_\parallel/v)f_{a1}$$
$$=\frac{2}{v_{Ta}}\xi x_a\left[u_{\parallel a}+\left(x_a^2-\frac{5}{2}\right)\frac{2}{5}\frac{q_{\parallel a}}{p_a}\right]f_{aM}+f_{a1}^{(l=1,j\geqslant 2)}, \quad (7)$$

where the coefficients of the first and second Laguerre polynomial components are given in terms of the parallel velocity $u_{\parallel a}\equiv n_a^{-1}\int d^3v\ f_{a1}v_\parallel$ and the parallel heat flow $q_{\parallel a}\equiv T_a\int d^3v\ f_{a1}v_\parallel(x_a^2-\frac{5}{2})$, respectively, and $f_{a1}^{(l=1,j\geqslant 2)}$ denotes the sum of the jth Laguerre polynomial components with $j\geqslant 2$. Integrating Eq. (3) multiplied by 1 and $\frac{1}{2}m_av^2$ over the velocity space, we obtain the incompressibility conditions

$$\nabla\cdot\mathbf{u}_a=\nabla\cdot\mathbf{q}_a=0, \quad (8)$$

where $\mathbf{u}_a=u_{\parallel a}\mathbf{b}+\mathbf{u}_{\perp a}$ and $\mathbf{q}_a=q_{\parallel a}\mathbf{b}+\mathbf{q}_{\perp a}$ with the diamagnetic perpendicular flows

$$\mathbf{u}_{\perp a}=\frac{cX_{a1}}{e_aB}\nabla s\times\mathbf{b},$$
$$\frac{\mathbf{q}_{\perp a}}{p_a}=\frac{5}{2}\frac{cX_{a2}}{e_aB}\nabla s\times\mathbf{b}. \quad (9)$$

Here, the thermodynamic forces X_{a1} and X_{a2} are defined by

$$X_{a1}\equiv-\frac{1}{n_a}\frac{\partial p_a}{\partial s}-e_a\frac{\partial\Phi}{\partial s}, \quad X_{a2}\equiv-\frac{\partial T_a}{\partial s}, \quad (10)$$

respectively, where the pressure $p_a\equiv n_aT_a$, the temperature T_a, and the electrostatic potential Φ are flux surface functions independent of θ and ζ. (Exactly speaking, $\nabla\cdot\mathbf{q}_a=0$ is valid to the lowest order of the small mass ratio $m_e/m_i\ll 1$.) Integrating the incompressibility conditions in Eq. (8) gives the local parallel flows as

$$u_{\parallel a}=\frac{\langle u_{\parallel a}B\rangle}{\langle B^2\rangle}B+\frac{cX_{a1}}{e_a}\tilde{U},$$
$$\frac{2}{5p_a}q_{\parallel a}=\frac{2}{5p_a}\frac{\langle q_{\parallel a}B\rangle}{\langle B^2\rangle}B+\frac{cX_{a2}}{e_a}\tilde{U}, \quad (11)$$

where $\tilde{U}$ is given as a solution of

$$\mathbf{B}\cdot\nabla\left(\frac{\tilde{U}}{B}\right)=\mathbf{B}\times\nabla s\cdot\nabla\left(\frac{1}{B^2}\right), \quad \langle B\tilde{U}\rangle=0. \quad (12)$$

As shown later in Eq. (20), $\tilde{U}$ is associated with the Pfirsch–Schlüter fluxes and its specific expressions are written in Eqs. (A4) and (A8).

Now, let us define g_a by

$$g_a=f_{a1}-f_{a1}^{(l=1)}. \quad (13)$$

The neoclassical viscosities which we are concerned with are derived from the $l=2$ component included in g_a. Substituting Eq. (13) into Eq. (3), we obtain

$$V_\parallel g_a-C_a^L(g_a)=H_a^{(l=1)}+H_a^{(l=2)}, \quad (14)$$

where the $l=1$ and $l=2$ Legendre component terms in the right-hand side are written as

$$H_a^{(l=1)}=\frac{e_a}{T_a}v_\parallel B\frac{\langle BE_\parallel\rangle}{\langle B^2\rangle}f_{aM}+C_a^L(f_{a1}^{(l=1)}) \quad (15)$$

and

$$H_a^{(l=2)}=\frac{f_{aM}}{T_a}\left[\sigma_{Ua}\left\{\frac{\langle u_{\parallel a}B\rangle}{\langle B^2\rangle}+\frac{2}{5p_a}\frac{\langle q_{\parallel a}B\rangle}{\langle B^2\rangle}\left(x_a^2-\frac{5}{2}\right)\right\}+\sigma_{Xa}\left\{X_{a1}+X_{a2}\left(x_a^2-\frac{5}{2}\right)\right\}\right]$$
$$=\frac{f_{aM}}{T_a}\frac{V'}{4\pi^2}\left[\frac{\sigma_{Pa}}{\chi'}\left\{\langle u_a^\theta\rangle+\frac{2}{5p_a}\langle q_a^\theta\rangle\left(x_a^2-\frac{5}{2}\right)\right\}+\frac{\sigma_{Ta}}{\psi'}\left\{\langle u_a^\zeta\rangle+\frac{2}{5p_a}\langle q_a^\zeta\rangle\left(x_a^2-\frac{5}{2}\right)\right\}\right], \quad (16)$$

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respectively. In deriving Eqs. (14)–(16), Eq. (11) is used and $f_{a1}^{(l=1,j\geqslant 2)}$ in Eq. (7) is neglected by employing the thirteen-moment (13M) approximation.[3] By including the high-order parallel flow variables corresponding to the $j=2,3,\ldots$ Laguerre polynomial components of $f_{a1}^{(l=1)}$, the formulation presented in this work can be extended straightforwardly to the cases with the higher-order (21M, 29M,...) approximations.[3] In Eq. (16), $u_a^\theta = \mathbf{u}_a\cdot\nabla\theta$ ($u_a^\zeta=\mathbf{u}_a\cdot\nabla\zeta$) and $q_a^\theta=\mathbf{q}_a\cdot\nabla\theta$ ($q_a^\zeta=\mathbf{q}_a\cdot\nabla\zeta$) are contravariant poloidal (toroidal) components of the flows, and σ_{Ua}, σ_{Xa}, σ_{Pa}, and σ_{Ta} are defined by

$$\sigma_{Ua}=-m_a v^2 P_2(\xi)\mathbf{B}\cdot\nabla\ln B=-V_\parallel(m_a v\xi B),$$
$$\sigma_{Xa}=-v^2P_2(\xi)\frac{B}{\Omega_a}\left(\tilde{U}\mathbf{b}+\frac{\nabla s\times\mathbf{b}}{B}\right)\cdot\nabla\ln B$$
$$=-v^2P_2(\xi)\frac{\mathbf{b}\cdot\nabla(B\tilde{U})}{2\Omega_a},$$
$$\sigma_{Pa}=-m_a v^2P_2(\xi)\mathbf{B}_P\cdot\nabla\ln B, \tag{17}$$
$$\sigma_{Ta}=-m_a v^2P_2(\xi)\mathbf{B}_T\cdot\nabla\ln B,$$

respectively, where $\mathbf{B}_P\equiv\chi'\nabla\zeta_H\times\nabla s$ and $\mathbf{B}_T\equiv\psi'\nabla s\times\nabla\theta_H$ are the poloidal and toroidal magnetic fields, respectively, represented by the Hamada coordinates (s,θ_H,ζ_H) (the subscript H is added to the angle variables whenever the Hamada coordinates should be used) and $\Omega_a\equiv e_aB/(m_ac)$ is the gyrofrequency. Then, we find that the parallel, poloidal, and toroidal neoclassical viscosities are written in terms of σ_{Ua}, σ_{Pa}, and σ_{Ta} as

$$\langle\mathbf{B}\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle=\left\langle\int d^3v\,g_a\sigma_{Ua}\right\rangle,$$
$$\langle\mathbf{B}\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle=\left\langle\int d^3v\,g_a\sigma_{Ua}\left(x_a^2-\frac{5}{2}\right)\right\rangle,$$
$$\langle\mathbf{B}_P\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle=\left\langle\int d^3v\,g_a\sigma_{Pa}\right\rangle,$$
$$\langle\mathbf{B}_P\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle=\left\langle\int d^3v\,g_a\sigma_{Pa}\left(x_a^2-\frac{5}{2}\right)\right\rangle, \tag{18}$$
$$\langle\mathbf{B}_T\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle=\left\langle\int d^3v\,g_a\sigma_{Ta}\right\rangle,$$
$$\langle\mathbf{B}_T\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle=\left\langle\int d^3v\,g_a\sigma_{Ta}\left(x_a^2-\frac{5}{2}\right)\right\rangle,$$

where $\boldsymbol{\pi}_a\equiv\int d^3v\,m_a(v_\parallel^2-\frac{1}{2}v_\perp^2)f_{a1}(\mathbf{bb}-\frac{1}{3}\mathbf{I})$ and $\boldsymbol{\Theta}_a\equiv\int d^3v\,m_a(v_\parallel^2-\frac{1}{2}v_\perp^2)(x_a^2-\frac{5}{2})f_{a1}(\mathbf{bb}-\frac{1}{3}\mathbf{I})$. We also note that the neoclassical radial particle and heat fluxes are written in terms of σ_{Xa} as

$$\Gamma_a=\left\langle\int d^3v\,g_a\mathbf{v}_{da}\cdot\nabla s\right\rangle=\left\langle\int d^3v\,g_a\sigma_{Xa}\right\rangle+\Gamma_a^{\rm PS},$$
$$\frac{q_a}{T_a}=\left\langle\int d^3v\,g_a\mathbf{v}_{da}\cdot\nabla s\left(x_a^2-\frac{5}{2}\right)\right\rangle$$
$$=\left\langle\int d^3v\,g_a\sigma_{Xa}\left(x_a^2-\frac{5}{2}\right)\right\rangle+\frac{q_a^{\rm PS}}{T_a}, \tag{19}$$

where $\Gamma_a^{\rm PS}$ and $q_a^{\rm PS}$ are the Pfirsch–Schlüter (PS) radial particle and heat fluxes defined by

$$\Gamma_a^{\rm PS}=-\frac{c}{e_a}\langle\tilde{U}F_{\parallel a1}\rangle,\quad \frac{q_a^{\rm PS}}{T_a}=-\frac{c}{e_a}\langle\tilde{U}F_{\parallel a2}\rangle, \tag{20}$$

respectively, with the parallel friction forces

$$F_{\parallel a1}=\int d^3v\,m_av_\parallel C_a^L(f_{a1}),$$
$$F_{\parallel a2}=\int d^3v\,m_av_\parallel\left(x_a^2-\frac{5}{2}\right)C_a^L(f_{a1}). \tag{21}$$

As shown in Refs. 29 and 33, $\langle\int d^3v\,g_a\sigma_{Xa}\rangle=\Gamma_a-\Gamma_a^{\rm PS}\equiv\Gamma_a^{\rm bn}$ and $T_a\langle\int d^3v\,g_a\sigma_{Xa}(x_a^2-5/2)\rangle=q_a-q_a^{\rm PS}\equiv q_a^{\rm bn}$ can be written as the sum of banana-plateau and nonaxisymmetric parts. Multiplying Eq. (14) by g_a/f_{aM}, integrating it in the velocity space, taking its flux surface average, and using Eqs. (16), (18), and (19), we can express the flux-surface-averaged entropy production rate[33,34] $\dot{S}_a$ associated with g_a per unit volume by the inner product of conjugate pairs of fluxes and forces as

$$T_a\dot{S}_a=-T_a\left\langle\int d^3v\frac{g_a}{f_{aM}}C_a^L(g_a)\right\rangle$$
$$=\langle\mathbf{B}\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle\frac{\langle u_{\parallel a}B\rangle}{\langle B^2\rangle}+\langle\mathbf{B}\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle\frac{2}{5p_a}$$
$$\times\frac{\langle q_{\parallel a}B\rangle}{\langle B^2\rangle}+\Gamma_a^{\rm bn}X_{a1}+\frac{q_a^{\rm bn}}{T_a}X_{a2}$$
$$=\frac{V'}{4\pi^2}\left[\langle\mathbf{B}_P\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle\frac{\langle u_a^\theta\rangle}{\chi'}\right.$$
$$+\langle\mathbf{B}_P\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle\frac{2}{5p_a}\frac{\langle q_a^\theta\rangle}{\chi'}+\langle\mathbf{B}_T\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle\frac{\langle u_a^\zeta\rangle}{\psi'}$$
$$\left.+\langle\mathbf{B}_T\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle\frac{2}{5p_a}\frac{\langle q_a^\zeta\rangle}{\psi'}\right]. \tag{22}$$

We find from Eq. (22) that the parallel viscosities $\langle\mathbf{B}\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle$ and $\langle\mathbf{B}\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle$ are transport fluxes conjugate to the parallel flows $\langle u_{\parallel a}B\rangle/\langle B^2\rangle$ and $(2/5p_a)\langle q_{\parallel a}B\rangle/\langle B^2\rangle$ as forces, respectively, and that the radial neoclassical fluxes $\Gamma_a^{\rm bn}$ and $q_a^{\rm bn}/T_a$ are conjugate to the radial gradient forces X_{a1} and X_{a2}, respectively. Also, as another choice, the poloidal viscosities $[\langle\mathbf{B}_P\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle,\langle\mathbf{B}_P\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle]$ and the toroidal viscosities $[\langle\mathbf{B}_T\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle,\langle\mathbf{B}_T\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle]$ can be regarded as transport fluxes conjugate to the poloidal flows $[\langle u_a^\theta\rangle/\chi',(2/5p_a)\langle q_a^\theta\rangle/\chi']$ and the toroidal flows $[\langle u_a^\zeta\rangle/\psi',(2/5p_a)\langle q_a^\zeta\rangle/\psi']$, respectively. Now, our main concern is how to obtain the transport matrices which connect these conjugate pairs of fluxes and forces. This is explained in the next section.

Here, we consider the $l=1$ Legendre component term $H_a^{(l=1)}$, which also makes a significant contribution to the solution g_a of Eq. (14) especially in the weakly collisional regime in order to insure $\int_{-1}^{+1}g_a d\xi=0$. Substituting Eq. (7) into Eq. (15) and using Eq. (11) and the rotational symmetry of the collision operator C_a^L, we can write $H_a^{(l=1)}$ in the following form:

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$$H_a^{(l=1)} = f_{aM} \frac{m_a \nu_D^a}{T_a} v \xi (B\alpha_a + \tilde{U}\gamma_a), \tag{23}$$

where α_a and γ_a are functions of (s,v) and are independent of (θ,ζ,ξ). We find in the next section that α_a is written in terms of the parallel flows and the radial gradient forces and that γ_a is unnecessary for calculation of neoclassical transport coefficients.

III. RESPONSE FUNCTIONS AND TRANSPORT COEFFICIENTS

Hereafter, as an approximation of the linearized collision operator in Eq. (14), we use the pitch-angle-scattering operator defined by

$$C_a^{\rm PAS} \equiv \frac{\nu_D^a}{2} \frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial}{\partial \xi}, \tag{24}$$

with the energy-dependent collision frequency ν_D^a given by[2]

$$\nu_D^a \equiv \sum_b \frac{3\sqrt{\pi}}{4} \tau_{ab}^{-1} x_a^{-3} \mathcal{H}(x_b), \tag{25}$$

where $(3\sqrt{\pi}/4)\tau_{ab}^{-1} \equiv 4\pi n_b e_a^2 e_b^2 \ln\Lambda/(m_a^2 v_{Ta}^3)$ ($\ln\Lambda$: The Coulomb logarithm) and $\mathcal{H}(x) \equiv [(2x^2-1)\Phi(x) + x\Phi'(x)]/(2x^2)$ [$\Phi(x) \equiv 2\pi^{-1/2}\int_0^x \exp(-t^2)dt$: The error function]. The use of $C_a^{\rm PAS}$ in Eq. (14) is considered to be a better approximation than that in Eq. (3) from the viewpoint of the momentum conservation because generally we have $\int d^3v\, m_a v_\parallel C_a^L(f_{a1}) \neq \int d^3v\, m_a v_\parallel C_a^{\rm PAS}(f_{a1})$ but $\int d^3v\, m_a v_\parallel C_a^L(f_{a1}-f_{a1}^{(l=1)}) = 0 = \int d^3v\, m_a v_\parallel C_a^{\rm PAS}(f_{a1}-f_{a1}^{(l=1)})$. Then, the formal solution of Eq. (14) with the source terms given by Eqs. (16) and (23) is written as

$$g_a = \frac{f_{aM}}{T_a}\left[G_{Ua}\left\{ \frac{\langle u_{\parallel a}B\rangle}{\langle B^2\rangle} + \frac{2}{5p_a}\frac{\langle q_{\parallel a}B\rangle}{\langle B^2\rangle}\left(x_a^2 - \frac{5}{2}\right)\right\} + G_{Xa}\left\{ X_{a1} + X_{a2}\left(x_a^2-\frac{5}{2}\right)\right\} + \alpha_a(G_{Ua} + m_a v\xi B) + m_a \nu_D^a \gamma_a \int^l \tilde{U} dl \right], \tag{26}$$

where G_{Ua} and G_{Xa} are defined as solutions of

$$(V_\parallel - C_a^{\rm PAS})\begin{bmatrix} G_{Ua} \\ G_{Xa}\end{bmatrix} = \begin{bmatrix} \sigma_{Ua} \\ \sigma_{Xa}\end{bmatrix}. \tag{27}$$

In deriving Eq. (26), we have also used the relation

$$(V_\parallel - C_a^{\rm PAS})(G_{Ua} + m_a v\xi B) = m_a \nu_D^a v \xi B. \tag{28}$$

We can easily find that the last term including γ_a in the right-hand side of Eq. (26), which is independent of ξ, makes no contribution to flows, viscosities, and radial transport fluxes.

Here, we define the inner product (F,G) for arbitrary functions $F(\theta,\zeta,\xi)$ and $G(\theta,\zeta,\xi)$ by

$$(F,G) \equiv \tfrac{1}{2}\int_{-1}^{1} d\xi \langle FG\rangle, \tag{29}$$

where $\langle\cdot\rangle$ denotes the flux surface average. With respect to this inner product, the operators $V_\parallel$ and $C_a^{\rm PAS}$ are found to be antisymmetric and symmetric, respectively,

$$(V_\parallel F, G) = -(F, V_\parallel G), \quad (C_a^{\rm PAS}F, G) = (F, C_a^{\rm PAS}G). \tag{30}$$

Then, substituting Eq. (26) into the relation $(v\xi B, C_a^{\rm PAS} g_a) = 0$, which is derived from the definition of g_a in Eq. (13), we can represent α_a by a linear combination of the parallel flows and the radial gradient forces as

$$\alpha_a = \frac{1}{\frac{2}{3} m_a T_a \nu_D^a x_a^2 \langle B^2\rangle - (\sigma_{Ua}, G_{Ua})}\left[(\sigma_{Ua}, G_{Ua}) \times \left\{ \frac{\langle u_{\parallel a}B\rangle}{\langle B^2\rangle} + \frac{2}{5p_a}\frac{\langle q_{\parallel a}B\rangle}{\langle B^2\rangle}\left(x_a^2-\frac{5}{2}\right)\right\} + (\sigma_{Ua}, G_{Xa}) \times \left\{ X_{a1} + X_{a2}\left(x_a^2 - \frac{5}{2}\right)\right\}\right]. \tag{31}$$

Substituting again Eq. (31) into Eq. (26), g_a is rewritten as

$$g_a = \frac{f_{aM}}{T_a}\left[\mathcal{G}_{Ua}\left\{ \frac{\langle u_{\parallel a}B\rangle}{\langle B^2\rangle} + \frac{2}{5p_a}\frac{\langle q_{\parallel a}B\rangle}{\langle B^2\rangle}\left(x_a^2-\frac{5}{2}\right)\right\} + \mathcal{G}_{Xa}\left\{ X_{a1} + X_{a2}\left(x_a^2-\frac{5}{2}\right)\right\} + m_a\nu_D^a\gamma_a\int^l \tilde{U}dl\right], \tag{32}$$

where $\mathcal{G}_{Ua}$ and $\mathcal{G}_{Xa}$ represent the responses of the distribution function g_a to the parallel flow $\langle u_{\parallel a}B\rangle/\langle B^2\rangle$ and the radial gradient force X_{a1}, respectively, which are defined by

$$\mathcal{G}_{Ua} = \left[1 - \frac{3(\sigma_{Ua}, G_{Ua})}{2m_a T_a \nu_D^a x_a^2 \langle B^2\rangle}\right]^{-1} \times \left[G_{Ua} + \frac{3(\sigma_{Ua}, G_{Ua})}{2m_a T_a \nu_D^a x_a^2\langle B^2\rangle} m_a v\xi B\right],$$

$$\mathcal{G}_{Xa} = G_{Xa} + \frac{3(\sigma_{Ua}, G_{Xa})}{2m_a T_a \nu_D^a x_a^2\langle B^2\rangle}\left[1 - \frac{3(\sigma_{Ua}, G_{Ua})}{2m_a T_a \nu_D^a x_a^2\langle B^2\rangle}\right]^{-1} \times (G_{Ua} + m_a v\xi B). \tag{33}$$

Using Eq. (30) and the definitions of σ's, G's, and $\mathcal{G}$'s, we can prove the following relations:

$$(\sigma_{Ua}, G_{Xa}) = (\sigma_{Xa}, G_{Ua}), \quad (\sigma_{Ua}, \mathcal{G}_{Xa}) = (\sigma_{Xa}, \mathcal{G}_{Ua}), \tag{34}$$

which are associated with the Onsager symmetry[33–35] of the transport coefficients.

Substituting Eq. (32) into Eqs. (18) and (19), we obtain the linear relations between the conjugate pairs of $[\langle \mathbf{B}\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle, \langle \mathbf{B}\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle, \Gamma_a^{\rm bn}, q_a^{\rm bn}/T_a]$ and $[\langle u_{\parallel a}B\rangle/\langle B^2\rangle, (2/5p_a)\langle q_{\parallel a}B\rangle/\langle B^2\rangle, X_{a1}, X_{a2}]$ as

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$$\begin{bmatrix} \langle \mathbf{B}\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle \\ \langle \mathbf{B}\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle \\ \Gamma_a^{\rm bn} \\ q_a^{\rm bn}/T_a \end{bmatrix} = \begin{bmatrix} M_{a1} & M_{a2} & N_{a1} & N_{a2} \\ M_{a2} & M_{a3} & N_{a2} & N_{a3} \\ N_{a1} & N_{a2} & L_{a1} & L_{a2} \\ N_{a2} & N_{a3} & L_{a2} & L_{a3} \end{bmatrix} \times \begin{bmatrix} \langle u_{\parallel a}B\rangle/\langle B^2\rangle \\ \dfrac{2}{5p_a}\langle q_{\parallel a}B\rangle/\langle B^2\rangle \\ X_{a1} \\ X_{a2} \end{bmatrix}. \tag{35}$$

Here, the coefficients M_{aj}, N_{aj}, and L_{aj} $(j=1,2,3)$ in the Onsager-symmetric matrix are written in the form of the energy integral

$$[M_{aj},N_{aj},L_{aj}] = n_a \frac{2}{\sqrt{\pi}}\int_0^\infty dK\sqrt{K}e^{-K}\left(K-\frac{5}{2}\right)^{j-1} \times [M_a(K),N_a(K),L_a(K)], \tag{36}$$

where $M_a(K)$, $N_a(K)$, and $L_a(K)$ represent contributions of monoenergetic particles with $K\equiv x_a^2\equiv m_a v^2/2T_a$ to M_{a1}, N_{a1}, and L_{a1}, respectively, which are given by the inner products of the source terms σ's and the response functions $\mathcal{G}$'s as

$$M_a(K) = \frac{1}{T_a}(\sigma_{Ua},\mathcal{G}_{Ua}) = \frac{1}{T_a}(\sigma_{Ua},G_{Ua})\left[1-\frac{3(\sigma_{Ua},G_{Ua})}{2m_aT_a\nu_D^a(K)K\langle B^2\rangle}\right]^{-1},$$

$$N_a(K) = \frac{1}{T_a}(\sigma_{Xa},\mathcal{G}_{Ua}) = \frac{1}{T_a}(\sigma_{Xa},G_{Ua})\left[1-\frac{3(\sigma_{Ua},G_{Ua})}{2m_aT_a\nu_D^a(K)K\langle B^2\rangle}\right]^{-1}, \tag{37}$$

$$L_a(K) = \frac{1}{T_a}(\sigma_{Xa},\mathcal{G}_{Xa}) = \frac{1}{T_a}(\sigma_{Xa},G_{Xa}) + \frac{3(\sigma_{Xa},G_{Ua})^2}{2m_aT_a^2\nu_D^a(K)K\langle B^2\rangle} \times \left[1-\frac{3(\sigma_{Ua},G_{Ua})}{2m_aT_a\nu_D^a(K)K\langle B^2\rangle}\right]^{-1}.$$

In the same way, we obtain the linear relations between the conjugate pairs of $[\langle \mathbf{B}_P\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle,\langle \mathbf{B}_P\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle,\langle \mathbf{B}_T\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle,\langle \mathbf{B}_T\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle]$ and $[\langle u_a^\theta\rangle/\chi',(2/5p_a)\langle q_a^\theta\rangle/\chi',\langle u_a^\zeta\rangle/\psi',(2/5p_a)\langle q_a^\zeta\rangle/\psi',]$ as

$$\begin{bmatrix} \langle \mathbf{B}_P\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle \\ \langle \mathbf{B}_P\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle \\ \langle \mathbf{B}_T\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle \\ \langle \mathbf{B}_T\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle \end{bmatrix} = \begin{bmatrix} M_{a1PP} & M_{a2PP} & M_{a1PT} & M_{a2PT} \\ M_{a2PP} & M_{a3PP} & M_{a2PT} & M_{a3PT} \\ M_{a1PT} & M_{a2PT} & M_{a1TT} & M_{a2TT} \\ M_{a2PT} & M_{a3PT} & M_{a2TT} & M_{a3TT} \end{bmatrix} \times \begin{bmatrix} \langle u_a^\theta\rangle/\chi' \\ \dfrac{2}{5p_a}\langle q_a^\theta\rangle/\chi' \\ \langle u_a^\zeta\rangle/\psi' \\ \dfrac{2}{5p_a}\langle q_a^\zeta\rangle/\psi' \end{bmatrix}. \tag{38}$$

Here, the Onsager-symmetric poloidal and toroidal viscosity coefficients M_{ajPP}, M_{ajPT}, and M_{ajTT} $(j=1,2,3)$ are also written in the form of the energy integral

$$[M_{ajPP},M_{ajPT},M_{ajTT}] = n_a\frac{2}{\sqrt{\pi}}\int_0^\infty dK\sqrt{K}e^{-K}\left(K-\frac{5}{2}\right)^{j-1} \times [M_{aPP}(K),M_{aPT}(K),M_{aTT}(K)], \tag{39}$$

where $M_{aPP}(K)$, $M_{aPT}(K)$, and $M_{TT}(K)$ represent contributions of monoenergetic particles to M_{ajPP}, M_{ajPT}, and M_{ajTT}, respectively, which are given in terms of $M_a(K)$, $N_a(K)$, and $L_a(K)$ as shown by Eq. (B5) in Appendix B.

Now, we need the solutions G_{Ua} and G_{Xa} of Eq. (27) in order to obtain the monoenergetic coefficients $[M_a(K),N_a(K),L_a(K)]$ in Eq. (37). Since, in the DKES[19,20] and other numerical codes for the neoclassical transport coefficients, the drift kinetic equation to be solved is not Eq. (14) but Eq. (3) with the pitch-angle-scattering operator, they appear at first to be irrelevant to calculation of the (G_{Ua},G_{Xa}) and $[M_a(K),N_a(K),L_a(K)]$. However, in fact, these codes can be made use of to obtain them as shown in the following.

When solving Eq. (3) by the DKES, the right-hand side of Eq. (3) are written as a linear combinations of the source terms σ_1^+ and σ_3^+ defined by[20]

$$\sigma_1^+ \equiv -\frac{v^2}{3\Omega_a}\left[1+\frac{1}{2}P_2(\xi)\right]\mathbf{b}\times\nabla\ln B\cdot\nabla s,$$
$$\sigma_3^+ \equiv \frac{v^2}{\nu_D^a}P_2(\xi)\mathbf{B}\cdot\nabla\ln B = V_\parallel\left(\frac{B}{\nu_D^a}v\xi\right), \tag{40}$$

which are associated with the radial particle flux and the bootstrap current, respectively. Here, it is noted that, since we have neglected $\mathbf{v}_E\cdot\nabla f_{a1}$ ($\mathbf{v}_E\equiv c\mathbf{E}\times\mathbf{B}/B^2$) in Eq. (3), σ_3^+ defined in Eq. (40) corresponds to $\sigma_3^+(E_s=0)$ in Rij and Hirshman.[20] This definition of σ_3^+ based on Ref. 20 differs from that in Ref. 19 in that the former excludes the Spitzer flow part from the total parallel flow. Effects of $\mathbf{v}_E\cdot\nabla f_{a1}$ such as nonlinear E_s-dependences of the neoclassical transport can be included by the procedures described in Appendix E. It is useful to find that σ_{Ua} and G_{Ua} are directly related to σ_3^+ and $(F_3^++F_3^-)$, respectively, by

$$\sigma_{Ua} = -m_a\nu_D^a\sigma_3^+,$$
$$G_{Ua} = -m_a\nu_D^a(F_3^++F_3^-), \tag{41}$$

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and that σ_{Xa} and G_{Xa} are written in terms of σ_1^+ and $(F_1^+ + F_1^-)$, respectively, as

$$\sigma_{Xa} = -\sigma_1^+ - V_\parallel\left(\frac{B}{\Omega_a} v\xi\tilde{U}\right),$$

$$G_{Xa} = -(F_1^+ + F_1^-) - \frac{B}{\Omega_a} v\xi\tilde{U} + \frac{\nu_D^a B}{\Omega_a}\int^l \tilde{U} dl. \tag{42}$$

Here, $(F_1^+ + F_1^-)$ and $(F_3^+ + F_3^-)$ represent the response functions associated with the source terms σ_1^+ and σ_3^+ for the case of $E_s = 0$ in Rij and Hirshman,[20] where the superscripts $+$ and $-$ denote the even and odd parts with respect to ξ, respectively. Then, substituting Eqs. (41) and (42) into Eq. (37), we have

$$M_a(K) = \frac{m_a^2}{T_a}[\nu_D^a(K)]^2 D_{33}(K)\left[1 - \frac{3m_a\nu_D^a(K)D_{33}(K)}{2T_aK\langle B^2\rangle}\right]^{-1},$$

$$N_a(K) = \frac{m_a}{T_a}\nu_D^a(K)D_{13}(K)\left[1 - \frac{3m_a\nu_D^a(K)D_{33}(K)}{2T_aK\langle B^2\rangle}\right]^{-1}, \tag{43}$$

$$L_a(K) = \frac{1}{T_a}\left(D_{11}(K) - \frac{B^2v^2\nu_D^a}{3\Omega_a^2}\langle\tilde{U}^2\rangle + \frac{3m_a\nu_D^a(K)[D_{13}(K)]^2}{2T_aK\langle B^2\rangle} \times\left[1 - \frac{3m_a\nu_D^a(K)D_{33}(K)}{2T_aK\langle B^2\rangle}\right]^{-1}\right),$$

where

$$D_{jk}(K) \equiv \tfrac{1}{2}\int_{-1}^{1} d\xi\langle\sigma_j^+ F_k^+\rangle \quad (j,k = 1,3), \tag{44}$$

represent the transport coefficients for monoenergetic particles which can be obtained as an output of the DKES[20] (for the case of $E_s = 0$). For collision frequencies in the banana and plateau regimes, the term $(B^2v^2\nu_D^a/3\Omega_a^2)\langle\tilde{U}^2\rangle$ of $L_a(K)$ in Eq. (43), which corresponds to the Pfirsch–Schlüter-flux part, is negligibly small.

Now, we have found that numerical solvers such as the DKES can be utilized to calculate the coefficients $[M_a(K), N_a(K), L_a(K)]$ by using Eq. (43). Once $[M_a(K), N_a(K), L_a(K)]$ are given, the monoenergetic poloidal and toroidal viscosity coefficients $[M_{aPP}(K), M_{aPT}(K), M_{aTT}(K)]$ are immediately derived from Eq. (B5) in Appendix B, and the energy-integrated coefficients (M_{aj}, N_{aj}, L_{aj}) and $(M_{ajPP}, M_{ajPT}, M_{ajTT})$ are obtained by Eqs. (36) and (39), respectively. Then, all the neoclassical transport coefficients for radial fluxes and parallel currents can be calculated from (M_{aj}, N_{aj}, L_{aj}) as shown in Appendix C. It should be noted that the parallel momentum balance equations and the friction-flow relations with collisional momentum conservation are used to derive the neoclassical transport coefficients in Appendix C. Therefore, these coefficients include the coupling effects between unlike-species particles as well as they recover the intrinsic ambipolarity of the radial particle fluxes in the symmetric limit. These properties are not obtained by only solving the drift kinetic equation (3) without the field particle collision term $C_{ab}(f_{aM}, f_{b1})$. For the symmetric case, $M_a(K)$, $N_a(K)$, and $L_a(K)$ are proportionally related to each other as shown by Eq. (D4) in Appendix D.

In the Pfirsch–Schlüter regime, $[\nu_D^a(K) \gg v_{Ta}\sqrt{K}/L_c$, ($L_c$: The characteristic length of magnetic ripples along the field line)], the plateau regime $[v_{Ta}\sqrt{K}/L_c \gg \nu_D^a(K) \gg (\delta B/B)^{3/2} v_{Ta}\sqrt{K}/L_c$, ($\delta B$: The field strength variation in the magnetic ripples)], and the banana regime $[\nu_D^a(K) \ll (\delta B/B)^{3/2} v_{Ta}\sqrt{K}/L_c]$, the monoenergetic coefficients $M_a(K)$ and $N_a(K)$ associated with the parallel viscosities can analytically be given by[33]

$$M_a(K) = \begin{cases} \frac{2}{5} m_a v_{Ta}^2 \tau_{aa}\langle(\mathbf{B}\cdot\nabla\ln B)^2\rangle K^2[\tau_{aa}\nu_T^a(K)]^{-1} & \text{(Pfirsch–Schlüter)} \\ \frac{1}{4}\pi m_a v_{Ta}\langle B^2\rangle^{1/2}(4\pi^2/V')\left(\sum_{(m,n)\neq(0,0)}|\beta_{mn}|^2|m\chi' - n\psi'|\right)K^{3/2} & \text{(plateau)} \\ \frac{2}{3} m_a \tau_{aa}^{-1}(f_t/f_c)\langle B^2\rangle K[\tau_{aa}\nu_D^a(K)] & \text{(banana)} \end{cases}$$

$$= m_a v_{Ta} K^{3/2} \times \begin{cases} \frac{2}{5}\langle(\mathbf{B}\cdot\nabla\ln B)^2\rangle[\nu_T^a(K)/v]^{-1} & \text{(Pfirsch–Schlüter)} \\ \frac{1}{4}\pi\langle B^2\rangle^{1/2}(4\pi^2/V')\left(\sum_{(m,n)\neq(0,0)}|\beta_{mn}|^2|m\chi' - n\psi'|\right) & \text{(plateau)} \\ \frac{2}{3}(f_t/f_c)\langle B^2\rangle[\nu_D^a(K)/v] & \text{(banana)} \end{cases} \tag{45}$$

and

$$N_a(K) = -\frac{cG_a^{(\mathrm{BS})}}{e_a\langle B^2\rangle} M_a(K), \tag{46}$$

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respectively. Here, $G_a^{(\mathrm{BS})}$ is a flux-surface function, which represents the geometrical factor associated with the bootstrap current[5,29–31,33,38] [see the paragraph after Eq. (49)], and is determined by the magnetic configuration as[33,38]

$$G_a^{(\mathrm{BS})} = \begin{cases} (4\pi^2/V')\langle(\mathbf{B}\cdot\nabla\ln B)^2\rangle^{-1}[B_\zeta^{(\mathrm{Boozer})}\langle(\partial\ln B/\partial\theta_H)(\mathbf{B}\cdot\nabla\ln B)\rangle - B_\theta^{(\mathrm{Boozer})}\langle(\partial\ln B/\partial\zeta_H)(\mathbf{B}\cdot\nabla\ln B)\rangle] & \text{(Pfirsch–Schlüter)} \\ \left(\sum_{(m,n)\neq(0,0)}|\beta_{mn}|^2|m\chi'-n\psi'|\right)^{-1}\sum_{(m,n)\neq(0,0)}|\beta_{mn}|^2(mB_\zeta^{(\mathrm{Boozer})}-nB_\theta^{(\mathrm{Boozer})})(m\chi'-n\psi')/|m\chi'-n\psi'| & \text{(plateau)} \end{cases} \quad (47)$$

and the analytical expression of $G_a^{(\mathrm{BS})}$ for the banana regime is given in Refs. 5 and 29–31. When we evaluate $G_a^{(\mathrm{BS})}$ for the Pfirsch–Schlüter regime given by Eq. (47), Eq. (A11) is useful. In Eq. (45), $f_t \equiv 1 - f_c$ and $f_c \equiv \frac{3}{4}\langle B^2\rangle\int_0^{B_{\max}^{-1}} d\lambda\, \lambda/\langle(1-\lambda B)^{1/2}\rangle$ represent the fractions of trapped and circulating particles, respectively, and $\nu_T^a(K)$ for the Pfirsch–Schlüter regime is given by[2] $\nu_T^a \equiv 3\nu_D^a + \nu_E^a \equiv (3\sqrt{\pi}/4)\Sigma_b \tau_{ab}^{-1}[\{\Phi(x_b) - 3G(x_b)\}/x_a^3 + 4(T_a/T_b)(1+m_b/m_a)G(x_b)/x_a]$ with $G(x) \equiv [\Phi(x) - x\Phi'(x)]/(2x^2)$. Thus, in order to correctly reproduce the viscosity coefficients for the Pfirsch–Schlüter regime, we should replace ν_D^a with $\nu_T^a/3$ when using the pitch-angle-scattering operator in Eq. (24) for that collisional region. In Eqs. (45) and (47), β_{mn} for the plateau regime are the coefficients in the Fourier expansion of B:

$$B = B_0\left(1 + \sum_{(m,n)\neq(0,0)} \beta_{mn}\exp[i(m\theta - n\zeta)]\right), \quad (48)$$

where we should note that the existence of the plateau regime requires $|\beta_{mn}| \ll 1$ and that it does not make a significant difference which of the flux-coordinate systems (s,θ,ζ) is used to calculate β_{mn} for the plateau regime. If all the particles in the velocity space are dominantly contained in either of the Pfirsch–Schlüter, plateau, and banana regimes, we obtain from Eqs. (35), (36), and (46),

$$\begin{bmatrix} \langle\mathbf{B}\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle \\ \langle\mathbf{B}\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle \end{bmatrix} = \langle B^2\rangle^{-1}\begin{bmatrix} M_{a1} & M_{a2} \\ M_{a2} & M_{a3}\end{bmatrix} \times\left(\begin{bmatrix} \langle u_{\parallel a}B\rangle \\ \dfrac{2}{5p_a}\langle q_{\parallel a}B\rangle \end{bmatrix} - G_a^{(\mathrm{BS})}\frac{c}{e_a}\begin{bmatrix} X_{a1} \\ X_{a2}\end{bmatrix}\right). \quad (49)$$

As shown by substituting Eq. (49) into the parallel momentum balance equations [see Eq. (C1) in Appendix C], $G_a^{(\mathrm{BS})}$ represents the geometrical factor which enters the coefficients relating the parallel flows to the thermodynamic forces. Also, it is directly confirmed from Eqs. (46), (C5), (C8) and (C11)–(C13) that the geometrical factor $G_a^{(\mathrm{BS})}$ appears in the neoclassical transport coefficients for the bootstrap current as well as in the nondiagonal coefficients connecting the electrons' fluxes (forces) with the ions' forces (fluxes). For symmetric systems described in Appendix D, the geometrical factor $G_a^{(\mathrm{BS})}$ defined by Eq. (46) is independent of the collision frequency [see Eq. (D5)] and, therefore, Eq. (49) is always satisfied. For example, $G_a^{(\mathrm{BS})} = B_\zeta^{(\mathrm{Boozer})}/\chi'$ in the axisymmetric case. However, for nonaxisymmetric systems, Eq. (49) is not generally valid (except for the limiting collision frequency regimes), and therefore, the two independent 2×2 matrices $[M_{aj}]$ and $[N_{aj}]$ obtained from the energy integral in Eq. (36) should be used instead for relating the parallel viscosities to the parallel flows and to the radial gradient forces.

We can analytically express the monoenergetic coefficient $L_a(K)$ for the Pfirsch–Schlüter and plateau regimes as[33]

$$L_a(K) = \frac{c^2 M_a(K)}{e_a^2\langle B^2\rangle^2}\begin{cases} (4\pi^2/V')^2\langle(\mathbf{B}\cdot\nabla\ln B)^2\rangle^{-1}\langle[B_\zeta^{(\mathrm{Boozer})}(\partial\ln B/\partial\theta_H) - B_\theta^{(\mathrm{Boozer})}(\partial\ln B/\partial\zeta_H)]^2\rangle & \text{(Pfirsch–Schlüter)} \\ \left(\sum_{(m,n)\neq(0,0)}|\beta_{mn}|^2|m\chi'-n\psi'|\right)^{-1}\sum_{(m,n)\neq(0,0)}|\beta_{mn}|^2(mB_\zeta^{(\mathrm{Boozer})}-nB_\theta^{(\mathrm{Boozer})})^2/|m\chi'-n\psi'| & \text{(plateau)} \end{cases}$$

$$= \frac{m_a c^2}{e_a^2} v_{Ta}K^{3/2}\begin{cases} \frac{2}{5}(4\pi^2/V')^2\langle B^2\rangle^{-2}\langle[B_\zeta^{(\mathrm{Boozer})}(\partial\ln B/\partial\theta_H) - B_\theta^{(\mathrm{Boozer})}(\partial\ln B/\partial\zeta_H)]^2\rangle[\nu_T^a(K)/v]^{-1} & \text{(Pfirsch–Schlüter)} \\ \frac{1}{4}\pi\langle B^2\rangle^{-3/2}(4\pi^2/V')\sum_{(m,n)\neq(0,0)}|\beta_{mn}|^2(mB_\zeta^{(\mathrm{Boozer})}-nB_\theta^{(\mathrm{Boozer})})^2/|m\chi'-n\psi'| & \text{(plateau)} \end{cases}, \quad (50)$$

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which shows that, for these regimes, $L_a(K)$ has the same dependence on the collision frequency and the energy as $M_a(K)$.

It is well-known that, for nonsymmetric systems, the centers of trapped-particle orbits move across magnetic surfaces and cause the neoclassical ripple transport in the weakly collisional sub-regime (so-called $1/\nu$ regime).[5–8] The bounce-averaged part of the distribution function $\langle f_{a1}\rangle_b \equiv (\oint f_{a1} dl/v_\parallel)/(\oint dl/v_\parallel)$ makes no contribution to the parallel viscosities[5,29] and consequently to $M_a(K)$ and $N_a(K)$, while it contributes dominantly to the radial particle and heat fluxes and to $L_a(K)$ in the $1/\nu$ regime. Using the analytical solution of the bounce-averaged drift kinetic equation by Shaing and Hokin,[7] we obtain $L_a(K)[\propto 1/\nu_D^a(K)]$ as

$$L_a(K)=\frac{1}{4\sqrt{2}\pi^2 T_a}\left(\frac{m_a c}{e_a\psi'}\right)^2 v_{Ta}^4\tau_{aa}\frac{K^2}{\tau_{aa}\nu_D^a(K)}G_a^{(1/\nu)}$$

$$=\frac{1}{2\sqrt{2}\pi^2}\frac{m_a c^2}{e_a^2}v_{Ta}K^{3/2}\frac{G_a^{(1/\nu)}}{(\psi')^2[\nu_D^a(K)/v]}$$

$$(\text{for the } 1/\nu \text{ regime}), \quad (51)$$

where $G_a^{(1/\nu)}$ represents the geometrical factor for the neoclassical ripple transport defined by[7]

$$G_a^{(1/\nu)}=\int_0^{2\pi}d\theta\ \epsilon_H^{3/2}\left[G_1\left(\frac{\partial\epsilon_T}{\partial\theta}\right)^2-2G_2\left(\frac{\partial\epsilon_T}{\partial\theta}\right)\left(\frac{\partial\epsilon_H}{\partial\theta}\right)^2+G_3\left(\frac{\partial\epsilon_H}{\partial\theta}\right)^2\right], \quad (52)$$

with $G_1=16/9$, $G_2=16/15$, and $G_3=0.684$ for the magnetic field strength $B=B_0[1+\epsilon_T(s,\theta)+\epsilon_H(s,\theta)\cos(l\theta-n\zeta)]$ $(|\epsilon_T|\ll 1, |\epsilon_H|\ll 1)$. Here, the safety factor $q(s)\equiv\psi'/\chi'$ is assumed to satisfy $nq(s)\gg l$. For this case, the $1/\nu$ regime is defined by $\langle\dot\theta\rangle_b\ll\nu_D^a(K)/\epsilon_H\ll\epsilon_H^{1/2}v_{Ta}/(R/n)$, where R denotes the major radius of the torus and $\langle\dot\theta\rangle_b$ represents the bounce-averaged poloidal angular velocity of helically trapped particles. [Note that, in the present study using Eq. (3) as the basic equation, we do not treat the case of $cE_r/(rB_0)\sim\langle\dot\theta\rangle_b\geq\nu_D^a(K)/\epsilon_H$ (r: the minor radius of the torus)]. In the $1/\nu$ regime, $M_a(\propto\nu_D^a)$ and $N_a(\propto\nu_D^a)$ make little contribution to the radial transport fluxes so that Eq. (35) gives $\Gamma_a^{\rm bn}\simeq L_{a1}X_{a1}+L_{a2}X_{a2}$ and $q_a^{\rm bn}/T_a\simeq L_{a2}X_{a1}+L_{a3}X_{a2}$, in which dependence on X_{b1} and X_{b2} with $b\neq a$ are negligible. This fact justifies conventional calculations of the neoclassical ripple transport using the pitch-angle-scattering collision model,[7,8] in which the collisional momentum conservation and the nondiagonal coupling between unlike-species particles are not taken into account. However, in general, we should use all elements M_{aj}, N_{aj}, and L_{aj} in Eq. (35) to calculate the total neoclassical transport fluxes, especially when the magnitude of the banana-plateau transport induced by the parallel viscosity is comparable to or larger than that of the ripple transport as is the case in quasi-symmetric systems.[10–12]

IV. NUMERICAL EXAMPLES

Here, in order to illustrate the validity of the procedures described in the previous sections, we present numerical results for the simple nonsymmetric system, in which the magnetic field strength is given by

$$B=B_0(s)[1-\epsilon_t(s)\cos\theta_B-\epsilon_h(s)\cos(l\theta_B-n\zeta_B)]. \quad (53)$$

The mean minor radius of the flux surface is used for the radial coordinate s. For simplicity, we consider a single flux surface of a large-aspect-ratio torus with the minor radius $s=0.4$ m and the major radius $R=4$ m. Then, parameters used for numerical calculations are determined as $B_0=1$ T, $\epsilon_t=0.1$, $0\leq\epsilon_h\leq 0.1$, $\psi'=0.4$ T·m, $\chi'=0.15$ T·m ($q\equiv\psi'/\chi'=2.6667$), $B_\zeta^{\rm (Boozer)}=4$ T·m, $B_\theta^{\rm (Boozer)}=0$ T·m (no net toroidal current), $l=2$, and $n=10$ (corresponding to the Large Helical Device[39]). Using these parameters and Eq. (53), we can calculate $\langle B^2\rangle=4\pi^2/(\int_0^{2\pi}d\theta_B\int_0^{2\pi}d\zeta_B B^{-2})$ and $V'\equiv 4\pi^2(\psi' B_\zeta^{\rm (Boozer)}+\chi' B_\theta^{\rm (Boozer)})/\langle B^2\rangle$. Hereafter, subscripts representing particle species are omitted.

The monoenergetic diffusion coefficients $[D_{11}(K), D_{13}(K), D_{33}(K)]$ are obtained by using the DKES. Figure 1 shows $D_{11}^*\equiv D_{11}(K)/[\frac{1}{2}v_T(Bv_T/\Omega)^2K^{3/2}]$, $D_{13}^*\equiv D_{13}(K)/[\frac{1}{2}v_T(Bv_T/\Omega)K]$, and $D_{33}^*\equiv D_{33}(K)/(\frac{1}{2}v_T K^{1/2})$ as a function of ν_D/v for $\epsilon_h=0$, 0.005, 0.01, 0.02, 0.05, and 0.1. Substituting these monoenergetic diffusion coefficients into Eq. (43) and using Eq. (B5) give other monoenergetic coefficients $[M(K), N(K), L(K)]$ and $[M_{PP}(K), M_{PT}(K), M_{TT}(K)]$, which are illustrated in Figs. 2–5.

Figure 2 shows $M^*\equiv M(K)/(mv_T K^{3/2})$ as a function of ν_D/v. Here, M^* is written in terms of D_{33}^* as

$$M^*=\frac{(\nu_D/v)^2 D_{33}^*}{1-\frac{3}{2}(\nu_D/v)D_{33}^*/\langle B^2\rangle}. \quad (54)$$

In Fig. 2, dotted curves with open circles and solid lines represent M^* obtained from numerical results of D_{33}^* in Fig. 1 and from the analytical formulas in Eq. (45), respectively. When the formula for the Pfirsch–Schlüter regime given by Eq. (45) is used in Fig. 2, ν_T is replaced with $3\nu_D$. However, as mentioned after Eq. (47), the correct functional form of $\nu_T(K)$ should be taken into account when we calculate the energy-integrated viscosity coefficients. We can see an excellent agreement between the numerical and analytical results except for transition regions between the banana, plateau, and Pfirsch–Schlüter regimes. A simple rational approximation,[2] which smoothly connects the three analytical expressions, would be useful for this case.

Figure 3 shows $L^*\equiv L(K)/[\frac{1}{2}(v_T/T)(Bv_T/\Omega)^2K^{3/2}]$ as a function of ν_D/v. Here, L^* is given in terms of D_{11}^* and D_{13}^* by

$$L^*=D_{11}^*-\frac{2}{3}(\nu_D/v)\langle\tilde U^2\rangle+\frac{\frac{3}{2}(\nu_D/v)(D_{13}^*)^2/\langle B^2\rangle}{1-\frac{3}{2}(\nu_D/v)D_{33}^*/\langle B^2\rangle}. \quad (55)$$

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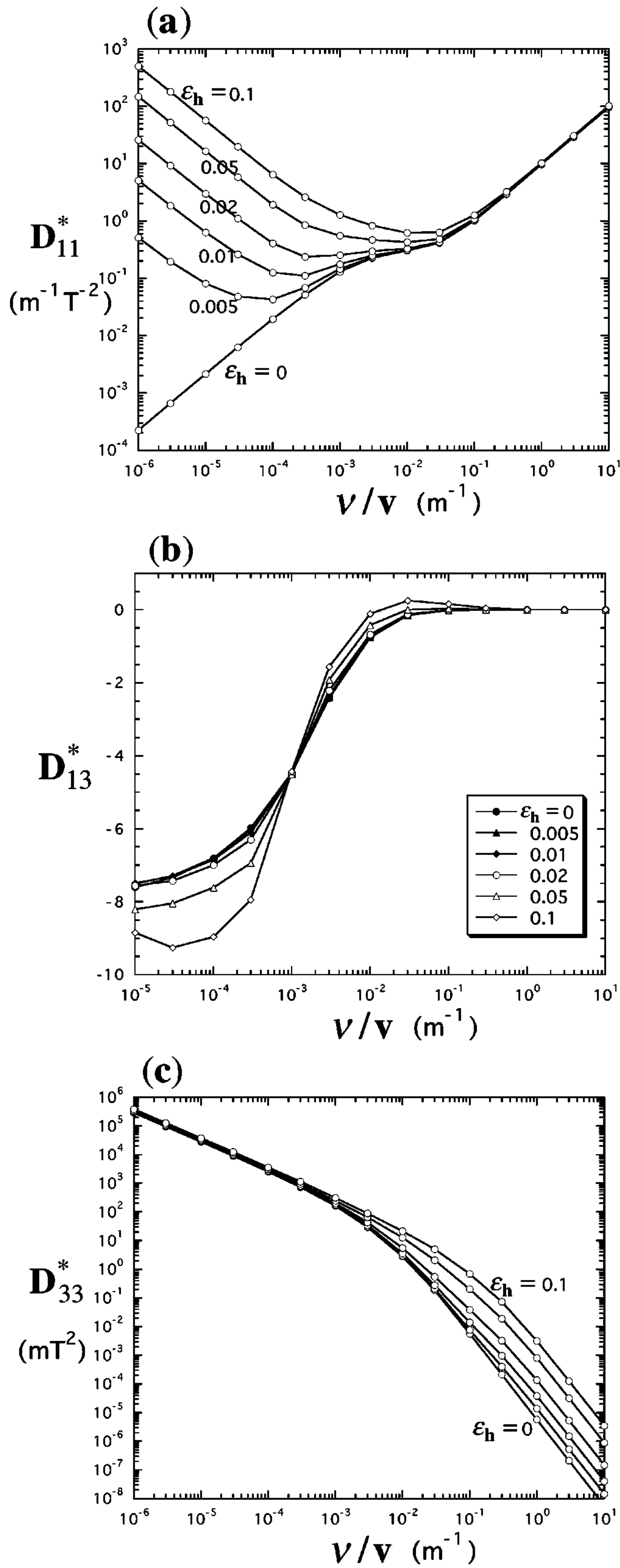

FIG. 1. $D_{11}^* \equiv D_{11}(K)/[\frac{1}{2}v_T(Bv_T/\Omega)^2K^{3/2}]$ (a), $D_{13}^* \equiv D_{13}(K)/[\frac{1}{2}v_T(Bv_T/\Omega)K]$ (b), and $D_{33}^* \equiv D_{33}(K)/(\frac{1}{2}v_TK^{1/2})$ (c) as a function of ν_D/v for $\epsilon_h = 0$, 0.005, 0.01, 0.02, 0.05, and 0.1 obtained by using the DKES.

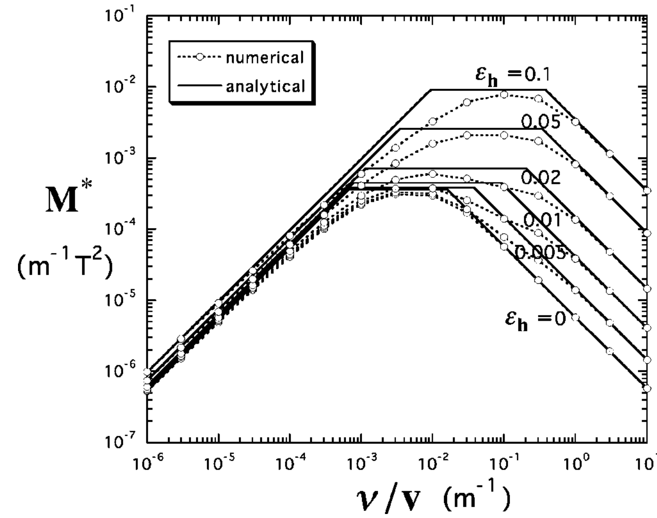

FIG. 2. $M^* \equiv M(K)/(mv_TK^{3/2})$ as a function of ν_D/v for $\epsilon_h = 0$, 0.005, 0.01, 0.02, 0.05, and 0.1. Dotted curves with open circles and solid lines represent M^* obtained from numerical results of D_{33}^* in Fig. 1 and from the analytical formulas in Eq. (45), respectively.

In the same way as in Fig. 2, dotted curves with open circles and solid lines in Fig. 3 represent L^* obtained from numerical results of D_{11}^*, D_{13}^*, and D_{33}^* in Fig. 1 and from the analytical formulas in Eqs. (50) and (51), respectively. We see that, in the $1/\nu$ regime with $\epsilon_h = 0.005$ and 0.01, numerically obtained L^* are significantly smaller than the analytical predictions. This is because, for such small ϵ_h's, the fraction of helically trapped particles are overestimated by the analytical formula in Eq. (51), where the lowest-order guiding-center motion is regarded as a toroidal one instead of a parallel one under the condition of $nq \gg l$. Recently, an improved formulation of the neoclassical ripple transport has been given by Beidler and Maaßberg.[8]

We plot the geometrical factor for the bootstrap current $G^{(\mathrm{BS})} \equiv -(e\langle B^2\rangle/c)N(K)/M(K)$ [see Eq. (46)] instead of $N(K)$ as a function of ν_D/v in Fig. 4. Here, $G^{(\mathrm{BS})}$ is written in terms of D_{13}^* and D_{33}^* as

$$G^{(\mathrm{BS})} = -\frac{\langle B^2\rangle D_{13}^*}{(\nu_D/v)D_{33}^*}. \tag{56}$$

In Fig. 4, dotted curves with open circles represent $G^{(\mathrm{BS})}$ obtained from numerical results of D_{13}^* and D_{33}^* in Fig. 1. The axisymmetric case with $\epsilon_h = 0$ is given by the constant, $G^{(\mathrm{BS})} = B^{(\mathrm{Boozer})}/\chi' = 26.667$. Analytical results given by Eq. (47) for the Pfirsch–Schlüter and plateau regimes are represented by thick line segments, which are in good agreement with the numerical results, although the latter do not show clear constancy in the plateau regime.

Figure 5 shows $[M_{PP}^*, -M_{PT}^*, M_{TT}^*] \equiv [M_{PP}(K), -M_{PT}(K), M_{TT}(K)]/[(4\pi^2/V')mv_T(\psi'\chi')^2K^{3/2}]$ as a function of ν_D/v. For $\epsilon_h \leqslant 0.02$, M_{TT} takes small negative values around the plateau regime, which are not plotted in Fig. 5. As ϵ_h increases in the Pfirsch–Schlüter and plateau regimes, the magnitude of the viscosity coefficients M_{PT} and M_{TT} increases more rapidly than M_{PP}. It is also seen that, in the $1/\nu$ regime, $M_{PP} \simeq -M_{PT} \simeq M_{TT} \propto 1/\nu_D$, which reflects

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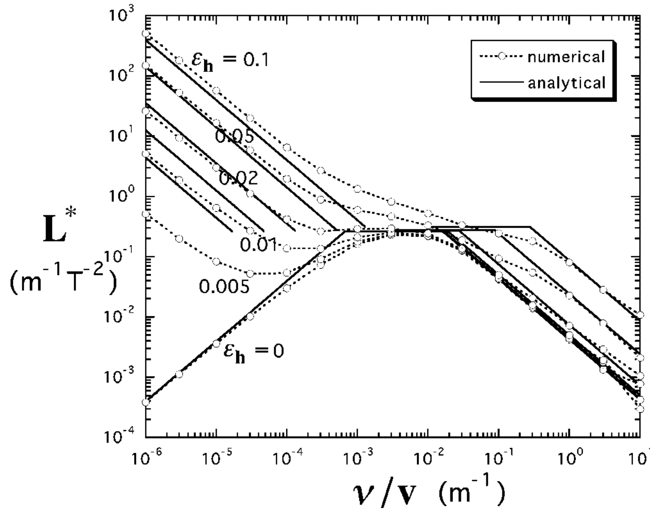

FIG. 3. $L^* \equiv L(K)/[\frac{1}{2}(v_T/T)(Bv_T/\Omega)^2 K^{3/2}]$ as a function of ν_D/v for $\epsilon_h = 0$, 0.005, 0.01, 0.02, 0.05, and 0.1. Dotted curves with circles and solid lines represent L^* obtained from numerical results of D_{11}^*, D_{13}^*, and D_{33}^* in Fig. 1 and from the analytical formulas in Eqs. (50) and (51), respectively.

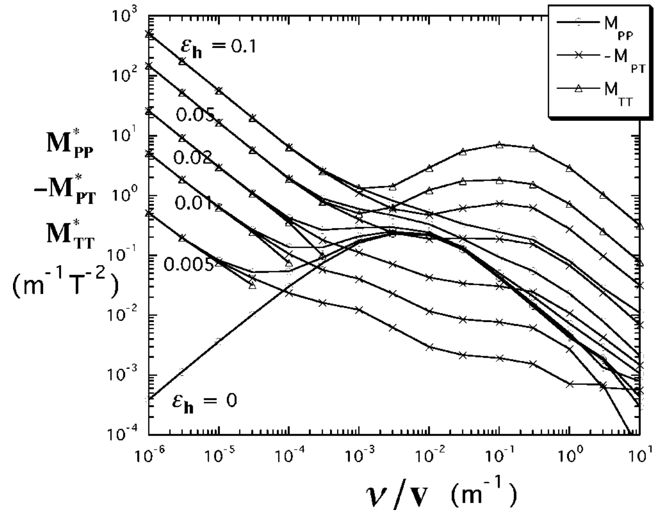

FIG. 5. Poloidal and toroidal viscosity coefficients as a function of ν_D/v for $\epsilon_h = 0$, 0.005, 0.01, 0.02, 0.05, and 0.1. Curves with circles, crosses, and triangles represent M_{PPT}^*, $-M_{PT}^*$, and M_{TT}^*, respectively.

from the fact that the parallel viscosity $\langle \mathbf{B} \cdot (\nabla \cdot \boldsymbol{\pi}_a) \rangle = \langle \mathbf{B}_P \cdot (\nabla \cdot \boldsymbol{\pi}_a) \rangle + \langle \mathbf{B}_T \cdot (\nabla \cdot \boldsymbol{\pi}_a) \rangle (\propto \nu_D)$ is much smaller than the viscosities in other directions $(\propto 1/\nu_D)$.

From the results shown above, it is confirmed that all neoclassical coefficients for the viscosities, the banana-plateau and nonsymmetric radial transport fluxes, and the geometrical factor associated with the bootstrap current are obtained straightforwardly by using our method.

V. CONCLUSIONS

In the present paper, we have presented two types of Onsager-symmetric matrices: One of them, with the elements

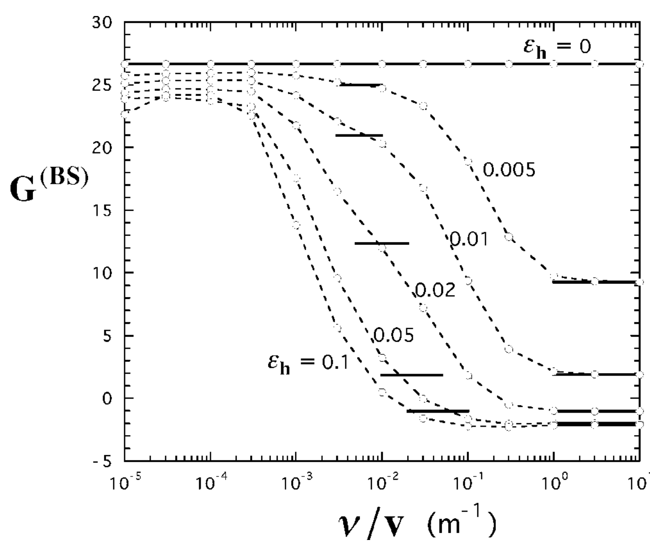

FIG. 4. The geometrical factor for the bootstrap current $G^{(\mathrm{BS})}$ as a function of ν_D/v for $\epsilon_h = 0$, 0.005, 0.01, 0.02, 0.05, and 0.1. Dotted curves with open circles represent $G^{(\mathrm{BS})}$ obtained from numerical results of D_{13}^* and D_{33}^* in Fig. 1. The axisymmetric case with $\epsilon_h = 0$ is given by the constant, $G^{(\mathrm{BS})} = B^{(\mathrm{Boozer})}/\chi' = 26.667$. Analytical results given by Eq. (47) for the Pfirsch–Schlüter and plateau regimes are represented by thick line segments.

(M_{aj}, N_{aj}, L_{aj}), relates the parallel viscosities and the radial fluxes to the parallel flows and the radial-gradient forces as in Eq. (35), and the other, represented by $(M_{ajPP}, M_{ajPT}, L_{ajTT})$, connects the poloidal and toroidal viscosities to the poloidal and toroidal flows as in Eq. (38). We have shown that the matrix elements (M_{aj}, N_{aj}, L_{aj}) can be obtained readily from the output of commonly used numerical codes such as the DKES and that the poloidal and toroidal viscosity coefficients $(M_{ajPP}, M_{ajPT}, L_{ajTT})$ can be derived directly from (M_{aj}, N_{aj}, L_{aj}). Using the matrix elements (M_{aj}, N_{aj}, L_{aj}) in the parallel momentum balance equations combined with the friction-flow relations yields the neoclassical transport coefficients for the radial particle and heat fluxes and the bootstrap current, which include the coupling effects between unlike-species particles as well as the intrinsic ambipolarity of the radial particle fluxes in the symmetric case. These procedures for accurate calculation of neoclassical viscosity and transport coefficients, the validity of which has been verified by numerical examples, are considered to be useful especially when evaluating how these neoclassical coefficients in quasi-symmetric toroidal systems such as quasi-axisymmetric systems deviate from those in exactly symmetric systems.

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APPENDIX A: BOOZER AND HAMADA COORDINATES

We consider general toroidal configurations, in which the magnetic field **B** is written as in Eq. (1) of Sec. II. In the

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Boozer[36] coordinates (s,θ_B,ζ_B), the covariant poloidal and toroidal components of the magnetic field $\mathbf{B}$ are flux-surface functions given by

$$B_\theta^{(\text{Boozer})}\equiv\mathbf{B}\cdot\frac{\partial\mathbf{x}}{\partial\theta_B}=\frac{2}{c}I_T(s),$$
$$B_\zeta^{(\text{Boozer})}\equiv\mathbf{B}\cdot\frac{\partial\mathbf{x}}{\partial\zeta_B}=\frac{2}{c}I_P^d(s). \tag{A1}$$

Here, the poloidal and toroidal currents are defined by $I_P^d(s)\equiv\int_{S_P^d(s)}\mathbf{B}\cdot d\mathbf{S}$ and $I_T(s)\equiv\int_{S_T(s)}\mathbf{B}\cdot d\mathbf{S}$, respectively, where $S_P^d(s)$ represents the part of a $\theta=$constant surface that lies *outside* the flux surface with the label s and $S_T(s)$ is the part of a $\zeta=$constant surface that lies *inside* the flux surface. The Jacobian for the Boozer coordinates is given by

$$\sqrt{g_B}\equiv[\nabla s\cdot(\nabla\theta_B\times\nabla\zeta_B)]^{-1}=\frac{V'(s)}{4\pi^2}\frac{\langle B^2\rangle}{B^2}. \tag{A2}$$

Next, in the Hamada[37] coordinates (s,θ_H,ζ_H), the contravariant poloidal and toroidal components of the magnetic field $\mathbf{B}$ and the Jacobian $\sqrt{g_H}$ are flux-surface functions written as

$$B^\theta_{(\text{Hamada})}\equiv\mathbf{B}\cdot\nabla\theta_H=\frac{4\pi^2}{V'(s)}\chi'(s),$$
$$B^\zeta_{(\text{Hamada})}\equiv\mathbf{B}\cdot\nabla\zeta_H=\frac{4\pi^2}{V'(s)}\psi'(s), \tag{A3}$$
$$\sqrt{g_H}\equiv[\nabla s\cdot(\nabla\theta_H\times\nabla\zeta_H)]^{-1}=\frac{V'(s)}{4\pi^2},$$

respectively. Here, the poloidal and toroidal fluxes are given by $2\pi\chi(s)=(2\pi)^{-1}\int_{V(s)}d^3x\mathbf{B}\cdot\nabla\theta$ and $2\pi\psi(s)=(2\pi)^{-1}\int_{V(s)}d^3x\mathbf{B}\cdot\nabla\zeta$, respectively, $V(s)$ represents the volume enclosed by the flux surface with the label s, and the derivative with respect to s is denoted by $'=d/ds$. Then, we find that, using the Hamada coordinates, Eq. (12) is easily solved to yield

$$\tilde{U}=\frac{B}{\chi'}\left(\frac{B_\zeta^{(\text{Hamada})}}{B^2}-\frac{\langle B_\zeta^{(\text{Hamada})}\rangle}{\langle B^2\rangle}\right)$$
$$=-\frac{B}{\psi'}\left(\frac{B_\theta^{(\text{Hamada})}}{B^2}-\frac{\langle B_\theta^{(\text{Hamada})}\rangle}{\langle B^2\rangle}\right), \tag{A4}$$

where $B_\theta^{(\text{Hamada})}\equiv\mathbf{B}\cdot(\partial\mathbf{x}/\partial\theta_H)$ and $B_\zeta^{(\text{Hamada})}\equiv\mathbf{B}\cdot(\partial\mathbf{x}/\partial\zeta_H)$. We should also note that $\langle B_\theta^{(\text{Hamada})}\rangle=B_\theta^{(\text{Boozer})}$ and $\langle B_\zeta^{(\text{Hamada})}\rangle=B_\zeta^{(\text{Boozer})}$.

The transformation from the Boozer to Hamada coordinates[40] are written in terms of the generating function G as

$$\theta_H=\theta_B+\chi' G(s,\theta_B,\zeta_B),$$
$$\zeta_H=\zeta_B+\psi' G(s,\theta_B,\zeta_B). \tag{A5}$$

Here, the generating function $G(s,\theta_B,\zeta_B)$ is periodic in θ_B and ζ_B and satisfies the magnetic differential equation

$$\mathbf{B}\cdot\nabla G=\frac{1}{\sqrt{g_H}}-\frac{1}{\sqrt{g_B}}, \tag{A6}$$

which is rewritten as

$$\left(\psi'\frac{\partial}{\partial\zeta_B}+\chi'\frac{\partial}{\partial\theta_B}\right)G=\frac{\langle B^2\rangle}{B^2}-1. \tag{A7}$$

Comparing Eq. (12) with (A7), we find that $\tilde{U}$ is related to G by

$$\tilde{U}=\frac{V'}{4\pi^2 B}\mathbf{B}\times\nabla s\cdot\nabla G$$
$$=\frac{B}{\langle B^2\rangle}\left(B_\zeta^{(\text{Boozer})}\frac{\partial G}{\partial\theta_B}-B_\theta^{(\text{Boozer})}\frac{\partial G}{\partial\zeta_B}\right)$$
$$=\frac{1}{B}\left(B_\zeta^{(\text{Hamada})}\frac{\partial G}{\partial\theta_H}-B_\theta^{(\text{Hamada})}\frac{\partial G}{\partial\zeta_H}\right). \tag{A8}$$

From Eq. (A5), we obtain

$$\frac{\partial\mathbf{x}}{\partial\theta_B}=\frac{\partial\mathbf{x}}{\partial\theta_H}+\frac{\partial G}{\partial\theta_B}\sqrt{g_H}\mathbf{B},$$
$$\frac{\partial\mathbf{x}}{\partial\zeta_B}=\frac{\partial\mathbf{x}}{\partial\zeta_H}+\frac{\partial G}{\partial\zeta_B}\sqrt{g_H}\mathbf{B}, \tag{A9}$$

and

$$\frac{\partial G}{\partial\theta_B}=\frac{\langle B^2\rangle}{B^2}\frac{\partial G}{\partial\theta_H},$$
$$\frac{\partial G}{\partial\zeta_B}=\frac{\langle B^2\rangle}{B^2}\frac{\partial G}{\partial\zeta_H}, \tag{A10}$$

where the partial derivatives $\partial/\partial\theta_H$ and $\partial/\partial\zeta_H$ are taken with the Hamada coordinates (s,θ_H,ζ_H) used as the independent variables. Using Eqs. (A9) and (A10), we have

$$B_\zeta^{(\text{Boozer})}\frac{\partial\ln B}{\partial\theta_H}-B_\theta^{(\text{Boozer})}\frac{\partial\ln B}{\partial\zeta_H}$$
$$=\left(B_\zeta^{(\text{Hamada})}+\frac{V'}{4\pi^2}\langle B^2\rangle\frac{\partial G}{\partial\zeta_H}\right)\frac{\partial\ln B}{\partial\theta_H}$$
$$-\left(B_\theta^{(\text{Hamada})}+\frac{V'}{4\pi^2}\langle B^2\rangle\frac{\partial G}{\partial\theta_H}\right)\frac{\partial\ln B}{\partial\zeta_H}$$
$$=\frac{V'}{4\pi^2}(\nabla s\times\nabla\ln B)\cdot\left(\mathbf{B}+\frac{V'}{4\pi^2}\langle B^2\rangle\nabla G\right)$$
$$=\frac{B^2}{\langle B^2\rangle}\left[\left(B_\zeta^{(\text{Boozer})}+\frac{V'}{4\pi^2}\langle B^2\rangle\frac{\partial G}{\partial\zeta_B}\right)\frac{\partial\ln B}{\partial\theta_B}\right.$$
$$\left.-\left(B_\theta^{(\text{Boozer})}+\frac{V'}{4\pi^2}\langle B^2\rangle\frac{\partial G}{\partial\theta_B}\right)\frac{\partial\ln B}{\partial\zeta_B}\right], \tag{A11}$$

which is useful when evaluating $G_a^{(\text{BS})}$ and L_a for the Pfirsch–Schlüter regime [see Eqs. (47) and (50)]. We also find from Eq. (A7) that

$$\mathbf{B}\cdot\nabla\frac{\partial G}{\partial\theta_B}=-\frac{2}{\sqrt{g_H}}\frac{\partial\ln B}{\partial\theta_B},$$
$$\mathbf{B}\cdot\nabla\frac{\partial G}{\partial\zeta_B}=-\frac{2}{\sqrt{g_H}}\frac{\partial\ln B}{\partial\zeta_B}. \tag{A12}$$

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Here, let us assume that $c_1\partial B/\partial\theta_B + c_2\partial B/\partial\zeta_B = 0$, where c_1 and c_2 are constants and $(c_1,c_2)\neq(0,0)$. This condition implies that the magnetic field strength is written as $B=B(s,c_2\theta_B - c_1\zeta_B)$, and it is satisfied approximately in quasi-symmetric systems, where the neoclassical ripple transport is suppressed. The axisymmetric, poloidally symmetric, and helically symmetric cases correspond to $c_1=0$, $c_2=0$, and $c_1\cdot c_2\neq 0$, respectively. Under this symmetry condition $c_1\partial B/\partial\theta_B + c_2\partial B/\partial\zeta_B=0$, we find from Eq. (A12) that $c_1\partial G/\partial\theta_B + c_2\partial G/\partial\zeta_B$ is a flux surface function, and therefore, $c_1\partial G/\partial\theta_B + c_2\partial G/\partial\zeta_B = \langle c_1\partial G/\partial\theta_B + c_2\partial G/\partial\zeta_B\rangle = 0$. Then, we also obtain $c_1\partial\mathbf{x}/\partial\theta_B + c_2\partial\mathbf{x}/\partial\zeta_B = c_1\partial\mathbf{x}/\partial\theta_H + c_2\partial\mathbf{x}/\partial\zeta_H$ and $c_1\partial G/\partial\theta_H + c_2\partial G/\partial\zeta_H=0$ from Eqs. (A9) and (A10), respectively. Consequently, we have $c_1\partial B/\partial\theta_H + c_2\partial B/\partial\zeta_H = c_1\partial B/\partial\theta_B + c_2\partial B/\partial\zeta_B = 0$. Inversely, if $c_1\partial B/\partial\theta_H + c_2\partial B/\partial\zeta_H=0$ is assumed, $c_1\partial B/\partial\theta_B + c_2\partial B/\partial\zeta_B=0$ is concluded. The equivalent conditions described above are summarized as

$$c_1\frac{\partial B}{\partial\theta_B}+c_2\frac{\partial B}{\partial\zeta_B}=0 \Leftrightarrow c_1\frac{\partial B}{\partial\theta_H}+c_2\frac{\partial B}{\partial\zeta_H}=0$$
$$\Leftrightarrow c_1\frac{\partial G}{\partial\theta_B}+c_2\frac{\partial G}{\partial\zeta_B}=0 \Leftrightarrow c_1\frac{\partial G}{\partial\theta_H}+c_2\frac{\partial G}{\partial\zeta_H}=0$$
$$\Leftrightarrow c_1\frac{\partial \mathbf{x}}{\partial\theta_B}+c_2\frac{\partial \mathbf{x}}{\partial\zeta_B}=c_1\frac{\partial \mathbf{x}}{\partial\theta_H}+c_2\frac{\partial \mathbf{x}}{\partial\zeta_H}. \tag{A13}$$

Thus, either Boozer or Hamada coordinates can be used to describe the symmetry condition for the magnetic field strength to suppress the neoclassical ripple transport.

APPENDIX B: POLOIDAL AND TOROIDAL VISCOSITY COEFFICIENTS

The poloidal and toroidal flows can be linearly related to the parallel flows and the radial gradient forces as

$$\begin{bmatrix}\langle u_a^\theta\rangle/\chi' \\ \langle u_a^\zeta\rangle/\psi'\end{bmatrix} = \frac{4\pi^2}{V'}\begin{bmatrix}1 & -cB_\zeta^{(\mathrm{Boozer})}/(e_a\chi'\langle B^2\rangle) \\ 1 & cB_\theta^{(\mathrm{Boozer})}/(e_a\psi'\langle B^2\rangle)\end{bmatrix} \times\begin{bmatrix}\langle u_{\|a}B\rangle/\langle B^2\rangle \\ X_{a1}\end{bmatrix},$$

$$\begin{bmatrix}\dfrac{2}{5p_a}\langle q_a^\theta\rangle/\chi' \\ \dfrac{2}{5p_a}\langle q_a^\zeta\rangle/\psi'\end{bmatrix} = \frac{4\pi^2}{V'}\begin{bmatrix}1 & -cB_\zeta^{(\mathrm{Boozer})}/(e_a\chi'\langle B^2\rangle) \\ 1 & cB_\theta^{(\mathrm{Boozer})}/(e_a\psi'\langle B^2\rangle)\end{bmatrix} \times\begin{bmatrix}\dfrac{2}{5p_a}\langle q_{\|a}B\rangle/\langle B^2\rangle \\ X_{a2}\end{bmatrix}, \tag{B1}$$

where the flux-surface-averaged poloidal and toroidal flows in the left-hand side do not depend on what flux coordinates (s,θ,ζ) are chosen. From Eq. (17), we obtain

$$\begin{bmatrix}\sigma_{Pa} \\ \sigma_{Ta}\end{bmatrix} = \frac{4\pi^2}{V'}\begin{bmatrix}\chi' B_\theta^{(\mathrm{Boozer})}/\langle B^2\rangle & -\dfrac{e_a}{c}\psi'\chi' \\ \psi' B_\zeta^{(\mathrm{Boozer})}/\langle B^2\rangle & \dfrac{e_a}{c}\psi'\chi'\end{bmatrix}\begin{bmatrix}\sigma_{Ua} \\ \sigma_{Xa}\end{bmatrix}. \tag{B2}$$

Then, we find from Eqs. (18), (19), and (B2) that the poloidal and toroidal viscosities are written in terms of the parallel viscosities and the radial fluxes as

$$\begin{bmatrix}\langle \mathbf{B}_P\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle & \langle \mathbf{B}_P\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle \\ \langle \mathbf{B}_T\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle & \langle \mathbf{B}_T\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle\end{bmatrix} = \frac{4\pi^2}{V'}\begin{bmatrix}\chi' B_\theta^{(\mathrm{Boozer})}/\langle B^2\rangle & -\dfrac{e_a}{c}\psi'\chi' \\ \psi' B_\zeta^{(\mathrm{Boozer})}/\langle B^2\rangle & \dfrac{e_a}{c}\psi'\chi'\end{bmatrix} \times\begin{bmatrix}\langle \mathbf{B}\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle & \langle \mathbf{B}\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle \\ \Gamma_a^{\mathrm{bn}} & q_a^{\mathrm{bn}}/T_a\end{bmatrix}. \tag{B3}$$

Using Eqs. (35), (38), (B1), and (B3), we obtain the relations between the poloidal and toroidal viscosity coefficients $(M_{ajPP},M_{ajPT},M_{ajTT})$ and the coefficients (M_{aj},N_{aj},L_{aj}) for the parallel viscosities and the radial fluxes

$$\begin{bmatrix}M_{ajPP} & M_{ajPT} \\ M_{ajPT} & M_{ajTT}\end{bmatrix} = \frac{4\pi^2}{V'}\begin{bmatrix}\chi' B_\theta^{(\mathrm{Boozer})}/\langle B^2\rangle & -\dfrac{e_a}{c}\psi'\chi' \\ \psi' B_\zeta^{(\mathrm{Boozer})}/\langle B^2\rangle & \dfrac{e_a}{c}\psi'\chi'\end{bmatrix}\begin{bmatrix}M_{aj} & N_{aj} \\ N_{aj} & L_{aj}\end{bmatrix} \times\begin{bmatrix}\chi' B_\theta^{(\mathrm{Boozer})}/\langle B^2\rangle & \psi' B_\zeta^{(\mathrm{Boozer})}/\langle B^2\rangle \\ -\dfrac{e_a}{c}\psi'\chi' & \dfrac{e_a}{c}\psi'\chi'\end{bmatrix}, \tag{B4}$$

and correspondingly those between the monoenergetic coefficients $[M_a(K),N_a(K),L_a(K)]$ and $[M_{aPP}(K),M_{aPT}(K), M_{aTT}(K)]$,

$$\begin{bmatrix}M_{aPP}(K) & M_{aPT}(K) \\ M_{aPT}(K) & M_{aTT}(K)\end{bmatrix} = \frac{4\pi^2}{V'}\begin{bmatrix}\chi' B_\theta^{(\mathrm{Boozer})}/\langle B^2\rangle & -\dfrac{e_a}{c}\psi'\chi' \\ \psi' B_\zeta^{(\mathrm{Boozer})}/\langle B^2\rangle & \dfrac{e_a}{c}\psi'\chi'\end{bmatrix} \times\begin{bmatrix}M_a(K) & N_a(K) \\ N_a(K) & L_a(K)\end{bmatrix} \times\begin{bmatrix}\chi' B_\theta^{(\mathrm{Boozer})}/\langle B^2\rangle & \psi' B_\zeta^{(\mathrm{Boozer})}/\langle B^2\rangle \\ -\dfrac{e_a}{c}\psi'\chi' & \dfrac{e_a}{c}\psi'\chi'\end{bmatrix}. \tag{B5}$$

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APPENDIX C: NEOCLASSICAL TRANSPORT COEFFICIENTS FOR RADIAL FLUXES AND PARALLEL CURRENTS

Integrating Eq. (3) multiplied by $m_a v_\parallel$ and $m_a v_\parallel(m_a v^2/2T_a - 5/2)$ and taking the flux surface average give the parallel momentum balance equations

$$\langle \mathbf{B}\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle - n_a e_a\langle BE_\parallel\rangle = \langle BF_{\parallel a1}\rangle,$$
$$\langle \mathbf{B}\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle = \langle BF_{\parallel a2}\rangle. \tag{C1}$$

The parallel friction forces $F_{\parallel a1}\equiv\int d^3v\, m_a v_\parallel C_a^L(f_{a1})$ and $F_{\parallel a2}\equiv\int d^3v\, m_a v_\parallel(m_a v^2/2T_a - 5/2)C_a^L(f_{a1})$ in the right-hand side of Eq. (C1) are related to the parallel flows $u_{\parallel a}$ and $q_{\parallel a}$ by the friction-flow relations (in the 13M approximation)

$$\begin{bmatrix}\langle BF_{\parallel a1}\rangle\\ \langle BF_{\parallel a2}\rangle\end{bmatrix} = \sum_b \begin{bmatrix} l_{11}^{ab} & -l_{12}^{ab}\\ -l_{21}^{ab} & l_{22}^{ab}\end{bmatrix}\begin{bmatrix}\langle Bu_{\parallel b}\rangle\\ \dfrac{2}{5p_b}\langle Bq_{\parallel b}\rangle\end{bmatrix}, \tag{C2}$$

where the coefficients l_{jk}^{ab} are defined by Eq. (4.4) in Hirshman and Sigmar,[2] and satisfy the conditions $l_{jk}^{ab}=l_{kj}^{ba}$ and $\Sigma_a l_{1k}^{ab}=0$, which are derived from the self-adjointness and the momentum conservation property of the linearized collision operator, respectively. The parallel viscosities $\langle \mathbf{B}\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle$ and $\langle \mathbf{B}\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle$ in the left-hand side of Eq. (C1) are written by Eq. (35) in terms of the parallel flows $\langle Bu_{\parallel a}\rangle$ and $\langle Bq_{\parallel a}\rangle$. Then, combining Eqs. (35), (C1), and (C2), we obtain

$$\sum_{b\neq e}\left(\frac{\delta_{ab}}{\langle B^2\rangle}\begin{bmatrix}M_{a1} & M_{a2}\\ M_{a2} & M_{a3}\end{bmatrix} - \begin{bmatrix} l_{11}^{ab} & -l_{12}^{ab}\\ -l_{21}^{ab} & l_{22}^{ab}\end{bmatrix}\right)\begin{bmatrix}\langle Bu_{\parallel b}\rangle\\ \dfrac{2}{5p_b}\langle Bq_{\parallel b}\rangle\end{bmatrix}$$
$$= -\begin{bmatrix}N_{a1} & N_{a2}\\ N_{a2} & N_{a3}\end{bmatrix}\begin{bmatrix}X_{a1}\\ X_{a2}\end{bmatrix} + \begin{bmatrix}n_a e_a\langle BE_\parallel\rangle\\ 0\end{bmatrix}$$
$$+\begin{bmatrix} l_{11}^{ae} & -l_{12}^{ae}\\ -l_{21}^{ae} & l_{22}^{ae}\end{bmatrix}\begin{bmatrix}\langle Bu_{\parallel e}\rangle\\ \dfrac{2}{5p_e}\langle Bq_{\parallel e}\rangle\end{bmatrix}$$
$$\simeq -\begin{bmatrix}N_{a1} & N_{a2}\\ N_{a2} & N_{a3}\end{bmatrix}\begin{bmatrix}X_{a1}\\ X_{a2}\end{bmatrix} \quad \text{for ion species } a(\neq e) \tag{C3}$$

and

$$\left(\frac{1}{\langle B^2\rangle}\begin{bmatrix}M_{e1} & M_{e2}\\ M_{e2} & M_{e3}\end{bmatrix} - \begin{bmatrix} l_{11}^{ee} & -l_{12}^{ee}\\ -l_{21}^{ee} & l_{22}^{ee}\end{bmatrix}\right)\begin{bmatrix}\langle Bu_{\parallel e}\rangle\\ \dfrac{2}{5p_e}\langle Bq_{\parallel e}\rangle\end{bmatrix}$$
$$= -\begin{bmatrix}N_{e1} & N_{e2}\\ N_{e2} & N_{e3}\end{bmatrix}\begin{bmatrix}X_{e1}\\ X_{e2}\end{bmatrix} - \begin{bmatrix}n_e e\langle BE_\parallel\rangle\\ 0\end{bmatrix}$$
$$+\sum_{a\neq e}\begin{bmatrix} l_{11}^{ea} & -l_{12}^{ea}\\ -l_{21}^{ea} & l_{22}^{ea}\end{bmatrix}\begin{bmatrix}\langle Bu_{\parallel a}\rangle\\ \dfrac{2}{5p_a}\langle Bq_{\parallel a}\rangle\end{bmatrix}. \tag{C4}$$

Here, general cases of multispecies of ions are considered. We should note that $m_e/m_a \ll 1$ for ion species a and that the parallel electric field term and the ion–electron friction term in Eq. (C3), are smaller than the other terms by a factor of $\mathcal{O}[(m_e/m_a)^{1/2}]$. Then, neglecting these $\mathcal{O}[(m_e/m_a)^{1/2}]$ terms in Eq. (C3), the lowest-order parallel flows $\langle Bu_{\parallel a}\rangle$ and $\langle Bq_{\parallel a}\rangle$ for ion species $a(\neq e)$ can be expressed as a linear combination of the thermodynamic forces X_{b1} and X_{b2} $(b\neq e)$, and these expressions are substituted into Eq. (C4) in order to write the electron parallel flows $\langle Bu_{\parallel e}\rangle$ and $\langle Bq_{\parallel e}\rangle$ in terms of the thermodynamic forces X_{e1}, X_{e2}, X_{b1}, X_{b2} $(b\neq e)$, and $\langle BE_\parallel\rangle$. Substituting these expressions of $\langle Bu_{\parallel e}\rangle$ and $\langle Bq_{\parallel e}\rangle$ in turn into Eq. (C3), the parallel ion flows $\langle Bu_{\parallel a}\rangle$ and $\langle Bq_{\parallel a}\rangle$ $(a\neq e)$ of the next order can be given in terms of the ion and electron thermodynamic forces. Once the relations of the parallel flows to the thermodynamic forces are obtained for all species a, substituting them into Eqs. (35) and (B1) and using Eq. (B3) yield the expressions of the radial neoclassical fluxes $[\Gamma_a^{\rm bn}, q_a^{\rm bn}]$, the parallel, poloidal, and toroidal viscosities $[\langle \mathbf{B}\cdot(\nabla\cdot\pi_a)\rangle, \langle \mathbf{B}\cdot(\nabla\cdot\Theta_a)\rangle, \langle \mathbf{B}_P\cdot(\nabla\cdot\pi_a)\rangle, \langle \mathbf{B}_P\cdot(\nabla\cdot\Theta_a)\rangle, \langle \mathbf{B}_T\cdot(\nabla\cdot\pi_a)\rangle, \langle \mathbf{B}_T\cdot(\nabla\cdot\Theta_a)\rangle]$, and the poloidal and toroidal flows $[\langle u_a^\theta\rangle, \langle q_a^\theta\rangle, \langle u_a^\zeta\rangle, \langle q_a^\zeta\rangle]$, in terms of the thermodynamic forces $[X_{e1}, X_{e2}, X_{b1}, X_{b2}(b\neq e), \langle BE_\parallel\rangle]$.

Applying the procedures described above to the case of a toroidal plasma consisting of electrons and a single species of ions, we can derive the following transport equations for the neoclassical radial fluxes of particles and heat and the neoclassical parallel electric current (bootstrap current)

$$\begin{bmatrix}\Gamma_e^{\rm bn}\\ q_e^{\rm bn}/T_e\\ \Gamma_i^{\rm bn}\\ q_i^{\rm bn}/T_i\\ J_E^{\rm BS}\end{bmatrix} = \begin{bmatrix} L_{11}^{ee} & L_{12}^{ee} & L_{11}^{ei} & L_{12}^{ei} & L_{1E}^{e}\\ L_{21}^{ee} & L_{22}^{ee} & L_{21}^{ei} & L_{22}^{ei} & L_{2E}^{e}\\ L_{11}^{ie} & L_{12}^{ie} & L_{11}^{ii} & L_{12}^{ii} & L_{1E}^{i}\\ L_{21}^{ie} & L_{22}^{ie} & L_{21}^{ii} & L_{22}^{ii} & L_{2E}^{i}\\ L_{E1}^{e} & L_{E2}^{e} & L_{E1}^{i} & L_{E2}^{i} & L_{EE}\end{bmatrix}\begin{bmatrix}X_{e1}\\ X_{e2}\\ X_{i1}\\ X_{i2}\\ X_E\end{bmatrix}, \tag{C5}$$

where the force X_E associated with the parallel electric field is denoted by

$$X_E \equiv \langle BE_\parallel\rangle/\langle B^2\rangle^{1/2}, \tag{C6}$$

and the bootstrap current $J_E^{\rm BS}$ is defined by the difference between the total parallel electric current J_E and the classical parallel electric current $J_E^{\rm cl}$,

$$J_E^{\rm BS} \equiv J_E - J_E^{\rm cl} \equiv n_e e\langle B(u_{\parallel i} - u_{\parallel e})\rangle/\langle B^2\rangle^{1/2} - \sigma_{\rm S} X_E, \tag{C7}$$

with the classical Spitzer conductivity $\sigma_{\rm S} \equiv (n_e e^2\tau_{ee}/m_e)\hat{l}_{22}^e/[\hat{l}_{11}^e\hat{l}_{22}^e - (\hat{l}_{12}^e)^2]$. Here, the dimensionless friction coefficients $\hat{l}_{ij}^a \equiv -(\tau_{aa}/n_a m_a)l_{ij}^{aa}$ are given by $\hat{l}_{11}^e = Z_i$, $\hat{l}_{12}^e = \frac{3}{2}Z_i$, $\hat{l}_{22}^e = \sqrt{2} + \frac{13}{4}Z_i$, and $\hat{l}_{22}^i = \sqrt{2}$ with the ion charge number Z_i, and high-order terms with respect to $(m_e/m_i)^{1/2}$ are neglected. Defining the 2×2 matrices for electrons and ions $(a=e,i)$ by

$$\mathcal{L}_a \equiv \begin{bmatrix}L_{a1} & L_{a2}\\ L_{a2} & L_{a3}\end{bmatrix}, \quad \mathcal{M}_a \equiv \frac{\tau_{aa}}{n_a m_a\langle B^2\rangle}\begin{bmatrix}M_{a1} & M_{a2}\\ M_{a2} & M_{a3}\end{bmatrix},$$
$$\mathcal{N}_a \equiv \frac{\tau_{aa}}{n_a m_a}\begin{bmatrix}N_{a1} & N_{a2}\\ N_{a2} & N_{a3}\end{bmatrix}, \quad \mathcal{E}_{11} \equiv \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \tag{C8}$$
$$\Lambda_e \equiv \begin{bmatrix}\hat{l}_{11}^e & -\hat{l}_{12}^e\\ -\hat{l}_{12}^e & \hat{l}_{22}^e\end{bmatrix}, \quad \Lambda_i \equiv \begin{bmatrix}0 & 0\\ 0 & \hat{l}_{22}^i\end{bmatrix},$$

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the transport coefficients in Eq. (C5) are explicitly given by

$$\begin{bmatrix} L_{11}^{aa} & L_{12}^{aa} \\ L_{21}^{aa} & L_{22}^{aa} \end{bmatrix} = \mathcal{L}_a - \frac{n_a m_a}{\tau_{aa}\langle B^2\rangle}\mathcal{N}_a(\mathcal{M}_a+\Lambda_a)^{-1}\mathcal{N}_a + \delta_{ai}\frac{n_e m_e}{\tau_{ee}\langle B^2\rangle}\mathcal{N}_i(\mathcal{M}_i+\Lambda_i)^{-1}\mathcal{E}_{11}(\mathcal{M}_e^{-1} + \Lambda_e^{-1})^{-1}\mathcal{E}_{11}(\mathcal{M}_i+\Lambda_i)^{-1}\mathcal{N}_i, \tag{C9}$$

$$\begin{bmatrix} L_{11}^{ei} & L_{12}^{ei} \\ L_{21}^{ei} & L_{22}^{ei} \end{bmatrix} = \begin{bmatrix} L_{11}^{ie} & L_{21}^{ie} \\ L_{12}^{ie} & L_{22}^{ie} \end{bmatrix} = -\frac{n_e m_e}{\tau_{ee}\langle B^2\rangle}\mathcal{N}_e(\mathcal{M}_e+\Lambda_e)^{-1}\Lambda_e\mathcal{E}_{11}(\mathcal{M}_i + \Lambda_i)^{-1}\mathcal{N}_i, \tag{C10}$$

$$[L_{E1}^e \;\; L_{E2}^e] = -[L_{1E}^e \;\; L_{2E}^e] = \frac{n_e e}{\langle B^2\rangle^{1/2}}[1 \;\; 0](\mathcal{M}_e+\Lambda_e)^{-1}\mathcal{N}_e, \tag{C11}$$

$$[L_{E1}^i \;\; L_{E2}^i] = -[L_{1E}^i \;\; L_{2E}^i] = -\frac{n_e e}{\langle B^2\rangle^{1/2}}[1 \;\; 0](\mathcal{M}_e+\Lambda_e)^{-1}\mathcal{M}_e\mathcal{E}_{11} \times(\mathcal{M}_i+\Lambda_i)^{-1}\mathcal{N}_i, \tag{C12}$$

$$L_{EE} = -\frac{n_e e^2 \tau_{ee}}{m_e}[1 \;\; 0]\{\Lambda_e^{-1} - (\mathcal{M}_e+\Lambda_e)^{-1}\}\begin{bmatrix}1\\0\end{bmatrix}. \tag{C13}$$

In the right-hand side of Eq. (C9), the term with δ_{ai} ($\equiv 1$ for $a=i$, 0 for $a=e$), which is of $\mathcal{O}[(m_e/m_i)^{1/2}]$, is kept in order to reproduce the intrinsic ambipolar particle fluxes $\Gamma_i = Z_i^{-1}\Gamma_e$ in the symmetric case (see Appendix D). It should be noted that the transport coefficients given in Eqs. (C9)–(C13) satisfy the Onsager relations

$$L_{jk}^{ab} = L_{kj}^{ba}, \quad L_{jE}^a = -L_{Ej}^a \quad (a,b=e,i; j,k=1,2). \tag{C14}$$

APPENDIX D: SYMMETRIC CASE

Here, we consider the symmetric case, in which $c_1\partial B/\partial\theta_B + c_2\partial B/\partial\zeta_B = 0$ holds. It should be recalled that the axisymmetric, poloidally symmetric, and helically symmetric cases correspond to $c_1=0$, $c_2=0$, and $c_1\cdot c_2\neq 0$, respectively. As shown in Eq. (A13), this case is also described by $c_1\partial B/\partial\theta_H + c_2\partial B/\partial\zeta_H = 0$. Then, Eqs. (17) and (B2) yield

$$c_1\frac{\sigma_{Pa}}{\chi'} + c_2\frac{\sigma_{Ta}}{\psi'} = 0,$$

$$\frac{c_1 B_\theta^{(\text{Boozer})} + c_2 B_\zeta^{(\text{Boozer})}}{\langle B^2\rangle}\sigma_{Ua} + (-c_1\psi' + c_2\chi')\frac{e_a}{c}\sigma_{Xa} = 0, \tag{D1}$$

$$\frac{c_1 B_\theta^{(\text{Boozer})} + c_2 B_\zeta^{(\text{Boozer})}}{\langle B^2\rangle}G_{Ua} + (-c_1\psi' + c_2\chi')\frac{e_a}{c}G_{Xa} = 0.$$

Thus, we find from Eq. (18) that the viscosities and the viscosity coefficients associated with the symmetry direction vanish

$$\frac{c_1}{\chi'}\langle \mathbf{B}_P\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle + \frac{c_2}{\psi'}\langle \mathbf{B}_T\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle = \frac{c_1}{\chi'}\langle \mathbf{B}_P\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle + \frac{c_2}{\psi'}\langle \mathbf{B}_T\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle = 0,$$

$$\frac{c_1}{\chi'}M_{ajPP} + \frac{c_2}{\psi'}M_{ajPT} = \frac{c_1}{\chi'}M_{ajPT} + \frac{c_2}{\psi'}M_{ajTT} = 0. \tag{D2}$$

The expressions for the banana-plateau particle and heat fluxes for the symmetric case in terms of the parallel viscosities are derived from Eqs. (18), (19), and (D1) as

$$\Gamma_a^{\text{bp}} = \frac{c(c_1 B_\theta^{(\text{Boozer})} + c_2 B_\zeta^{(\text{Boozer})})}{e_a(c_1\psi' - c_2\chi')\langle B^2\rangle}\langle \mathbf{B}\cdot(\nabla\cdot\boldsymbol{\pi}_a)\rangle,$$

$$\frac{q_a^{\text{bp}}}{T_a} = \frac{c(c_1 B_\theta^{(\text{Boozer})} + c_2 B_\zeta^{(\text{Boozer})})}{e_a(c_1\psi' - c_2\chi')\langle B^2\rangle}\langle \mathbf{B}\cdot(\nabla\cdot\boldsymbol{\Theta}_a)\rangle. \tag{D3}$$

Using Eq. (D3) and the parallel momentum balance in Eq. (C1) with the charge neutrality condition $\Sigma_a n_a e_a = 0$, we obtain the well-known intrinsic ambipolarity condition that, in the symmetric case, $\Sigma_a e_a \Gamma_a^{\text{bp}} = 0$ is satisfied for arbitrary values of the thermodynamic forces.

Equations (36), (37), (46), and (D1) give the relations of the coefficients $[M_{aj}, N_{aj}, L_{aj}]$ and $[M_a(K), N_a(K), L_a(K)]$,

$$\frac{N_{aj}}{M_{aj}} = \frac{L_{aj}}{N_{aj}} = \frac{N_a(K)}{M_a(K)} = \frac{L_a(K)}{N_a(K)} = \frac{c(c_1 B_\theta^{(\text{Boozer})} + c_2 B_\zeta^{(\text{Boozer})})}{e_a(c_1\psi' - c_2\chi')\langle B^2\rangle}, \tag{D4}$$

and the geometric factor $G_a^{(\text{BS})}$,

$$G_a^{(\text{BS})} = \frac{c_1 B_\theta^{(\text{Boozer})} + c_2 B_\zeta^{(\text{Boozer})}}{-c_1\psi' + c_2\chi'}, \tag{D5}$$

for the symmetric case.

APPENDIX E: EFFECTS OF THE E×B DRIFT

In the left-hand side of Eq. (3), the collisionless orbit operator $V_\parallel$ contains only the part of particles' parallel motion because other drift motions are neglected as higher-order terms in the gyroradius expansion. Here, in order to consider additional effects of the $\mathbf{E}\times\mathbf{B}$ drift on the neoclassical transport coefficients, we use the drift kinetic equation given by

$$Vf_{a1} - C_a^L(f_{a1}) = -\mathbf{v}_{da}\cdot\nabla f_{aM} + \frac{e_a}{T_a}v_\parallel B\frac{\langle BE_\parallel\rangle}{\langle B^2\rangle}f_{aM}, \tag{E1}$$

where the operator $V \equiv V_\parallel + V_E$ consists of the parallel motion part $V_\parallel$ given by Eq. (5) and the $\mathbf{E}\times\mathbf{B}$ drift part V_E defined by

$$V_E \equiv \mathbf{v}_E\cdot\nabla \equiv \frac{cE_s}{\langle B^2\rangle}\nabla s\times\mathbf{B}\cdot\nabla, \tag{E2}$$

with ∇ taken for (v,ξ) being fixed. The $\mathbf{E}\times\mathbf{B}$ drift operator V_E given by Eq. (E2) has the same form as employed in the DKES[19,20] and by Taguchi.[32] Here, following Taguchi,[32] we

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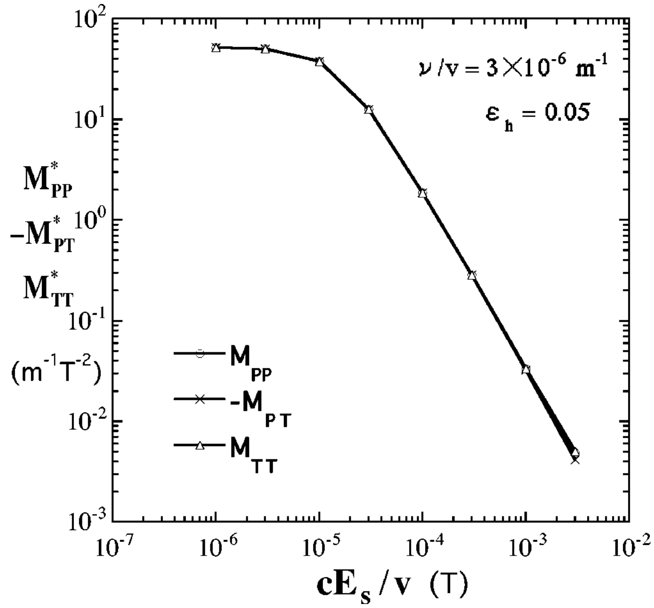

FIG. 6. Poloidal and toroidal viscosity coefficients as a function of cE_s/v for $\epsilon_h=0.05$ and $\nu_D/v=3\times10^{-6}$. Other parameters are the same as in Sec. IV. Curves with circles, crosses, and triangles represent M^*_{PP}, $-M^*_{PT}$, and M^*_{PP}, respectively.

neglect effects of $\mathbf{v}_E\cdot\nabla\{\int d^3v f_{a1}[1,(\frac{1}{2}m_a v^2-\frac{5}{2}T_a)]\}$ in the density and energy balance equations, and accordingly assume that the incompressibility conditions in Eq. (8) and the expressions for the local parallel flows in Eq. (11) are still valid. Then, using Eq. (13), Eq. (E1) is rewritten as

$$Vg_a - C^L_a(g_a) = H^-_a + H^+_a . \tag{E3}$$

Here, H^-_a is equal to $H^{(l=1)}_a$ given by Eq. (15) and H^+_a is written in the same expression as in the right-hand side of Eq. (16) if we note that σ_{Ua}, σ_{Xa}, σ_{Pa}, and σ_{Ta} now include the $\mathbf{E}\times\mathbf{B}$ terms and are redefined by

$$\begin{aligned}
\sigma_{Ua} &= -m_a v^2 P_2(\xi)\mathbf{B}\cdot\nabla\ln B - V_E(m_a v\xi B)\\
&= -V(m_a v\xi B),\\
\sigma_{Xa} &= -v^2 P_2(\xi)\frac{\mathbf{b}\cdot\nabla(B\widetilde{U})}{2\Omega_a} - V_E\left(\frac{B}{\Omega_a}v\xi\widetilde{U}\right),\\
\sigma_{Pa} &= -m_a v^2 P_2(\xi)\mathbf{B}_P\cdot\nabla\ln B - V_E(m_a v\xi\mathbf{B}_P\cdot\mathbf{b}),\\
\sigma_{Ta} &= -m_a v^2 P_2(\xi)\mathbf{B}_T\cdot\nabla\ln B - V_E(m_a v\xi\mathbf{B}_P\cdot\mathbf{b}),
\end{aligned} \tag{E4}$$

respectively. The superscripts + and − in H^+_a and H^-_a represent the symmetric and anti-symmetric parts with respect to the transformation $(\xi,E_s)\to(-\xi,-E_s)$, respectively. In this appendix, we also assume the stellarator symmetry, $B(s,\theta,\zeta)=B(s,-\theta,-\zeta)$, which is satisfied by practically all helical devices. Then, all neoclassical transport coefficients are even functions of E_s.

Using what we have noted above, we can show that, even if $\mathbf{E}\times\mathbf{B}$ drift term is included, Eqs. (18)–(44) in Secs. II and III are still valid by replacing $V_\parallel$ with $V\equiv V_\parallel+V_E$ where it appears. [Note that, by doing this replacement in Eq. (40), definitions of both σ^+_1 and σ^+_3 coincide with those in Rij and Hirshman[20] even for $E_s\neq0$.] Also, Eqs. (46) and (54)–(56) are available. Thus, using these formulas, we can calculate dependence of the neoclassical coefficients on the radial electric field. Figure 6 shows the normalized monoenergetic neoclassical viscosity coefficients M^*_{PP}, M^*_{PT}, and M^*_{TT} as a function of cE_s/v, which are numerically obtained in the same way as in Fig. 5. Here, $\epsilon_h=0.05$ and $\nu_D/v=3\times10^{-6}$ are used while other parameters are the same as in Sec. IV. These parameters correspond to the $1/\nu$ regime for the case of $E_s=0$. In Fig. 6, $M_{PP}\simeq-M_{PT}\simeq M_{TT}$ and their reduction with increasing cE_s/v are clearly seen. The E_s-dependent neoclassical transport coefficients for radial fluxes and parallel currents can also be calculated in the same way as in Appendix C.

[1] F. L. Hinton and R. D. Hazeltine, Rev. Mod. Phys. **48**, 239 (1976).
[2] S. P. Hirshman and D. J. Sigmar, Nucl. Fusion **21**, 1079 (1981).
[3] R. Balescu, *Transport Processes in Plasmas* (2 volumes) (North-Holland, Amsterdam, 1988).
[4] W. Horton, Rev. Mod. Phys. **71**, 735 (1999).
[5] M. Wakatani, *Stellarator and Heliotron Devices* (Oxford University Press, Oxford, 1998), Chap. 7.
[6] L. M. Kovrizhnykh, Nucl. Fusion **24**, 851 (1984).
[7] K. C. Shaing and S. A. Hokin, Phys. Fluids **26**, 2136 (1983).
[8] C. D. Beidler and H. Maaßberg, Plasma Phys. Controlled Fusion **43**, 1131 (2001).
[9] D. E. Hastings, W. A. Houlberg, and K. C. Shaing, Nucl. Fusion **25**, 445 (1985).
[10] J. Nührenberg and R. Zille, Phys. Lett. A **129**, 113 (1988).
[11] P. Garabedian and L.-P. Ku, Phys. Plasmas **6**, 645 (1999).
[12] S. Okamura, K. Matsuoka, S. Nishimura *et al.*, Nucl. Fusion **41**, 1865 (2001).
[13] R. D. Hazeltine, Phys. Plasmas **15**, 77 (1973).
[14] R. D. Hazeltine and J. D. Meiss, *Plasma Confinement* (Addison-Wesley, Redwood City, CA, 1992), p. 113.
[15] A. H. Boozer and G. Kuo-Petravic, Phys. Fluids **24**, 851 (1981).
[16] R. H. Fowler, J. A. Rome, and J. F. Lyon, Phys. Fluids **28**, 338 (1985).
[17] H. E. Mynick and W. N. G. Hitchon, Nucl. Fusion **26**, 425 (1986).
[18] C. D. Beidler, W. N. G. Hitchon, and J. L. Shoet, J. Comput. Phys. **72**, 220 (1987).
[19] S. P. Hirshman, K. C. Shaing, W. I. van Rij, C. O. Beasley, Jr., and E. C. Crume, Jr., Phys. Fluids **29**, 2951 (1986).
[20] W. I. van Rij and S. P. Hirshman, Phys. Fluids B **1**, 563 (1989).
[21] J. Todoroki, J. Phys. Soc. Jpn. **59**, 2758 (1990).
[22] Y. Ogawa, T. Amano, N. Nakajima, N. Ohyabu, K. Yamazaki, S. P. Hirshman, W. I. van Rij, and K. C. Shaing, Nucl. Fusion **32**, 119 (1992).
[23] R. Kanno, N. Nakajima, H. Sugama, M. Okamoto, and Y. Ogawa, Nucl. Fusion **37**, 1463 (1997).
[24] See, for example, p. 164 in Ref. 14.
[25] E. C. Crume, Jr., C. O. Beasley, S. P. Hirshman, and W. I. van Rij, Phys. Fluids **30**, 1152 (1987).
[26] K. C. Shaing, M. Yokoyama, M. Wakatani, and C. T. Hsu, Phys. Plasmas **3**, 965 (1996).
[27] K. C. Shaing, S. P. Hirshman, and J. D. Callen, Phys. Fluids **29**, 521 (1986).
[28] M. Coronado and H. Wobig, Phys. Fluids **29**, 527 (1986).
[29] K. C. Shaing and J. D. Callen, Phys. Fluids **26**, 3315 (1983).
[30] K. C. Shaing, S. P. Hirshman, and J. S. Tolliver, Phys. Fluids **29**, 2548 (1986).
[31] N. Nakajima and M. Okamoto, J. Phys. Soc. Jpn. **61**, 833 (1992).
[32] M. Taguchi, Phys. Fluids B **4**, 3638 (1992).

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[33]H. Sugama and W. Horton, Phys. Plasmas **3**, 304 (1996).
[34]H. Sugama, M. Okamoto, W. Horton, and M. Wakatani, Phys. Plasmas **3**, 2379 (1996).
[35]S. R. de Groot and P. Mazur, *Nonequilibrium Thermodynamics* (North-Holland, Amsterdam, 1962), Chap. 4.
[36]A. H. Boozer, Phys. Fluids **26**, 1288 (1983).
[37]S. Hamada, Nucl. Fusion **2**, 23 (1962).
[38]K. Y. Watanabe, N. Nakajima, M. Okamoto, K. Yamazaki, Y. Nakamura, and M. Wakatani, Nucl. Fusion **35**, 335 (1995).
[39]O. Motojima, H. Yamada, A. Komori *et al.*, Phys. Plasmas **6**, 1843 (1999).
[40]N. Nakajima, J. Todoroki, and M. Okamoto, J. Plasma Fusion Res. **68**, 395 (1992).

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