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Extended gyrokinetic field theory for time-dependent magnetic confinement fields

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A gyrokinetic system of equations for turbulent toroidal plasmas in time-dependent axisymmetric background magnetic fields is derived from the variational principle. Besides governing equations for gyrocenter distribution functions and turbulent electromagnetic fields, the conditions which self-consistently determine the background magnetic fields varying on a transport time scale are obtained by using the Lagrangian, which includes the constraint on the background fields. Conservation laws for energy and toroidal angular momentum of the whole system in the time-dependent background magnetic fields are naturally derived by applying Noether's theorem. It is shown that the ensemble-averaged transport equations of particles, energy, and toroidal momentum given in the present work agree with the results from the conventional recursive formulation with the WKB representation except that collisional effects are disregarded here. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4863426]

I. INTRODUCTION

Numerous studies have so far been done based on gyrokinetic theories and simulations in order to investigate microinstabilities, turbulence, and transport processes in magnetically confined plasmas.[1–4] In conventional gyrokinetic studies, the gyrocenter phase-space variables are defined by using the background magnetic confinement field that is assumed to be independent of time. Recently, several works have been trying to perform long-time gyrokinetic turbulent transport simulations including evolutions of equilibrium profiles[5,6] although they still use the above-mentioned assumption. However, the background or equilibrium magnetic field changes along with the pressure profile on the transport time scale. Therefore, in order to accurately describe the long-time behaviors of the gyrokinetic turbulence, we need to treat the time-dependent background field and show how to determine its time dependence. In this work, the gyrokinetic field theory[7] is extended to derive the conditions which determine the time-dependent magnetic confinement fields in axisymmetric toroidal systems.

Basic equations for a wide range of physical systems including plasmas can be derived from the variational principle, which is useful to elucidate conservation properties.[8] Noting that the gyrokinetic model is an approximate representation of the Vlasov-Poisson-Ampère equations, their conservation laws were investigated by the variational principle in our previous work,[9] where it was shown how they differ from those for the full Vlasov-Maxwell system. In the gyrokinetic field theory, all equations which govern gyrocenter distribution functions and electromagnetic fields, are derived by applying the variation principle to the action integral of the Lagrangian, for the turbulent magnetized plasma system consisting of particles and fields.[7,10] Therefore, Noether's theorem[8] can be utilized to elegantly derive various conservation laws from the symmetry properties of the system. Especially, the toroidal momentum conservation law has been actively investigated in recent works based on the gyrokinetic field theory because the toroidal momentum transport is deeply connected to one of critical issues for plasma confinement studies, which is how to accurately predict profiles of toroidal flows and radial electric fields in tokamaks.[4,11–17] Regarding these conservation laws, they have also been derived from the conventional drift kinetic and gyrokinetic equations based on the recursive formulation.[14–16,18] In the present paper, not only the particle, energy, and toroidal momentum conservation laws in the time-dependent background fields are naturally derived from the extended gyrokinetic field theory but also their ensemble averages are taken with the help of the WKB representation[19] in order to elucidate the consistency between the present results and those from the conventional recursive formulation.

The rest of this paper is organized as follows. Section II presents the action integral of the Lagrangian, from which all governing equations for the gyrocenter motion, distribution functions, turbulent electromagnetic fields, and the time-dependent equilibrium field are derived in Sec. III using the variational principle. It should be noted that, in the present formulation as well as in other Lagrangian and Hamiltonian formulations, we do not treat effects of collisions and external sources such as heating and torque terms, which remain as future subjects. In Sec. IV, useful formulas for the gyrocenter densities and the polarization density are derived from the gyrokinetic Vlasov and Poisson equations obtained in Sec. III. In Sec. V, we consider general infinitesimal transformations of all variables included in the Lagrangian, and find the expression for the resultant variation of the action integral, which gives a general form of conservation laws as a result of Noether's theorem. Then, as specific examples, conservation laws of energy and toroidal angular momentum are derived from the invariances of the system under the time translation and the toroidal rotation. These conservation laws are ensemble-averaged in Sec. VI with the scale separation technique using the WKB representation. Then, the resultant ensemble-averaged particle, energy, and toroidal moment transport equations, which are of the second order in the normalized gyroradius, are shown to agree with the conventional results except that the collisional effects are disregarded in the present results. Finally, conclusions are given in Sec. VII.

II. LAGRANGIAN

All governing equations for the gyrokinetic system considered here is derived from the variational principle

$$\delta\mathcal{I} \equiv \delta\int_{t_1}^{t_2} L dt = 0, \tag{1}$$

where $\mathcal{I}$ denotes the action integral and δ represents the variation. The Lagrangian L is written as

$$\begin{aligned} L = &\sum_a \int d^6\mathbf{Z}_0\, D_a(\mathbf{Z}_0) F_a(\mathbf{Z}_0, t_0) \\ &\times L_a[\mathbf{Z}_a(\mathbf{Z}_0, t_0; t), \dot{\mathbf{Z}}_a(\mathbf{Z}_0, t_0; t); \{\phi, \mathbf{A}_0, \mathbf{A}_1\}] \\ &+ \int d^3\mathbf{x}\, \mathcal{L}_f, \end{aligned} \tag{2}$$

where $\mathcal{L}_f$ represents the Lagrangian density associated with electromagnetic fields [see Eq. (14)]. The single-particle Lagrangian L_a for particle species a is written in terms of the gyrocenter coordinates $\mathbf{Z}_a = (Z_a^i)_{i=1,\cdots,6} = (\mathbf{X}_a, U_a, \mu_a, \xi_a)$ as

$$L_a(\mathbf{Z}, \dot{\mathbf{Z}}; \{\phi, \mathbf{A}_0, \mathbf{A}_1\}) = \frac{e_a}{c}\mathbf{A}_a^* \cdot \dot{\mathbf{X}}_a + \frac{m_a c}{e_a}\mu_a \dot{\xi}_a - H_a, \tag{3}$$

where $\mathbf{X}_a$, U_a, μ_a, and ξ_a denote the gyrocenter position, parallel velocity, magnetic moment, and gyrophase angle, respectively, $\dot{} \equiv d/dt$ represents the time derivative, and $\mathbf{A}_a^*$ is defined by

$$\mathbf{A}_a^* = \mathbf{A}_0(\mathbf{X}_a, t) + \frac{m_a c}{e_a} U_a \mathbf{b}(\mathbf{X}_a, t). \tag{4}$$

[It is noted that, in Ref. 7, the gyrocenter coordinates are denoted by $\overline{\mathbf{Z}}_a = (\overline{\mathbf{X}}_a, \overline{U}_a, \overline{\mu}_a, \overline{\xi}_a)$ instead of $\mathbf{Z}_a = (\mathbf{X}_a, U_a, \mu_a, \xi_a)$.] Here, the vector potential $\mathbf{A}_0$ is associated with the equilibrium magnetic field $\mathbf{B}_0 = \nabla \times \mathbf{A}_0$, which is assumed to be time-dependent, and the gyrocenter Hamiltonian, which is independent of ξ_a, given by

$$H_a = \frac{1}{2} m_a U_a^2 + \mu_a B_0 + e_a \Psi_a, \tag{5}$$

with

$$\begin{aligned} \Psi_a = &\langle \psi_a(\mathbf{Z}_a, t)\rangle_{\xi_a} + \frac{e_a}{2m_a c^2}\langle |\mathbf{A}_1(\mathbf{X}_a + \boldsymbol{\rho}_a, t)|^2\rangle_{\xi_a} \\ &- \frac{e_a}{2B_0}\frac{\partial}{\partial\mu}\langle [\tilde{\psi}_a(\mathbf{Z}_a, t)]^2\rangle_{\xi_a}. \end{aligned} \tag{6}$$

Here and hereafter, the gyrophase-average and gyrophase-dependent parts of an arbitrary periodic gyrophase function $Q(\xi_a)$ are written as

$$\langle Q\rangle_{\xi_a} \equiv \oint \frac{d\xi_a}{2\pi} Q(\xi_a) \quad \text{and} \quad \tilde{Q} \equiv Q - \langle Q\rangle_{\xi_a}, \tag{7}$$

respectively. On the right-hand side of Eq. (6), the gyroradius vector is given by $\boldsymbol{\rho}_a = \mathbf{b}(\mathbf{X}_a, t) \times \mathbf{v}_{a0}(\mathbf{Z}_a, t)/\Omega_a(\mathbf{X}_a, t)$ with the gyrofrequency $\Omega_a = e_a B_0/(m_a c)$, and the field variable ψ_a is defined by

$$\psi_a(\mathbf{Z}_a, t) = \phi(\mathbf{X}_a + \boldsymbol{\rho}_a, t) - \frac{1}{c}\mathbf{v}_{a0}(\mathbf{Z}_a, t)\cdot \mathbf{A}_1(\mathbf{X}_a + \boldsymbol{\rho}_a, t), \tag{8}$$

where ϕ and $\mathbf{A}_1$ denote the electrostatic potential and the perturbation part of the vector potential, respectively. The zeroth-order particle velocity $\mathbf{v}_{a0}$ is written in terms of the gyrocenter coordinates as

$$\begin{aligned} \mathbf{v}_{a0}(\mathbf{Z}_a, t) = &U_a \mathbf{b}(\mathbf{X}_a, t) - [2\mu_a B_0(\mathbf{X}_a)/m_a]^{1/2} \\ &\times [\sin\xi_a\, \mathbf{e}_1(\mathbf{X}_a, t) + \cos\xi_a\, \mathbf{e}_2(\mathbf{X}_a, t)], \end{aligned} \tag{9}$$

where the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{b} \equiv \mathbf{B}_0/B_0$ form a right-handed orthogonal system.

On the right-hand side of Eq. (2), $\int d^6\mathbf{Z}_0 \equiv \int d^3\mathbf{X}_0 \int_{-\infty}^{\infty} dU_0 \int_0^{\infty} d\mu_0 \int_0^{2\pi} d\xi_0$ represents the integral with respect to the initial gyrocenter coordinates $\mathbf{Z}_0 \equiv (\mathbf{X}_0, U_0, \mu_0, \xi_0)$, $F_a(\mathbf{Z}_0, t_0)$ denotes the distribution function for species a at an arbitrarily specified initial time t_0, and the Jacobian is given by

$$D_a(\mathbf{Z}_0, t_0) \equiv B_{a\parallel}^*(\mathbf{Z}_0, t_0)/m_a, \tag{10}$$

where $B_{a\parallel}^* \equiv \mathbf{B}_a^* \cdot \mathbf{b}$ and $\mathbf{B}_a^*$ is defined by

$$\mathbf{B}_a^*(\mathbf{Z}_a, t) \equiv \nabla \times \mathbf{A}_a^* = \mathbf{B}_0(\mathbf{X}_a, t) + (m_a c/e_a) U_a \nabla \times \mathbf{b}(\mathbf{X}_a, t), \tag{11}$$

with $\nabla = \partial/\partial\mathbf{X}_a$. The gyrocenter coordinates of the particle at the time t are denoted by $\mathbf{Z}_a(\mathbf{Z}_0, t_0; t)$ which satisfy the initial condition

$$\mathbf{Z}_a(\mathbf{Z}_0, t_0; t_0) = \mathbf{Z}_0. \tag{12}$$

Poisson brackets are determined from the single-particle Lagrangian in Eq. (3). The nonvanishing components of the Poisson brackets for pairs of the gyrocenter coordinates are given by

$$\begin{aligned} &\{\mathbf{X}_a, \mathbf{X}_a\} = \frac{c}{e_a B_{a\parallel}^*}\mathbf{b} \times \mathbf{I}, \quad \{\mathbf{X}_a, U_a\} = \frac{\mathbf{B}_a^*}{m_a B_{a\parallel}^*}, \\ &\{\xi_a, \mu_a\} = \frac{e_a}{m_a c}, \end{aligned} \tag{13}$$

where $\mathbf{I} \equiv \mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{b}\mathbf{b}$ represents the unit dyadic.

The last integral term on the right-hand side of Eq. (2) is associated with the electromagnetic fields and the Lagrangian density $\mathcal{L}_f$ is defined by

$$\begin{aligned} \mathcal{L}_f = &\frac{1}{8\pi}\left(|\nabla\phi(\mathbf{x})|^2 - |\nabla \times [\mathbf{A}_0(\mathbf{x}, t) + \mathbf{A}_1(\mathbf{x}, t)]|^2\right) \\ &+ \frac{1}{4\pi c}\lambda(\mathbf{x}, t)\nabla \cdot \mathbf{A}_1(\mathbf{x}, t) + \mathcal{L}_{B0}, \end{aligned} \tag{14}$$

where

$$\mathcal{L}_{B0} = \frac{1}{4\pi}\mathbf{A}(\mathbf{x},t)\cdot[\mathbf{B}_0(\mathbf{x},t) - I\nabla\zeta - \nabla\zeta\times\nabla\chi] + \frac{1}{4\pi c}\alpha(\mathbf{x},t)\nabla\cdot\mathbf{A}_0(\mathbf{x},t) \tag{15}$$

is newly introduced to impose constraint conditions on the equilibrium magnetic field that is axisymmetric and time-dependent [see also Eq. (30)]. Here, I and χ represent the covariant toroidal component of $\mathbf{B}_0$ and poloidal magnetic flux divided by 2π, respectively, and ζ is the toroidal angle coordinate. Here, I and χ are both independent of ζ and they are written as $I = I(\chi, t)$ and $\chi = \chi(R, z)$ where the right-handed cylindrical spatial coordinates (R, z, ζ) are employed.

The last two terms in Eq. (6) give the perturbation to the Hamiltonian on the second order in the parameter $\delta = \rho/L$ given by the ratio of the gyroradius to the equilibrium scale length L. We retain these second-order perturbation terms here because they influence the gyrokinetic Poisson equation and/or Ampère's law derived in Sec. III to the lowest order in δ. However, in this work, we neglect all other second-order terms. The second-order correction terms[21,22] to define the difference between the particle and gyrocenter positions are not considered here. In order to avoid a secular deviation of the particle position from the gyrocenter in a long time gyrokinetic simulation, Wang and Hahm[20] considered the correction due to the fluctuating $\mathbf{E}\times\mathbf{B}$ velocity in the definition of the gyrocenter position and included the polarization drift in the gyrocenter equations of motion, which are not retained in this work either. Besides, $\mathbf{A}_a^*$ defined in Eq. (4) does not contain the gyrogauge-dependent term which is of the second order in δ.[21] In spite of these facts, we see that the second-order transport equations for particles, energy, and toroidal momentum shown in Sec. VI are not influenced by these second-order terms neglected in the present paper.

III. GYROKINETIC EQUATIONS

In this section, governing equations for the gyrocenter motion, distribution functions, turbulent electromagnetic fields, and the time-dependent equilibrium field are all derived from the variational principle using the Lagrangian shown in Eq. (2). In the derivation, variational variables are assumed to be fixed at the boundaries of the integral regions.

A. Gyrocenter motion equations and gyrokinetic Vlasov equation

The gyrocenter motion equations are obtained from $\delta\mathcal{I}/\delta\mathbf{Z}_a = 0$ as

$$\frac{d\mathbf{Z}_a}{dt} = \{\mathbf{Z}_a, H_a\} + \{\mathbf{Z}_a, \mathbf{X}_a\}\cdot\frac{e_a}{c}\frac{\partial\mathbf{A}_a^*}{\partial t}, \tag{16}$$

which are rewritten as

$$\frac{d\mathbf{X}_a}{dt} = \frac{1}{B_{a\parallel}^*}\left[\left(U_a + \frac{e_a}{m_a}\frac{\partial\Psi_a}{\partial U_a}\right)\mathbf{B}_a^* + c\mathbf{b}\times\left(\frac{\mu_a}{e_a}\nabla B_0 + \nabla\Psi_a + \frac{1}{c}\frac{\partial\mathbf{A}_a^*}{\partial t}\right)\right], \tag{17}$$

$$\frac{dU_a}{dt} = -\frac{\mathbf{B}_a^*}{m_a B_{a\parallel}^*}\cdot\left[\mu_a\nabla B_0 + e_a\left(\nabla\Psi_a + \frac{1}{c}\frac{\partial\mathbf{A}_a^*}{\partial t}\right)\right], \tag{18}$$

$$\frac{d\mu_a}{dt} = 0, \tag{19}$$

and

$$\frac{d\xi_a}{dt} = \Omega_a + \frac{e_a^2}{m_a c}\frac{\partial\Psi_a}{\partial\mu_a}, \tag{20}$$

where the effects of the vector potential for the time-dependent background magnetic field appear through the terms proportional to $\partial\mathbf{A}_a^*/\partial t$ and the fluctuating electromagnetic fields are included in the potential Ψ_a.

The distribution function $F_a(\mathbf{Z}, t)$ for the time t is determined by

$$D_a(\mathbf{Z},t)F_a(\mathbf{Z},t) = \int d^6\mathbf{Z}_0\, D_a(\mathbf{Z}_0,t_0)F_a(\mathbf{Z}_0,t_0) \times \delta^6[\mathbf{Z} - \mathbf{Z}_a(\mathbf{Z}_0,t_0;t)], \tag{21}$$

where $\delta^6(\mathbf{Z}-\mathbf{Z}_a) = \delta^3(\mathbf{X}-\mathbf{X}_a)\delta(U-U_a)\delta(\mu-\mu_a)\delta[\xi-\xi_a\ (\mathrm{mod}2\pi)]$.

Since Eqs. (16)–(20) are independent of the gyrophase ξ_a, $\mathbf{X}_a(\mathbf{Z}_0, t_0; t)$, $U_a(\mathbf{Z}_0, t_0; t)$, and $\mu_a(\mathbf{Z}_0, t_0; t)$ are all independent of the initial gyrophase ξ_0. The Jacobian D_a is also gyrophase-independent. Then, we find from Eq. (21) that, if F_a is initially gyrophase-independent, it is gyrophase-independent at any time. Hereafter, we assume without loss of generality that F_a is gyrophase-independent, $\partial F_a(\mathbf{Z},t)/\partial\xi = 0$. Noting that the Jacobian $D_a \equiv B_{a\parallel}^*/m_a$ is time-dependent, we see that the gyrocenter phase-space conservation law is given by

$$\frac{\partial D_a(\mathbf{Z},t)}{\partial t} + \frac{\partial}{\partial\mathbf{Z}}\cdot\left(D_a(\mathbf{Z},t)\frac{d\mathbf{Z}_a}{dt}(\mathbf{Z},t)\right) = 0, \tag{22}$$

where $(d\mathbf{Z}_a/dt)(\mathbf{Z},t)$ represents the value of the right-hand side of Eq. (16) evaluated at the gyrocenter position $\mathbf{Z}$ and the time t. From Eqs. (16) and (21), we have the gyrokinetic Vlasov equation in the conservation form

$$\frac{\partial}{\partial t}(D_a F_a) + \frac{\partial}{\partial\mathbf{Z}}\cdot\left(D_a F_a\frac{d\mathbf{Z}_a}{dt}\right) = 0, \tag{23}$$

which is rewritten with the help of Eq. (22) in the convection form

$$\left(\frac{\partial}{\partial t} + \frac{d\mathbf{Z}_a}{dt}\cdot\frac{\partial}{\partial\mathbf{Z}}\right)F_a(\mathbf{Z},t) = 0. \tag{24}$$

B. Equations for electromagnetic fields

The Coulomb gauge conditions $\nabla\cdot\mathbf{A}_1 = 0$ and $\nabla\cdot\mathbf{A}_0 = 0$ for the perturbation and equilibrium parts of the vector potential are derived from $\delta\mathcal{I}/\delta\lambda = 0$ and $\delta\mathcal{I}/\delta\alpha = 0$, respectively. The gyrokinetic Poisson equation is obtained from $\delta\mathcal{I}/\delta\phi = 0$ as

$$\nabla^2\phi(\mathbf{x},t) = -4\pi\sum_a e_a\int d^6\mathbf{Z}\,D_a(\mathbf{Z},t)\delta^3(\mathbf{X}+\boldsymbol{\rho}_a-\mathbf{x})\times\left[F_a(\mathbf{Z},t)+\frac{e_a\tilde{\psi}_a}{B_0}\frac{\partial F_a}{\partial\mu}\right]. \tag{25}$$

From $\delta\mathcal{I}/\delta\mathbf{A}_1 = 0$, we obtain

$$\nabla^2(\mathbf{A}_0+\mathbf{A}_1) - \frac{1}{c}\nabla\lambda = -\frac{4\pi}{c}\mathbf{j}_G, \tag{26}$$

where the gyrokinetic current density is defined by

$$\mathbf{j}_G \equiv \sum_a e_a\int d^6\mathbf{Z}D_a(\mathbf{Z})\delta^3[\mathbf{X}+\boldsymbol{\rho}_a(\mathbf{Z})-\mathbf{x}]\times\left(F_a(\mathbf{Z},t)\left[\mathbf{v}_{a0}(\mathbf{Z})-\frac{e_a}{m_a c}\mathbf{A}_1(\mathbf{X}+\boldsymbol{\rho}_a(\mathbf{Z}),t)\right]+\frac{e_a\tilde{\psi}_a}{B_0}\frac{\partial F_a}{\partial\mu}\mathbf{v}_{a0}(\mathbf{Z})\right). \tag{27}$$

Note that any vector field $\mathbf{a}$ can be expressed as $\mathbf{a} = \mathbf{a}_L + \mathbf{a}_T$, where $\mathbf{a}_L \equiv -(4\pi)^{-1}\nabla\int d^3\mathbf{x}'(\nabla'\cdot\mathbf{a})/|\mathbf{x}-\mathbf{x}'|$ and $\mathbf{a}_T \equiv (4\pi)^{-1}\nabla\times(\nabla\times\int d^3\mathbf{x}'\mathbf{a}/|\mathbf{x}-\mathbf{x}'|)$ represent the longitudinal (or irrotational) and transverse (or solenoidal) parts, respectively.[23] Then, the longitudinal and transverse parts of Eq. (27) are written as

$$\nabla\lambda = 4\pi(\mathbf{j}_G)_L \tag{28}$$

and

$$\nabla^2(\mathbf{A}_0+\mathbf{A}_1) = -\frac{4\pi}{c}(\mathbf{j}_G)_T, \tag{29}$$

respectively. Equation (29) represents the gyrokinetic Ampère's law.

From $\delta\mathcal{I}/\delta\boldsymbol{\Lambda} = 0$, the equilibrium magnetic field $\mathbf{B}_0$ is given in the axisymmetric form as

$$\mathbf{B}_0 = I\nabla\zeta + \nabla\zeta\times\nabla\chi. \tag{30}$$

The equilibrium vector potential $\mathbf{A}_0$, which satisfies the Coulomb gauge condition and Eq. (30) with $\mathbf{B}_0 = \nabla\times\mathbf{A}_0$, is given by

$$\mathbf{A}_0 = -\chi\nabla\zeta + \mathbf{A}_{P0}, \tag{31}$$

with

$$\mathbf{A}_{P0} = \nabla\zeta\times\nabla\eta, \tag{32}$$

where $\eta = \eta(R,Z)$ is the solution of

$$\Delta_*\eta \equiv R^2\nabla\cdot(R^{-2}\nabla\eta) = I. \tag{33}$$

The conditions for $\boldsymbol{\Lambda}$ are derived from $\delta\mathcal{I}/\delta\chi = 0$ and $\delta\mathcal{I}/\delta I = 0$ as

$$\overline{(\nabla\times\boldsymbol{\Lambda})^\zeta} = \frac{\partial I}{\partial\chi}\overline{\Lambda^\zeta}, \tag{34}$$

and

$$\langle\Lambda^\zeta\rangle = 0, \tag{35}$$

respectively, where the toroidal-angle and flux-surface averages are defined by $\overline{\cdots} \equiv (2\pi)^{-1}\oint\cdots d\zeta$ and $\langle\cdots\rangle \equiv (2\pi)^{-2}\oint\sqrt{g}\oint\cdots d\theta d\zeta$, respectively, and the Jacobian for the flux coordinates (χ,θ,ζ) is given by

$$\sqrt{g} = [\nabla\chi\cdot(\nabla\theta\times\nabla\zeta)]^{-1} = \frac{R^2 q}{I}. \tag{36}$$

Here, θ denotes the poloidal angle and $q \equiv \mathbf{B}_0\cdot\nabla\zeta/\mathbf{B}_0\cdot\nabla\theta$ is the safety factor. We use the superscript $(\cdots)^\zeta$ and the subscript $(\cdots)_\zeta$ to represent the contravariant and covariant toroidal components, respectively: $\Lambda^\zeta \equiv \boldsymbol{\Lambda}\cdot\nabla\zeta$ and $\Lambda_\zeta \equiv \boldsymbol{\Lambda}\cdot\partial\mathbf{x}/\partial\zeta \equiv \boldsymbol{\Lambda}\cdot R^2\nabla\zeta$.

We also find that $\delta\mathcal{I}/\delta\mathbf{A}_0 = 0$ yields

$$\nabla^2(\mathbf{A}_0+\mathbf{A}_1) + \nabla\times\boldsymbol{\Lambda} - \frac{1}{c}\nabla\alpha + \frac{4\pi}{c}(\mathbf{j}^{(\mathrm{gc})}+\nabla\times\mathbf{M}) = 0, \tag{37}$$

where the gyrocenter current is written as

$$\mathbf{j}^{(\mathrm{gc})} = \sum_a e_a n_a^{(\mathrm{gc})}\mathbf{u}_a^{(\mathrm{gc})}. \tag{38}$$

Here, the gyrocenter density $n_a^{(\mathrm{gc})}$ and the gyrocenter fluid velocity $\mathbf{u}_a^{(\mathrm{gc})}$ are defined by

$$n_a^{(\mathrm{gc})}(\mathbf{X},t) = 2\pi\int dU\int d\mu\,D_a(\mathbf{X},U,t)F_a(\mathbf{X},U,\mu,t), \tag{39}$$

and

$$n_a^{(\mathrm{gc})}\mathbf{u}_a^{(\mathrm{gc})} = 2\pi\int dU\int d\mu\,D_a F_a\mathbf{v}_a^{(\mathrm{gc})}, \tag{40}$$

where the gyrocenter drift velocity $\mathbf{v}_a^{(\mathrm{gc})} = d\mathbf{X}_a/dt$ is given by evaluating the right-hand side of Eq. (17) at $(\mathbf{X}, U, \mu)$. The last term on the left-hand side of Eq. (37) represents the magnetization current. The magnetization is defined by

$$\mathbf{M} = \sum_a \mathbf{M}_a, \tag{41}$$

with

$$\mathbf{M}_a = c\int dU\int d\mu\int d\xi\,D_a F_a\left(-\mu\mathbf{b} + \frac{m_a U}{B_0}(\mathbf{v}_a^{(\mathrm{gc})})_\perp - \mathbf{N}_a\right), \tag{42}$$

where

$$\begin{aligned}\mathbf{N}_a &= e_a\langle\mathbf{D}_B\psi_a\rangle_\xi + \frac{e_a^2}{2m_a c^2}\langle\mathbf{D}_B(|\mathbf{A}_1|^2)\rangle_\xi + \frac{e_a^2}{2B_0^2}\mathbf{b}\frac{\partial}{\partial\mu}\langle(\tilde{\psi}_a)^2\rangle_\xi - \frac{e_a^2}{B_0}\frac{\partial}{\partial\mu}\langle\tilde{\psi}_a\mathbf{D}_B\psi_a\rangle_\xi,\\ \mathbf{D}_B\psi_a &= -\frac{1}{B_0}\left[\left(\frac{1}{2}\mathbf{b}\boldsymbol{\rho}_a + \boldsymbol{\rho}_a\mathbf{b}\right)\cdot\left(\nabla\phi - \nabla\mathbf{A}_1\cdot\frac{\mathbf{v}_{a0}}{c}\right) + \frac{U}{c}\mathbf{A}_{1\perp} + \frac{1}{c}\left(\frac{1}{2}\mathbf{b}\mathbf{v}_{a0\perp} - \mathbf{v}_{a0\perp}\mathbf{b}\right)\cdot\mathbf{A}_1\right],\\ \mathbf{D}_B(|\mathbf{A}_1|^2) &= -\frac{1}{B_0}\left(\frac{1}{2}\mathbf{b}\boldsymbol{\rho}_a + \boldsymbol{\rho}_a\mathbf{b}\right)\cdot\nabla(|\mathbf{A}_1|^2),\end{aligned} \tag{43}$$

and the perpendicular component of an arbitrary vector $\mathbf{a}$ is denoted by $\mathbf{a}_\perp \equiv (\mathbf{b}\times\mathbf{a})\times\mathbf{b}$.

In the same way as in Eqs. (26)–(29), Eq. (37) is divided into the longitudinal part

$$-\frac{1}{c}\nabla\alpha + \frac{4\pi}{c}(\mathbf{j}^{(\mathrm{gc})})_L = 0, \tag{44}$$

and the transverse part

$$\nabla^2(\mathbf{A}_0+\mathbf{A}_1) + \nabla\times\boldsymbol{\Lambda} + \frac{4\pi}{c}((\mathbf{j}^{(\mathrm{gc})})_T + \nabla\times\mathbf{M}) = 0. \tag{45}$$

Equation (45) gives the gyrokinetic Ampère's law in a different form from Eq. (29). The two different expressions of the gyrokinetic Ampère's law are necessary to determine the equilibrium and perturbation parts of the magnetic field separately. We define $\mathbf{B}^{(\mathrm{gc})}$ as the magnetic field produced by $(\mathbf{j}^{(\mathrm{gc})})_T$

$$\nabla\times\mathbf{B}^{(\mathrm{gc})} = \frac{4\pi}{c}(\mathbf{j}^{(\mathrm{gc})})_T, \tag{46}$$

from which we obtain

$$\begin{aligned}\frac{1}{\sqrt{g}}\frac{\partial\overline{(B^{(\mathrm{gc})})_\zeta}}{\partial\theta} &= \frac{4\pi}{c}\overline{(\mathbf{j}^{(\mathrm{gc})})_T\cdot\nabla\chi},\\ \frac{1}{\sqrt{g}}\frac{\partial\overline{(B^{(\mathrm{gc})})_\zeta}}{\partial\chi} &= -\frac{4\pi}{c}\overline{(\mathbf{j}^{(\mathrm{gc})})_T\cdot\nabla\theta}.\end{aligned} \tag{47}$$

Then, using Eqs. (30) and (45), we have

$$\overline{\Lambda_\zeta} = I - \frac{4\pi}{c}\overline{M_\zeta} - \overline{(B^{(\mathrm{gc})})_\zeta} + \overline{B_{1\zeta}}, \tag{48}$$

which is combined with Eqs. (35) and (36) to obtain

$$I = \oint\frac{d\theta}{2\pi}\left[\frac{4\pi}{c}\overline{M_\zeta} + \overline{(B^{(\mathrm{gc})})_\zeta} - \overline{B_{1\zeta}}\right]. \tag{49}$$

Using Eq. (34) and taking the ζ-average of the toroidal component of Eq. (45) gives

$$\Delta_*\chi = \overline{\left(\frac{4\pi}{c}[(\mathbf{j}^{(\mathrm{gc})})_T + \nabla\times\mathbf{M}] - \nabla\times\mathbf{B}_1\right)\cdot R^2\nabla\zeta} + \frac{\partial I}{\partial\chi}\overline{\Lambda_\zeta}. \tag{50}$$

The time-dependent axisymmetric background field $\mathbf{B}_0$ [see Eq. (30)] can be self-consistently determined by using Eqs. (49) and (50), in which effects of the turbulent current and fields are included. The well-known Grad-Shafranov equation [see, for example, Sec. 3.10 of Ref. 19] corresponds to the lowest-order part of Eq. (50) in the δ-expansion. In fact, on the zeroth order, turbulent fluctuations are neglected, $\overline{\Lambda_\zeta} = 0$ holds, and the zeroth-order current $\mathbf{j}_0 = (\mathbf{j}^{(\mathrm{gc})})_T + \nabla\times\mathbf{M}$ satisfies the MHD equilibrium condition, $c^{-1}\mathbf{j}_0 \times\mathbf{B}_0 = \nabla(\sum_a n_{a0}T_{a0})$, where n_{a0} and T_{a0} denote the zeroth-order density and temperature for species a [see Eq. (116)], respectively. Then, using Eqs. (46)–(50) to express $\mathbf{j}_0\cdot\nabla\theta$ and $\mathbf{j}_0\cdot\nabla\zeta$ in terms of $\partial I/\partial\chi$ and $\Delta_*\chi$, respectively, and substituting them into the radial component of the MHD equilibrium condition yield the Grad-Shafranov equation. This derivation of the Grad-Shafranov equation is based on the low-flow ordering used in the present work; the magnitude of the background $\mathbf{E}\times\mathbf{B}$ drift velocity $\mathbf{u}_E$ is assumed to be on the same order as that of the diamagnetic drift velocity given by the thermal velocity v_T times $\delta = \rho/L$. When the high-flow ordering $\mathbf{u}_E = \mathcal{O}(v_T)$ is used, the large-amplitude radial electric field modifies the gyrokinetic equations,[24–26] and accordingly makes the momentum conservation law different from the one shown in the present work.

In summary, for the present model, Eqs. (24), (25), (26), (49), and (50) constitute the closed system of governing equations which determine F_a, ϕ, $\mathbf{A}_1$, I, and χ ($\mathbf{A}_0$ and $\mathbf{B}_0$ are determined from I and χ). It should be noted that these governing equations do not contain the other field variables λ, α, and $\boldsymbol{\Lambda}$, which are included in the Lagrangian, Eq. (2), as the Lagrange undetermined multipliers associated with the constraint conditions for $\mathbf{A}_1$, $\mathbf{A}_0$, and $\mathbf{B}_0$. If we fix the background magnetic field, we can eliminate Eqs. (49)–(50), and Eqs. (24)–(26) form the closed system equations for F_a, ϕ, and $\mathbf{A}_1$ as obtained in the previous gyrokinetic formulations. For the case of the electrostatic turbulence, $\mathbf{A}_1$ is neglected, Eq. (26) is not used, and the reduced set of equations is given by Eqs. (24), (25), (49), and (50) which determine F_a, ϕ, I, and χ. These equations can be used to describe the gyrokinetic system, in which the time evolutions of equilibrium profiles are dominated by the electrostatic turbulent transport while there are slow variations of the background magnetic field to be consistent with the evolving profiles.

IV. GYROCENTER DENSITIES AND POLARIZATION

Integrating the gyrokinetic Vlasov equation, Eq. (23), with respect to the velocity-space coordinates (U, μ, ξ), we immediately obtain

$$\frac{\partial n_a^{(\mathrm{gc})}}{\partial t} + \nabla\cdot\left(n_a^{(\mathrm{gc})}\mathbf{u}_a^{(\mathrm{gc})}\right) = 0, \tag{51}$$

where $n_a^{(\mathrm{gc})}$ and $\mathbf{u}_a^{(\mathrm{gc})}$ are defined in Eqs. (39) and (40). The delta-function part appearing in Eq. (25) is rewritten as

$$\delta^3(\mathbf{X}+\boldsymbol{\rho}_a-\mathbf{x}) = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{i_1,\cdots,i_n}\rho_{ai_1}\cdots\rho_{ai_n}\frac{\partial^n\delta^3(\mathbf{X}-\mathbf{x})}{\partial X_{i_1}\cdots\partial X_{i_n}}, \tag{52}$$

which is a useful formula to represent effects of finite gyroradii. Several numerical schemes to evaluate the phase-space integral including $\delta^3(\mathbf{X}+\boldsymbol{\rho}_a-\mathbf{x})$ as seen in Eq. (25) have been devised for gyrokinetic turbulence simulation.[27–29] Substituting Eq. (52) into Eq. (25) and rewriting $\mathbf{x}$ as $\mathbf{X}$, the gyrokinetic Poisson equation is rewritten as

$$\nabla\cdot\mathbf{E}_L = 4\pi\left(\sum_a e_a n_a^{(\mathrm{gc})} - \nabla\cdot\mathbf{P}^{(\mathrm{pol})}\right), \tag{53}$$

where $\mathbf{E}_L = -\nabla\phi$, $\nabla = \partial/\partial\mathbf{X}$, and $\mathbf{P}^{(\mathrm{pol})}$ represent the polarization density defined by

$$\mathbf{P}^{(\mathrm{pol})} = \sum_a e_a \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \sum_{i_1,\cdots,i_n} \int dU \int d\mu \int d\xi \times \frac{\partial^{n-1}(D_a F_a^* \boldsymbol{\rho}_a \rho_{ai_1} \cdots \rho_{ai_{n-1}})}{\partial X_{i_1} \cdots \partial X_{i_{n-1}}}. \tag{54}$$

Here, ρ_{ai} denotes the ith Cartesian component of $\boldsymbol{\rho}_a = \mathbf{b}(\mathbf{X},t) \times \mathbf{v}_{a0}(\mathbf{Z},t)/\Omega_a(\mathbf{X},t)$, and

$$F_a^* = F_a + \frac{e_a \tilde{\psi}_a}{B_0} \frac{\partial F_a}{\partial \mu}. \tag{55}$$

We can also rewrite $\mathbf{P}^{(\mathrm{pol})}$ as

$$\mathbf{P}^{(\mathrm{pol})} = \mathbf{P}_g + \mathbf{P}_\psi, \tag{56}$$

where

$$\mathbf{P}_g = -\sum_a e_a \sum_{l=1}^{\infty} \frac{1}{(2l)!} \sum_{i_1,\cdots,i_{2l-1}} \int dU \int d\mu \int d\xi \times \frac{\partial^{2l-1}(D_a F_a \boldsymbol{\rho}_a \rho_{ai_1} \cdots \rho_{ai_{2l-1}})}{\partial X_{i_1} \cdots \partial X_{i_{2l-1}}} \tag{57}$$

and

$$\mathbf{P}_\psi = \sum_a \frac{e_a^2}{B_0} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \sum_{i_1,\cdots,i_n} \int dU \int d\mu \int d\xi \times \frac{\partial^{n-1}[D_a \tilde{\psi}_a (\partial F_a/\partial\mu) \boldsymbol{\rho}_a \rho_{ai_1} \cdots \rho_{ai_{n-1}}]}{\partial X_{i_1} \cdots \partial X_{i_{n-1}}}. \tag{58}$$

We see that the $\mathbf{P}_\psi$ represents the polarization caused by the field ψ and that the charge density (at the position X) for the case of $\psi = 0$ is given by

$$\sum_a e_a n_a^{(\mathrm{gc})} - \nabla \cdot \mathbf{P}_g = \sum_a e_a \int d^6\mathbf{Z}' \, D_a(\mathbf{Z}',t) F_a(\mathbf{Z}',t) \times \delta^3(\mathbf{X}' + \boldsymbol{\rho}_a - \mathbf{X}), \tag{59}$$

which shows that the *particle* charge density should be evaluated from the *gyrocenter* charge density with keeping the corrections due to finite gyroradii.

Using Eqs. (38) and (51), we find

$$\frac{\partial}{\partial t}\left(\sum_a e_a n_a^{(\mathrm{gc})}\right) + \nabla \cdot \mathbf{j}^{(\mathrm{gc})} = 0. \tag{60}$$

Equation (53) is rewritten as

$$\sum_a e_a n_a^{(\mathrm{gc})} = \nabla \cdot \left(\frac{\mathbf{E}_L}{4\pi} + \mathbf{P}^{(\mathrm{pol})}\right), \tag{61}$$

which is substituted into Eq. (60) to obtain

$$\mathbf{j}_L^{(\mathrm{gc})} = -\frac{\partial}{\partial t}\left(\frac{\mathbf{E}_L}{4\pi} + \mathbf{P}_L^{(\mathrm{pol})}\right). \tag{62}$$

Thus, the longitudinal part $\mathbf{j}_L^{(\mathrm{gc})}$ of the gyrocenter current is equal to the minus sign of the longitudinal part of the displacement current plus the polarization current. Then, using Eqs. (60) and (62), we find that the useful formula

$$\begin{aligned} &\left\langle \frac{\partial}{\partial t}\left(\mathcal{A}\sum_a e_a n_a^{(\mathrm{gc})}\right)\right\rangle + \left\langle \nabla \cdot \left(\mathcal{A}\mathbf{j}_L^{(\mathrm{gc})}\right)\right\rangle \\ &= \left\langle \frac{\partial \mathcal{A}}{\partial t}\sum_a e_a n_a^{(\mathrm{gc})}\right\rangle + \langle \mathbf{j}_L^{(\mathrm{gc})} \cdot \nabla\mathcal{A}\rangle, \\ &= \left\langle \nabla \cdot \left[\frac{\partial \mathcal{A}}{\partial t}\left(\frac{\mathbf{E}_L}{4\pi} + \mathbf{P}_L^{(\mathrm{pol})}\right)\right]\right\rangle \\ &\quad - \left\langle \frac{\partial}{\partial t}\left[\left(\frac{\mathbf{E}_L}{4\pi} + \mathbf{P}_L^{(\mathrm{pol})}\right) \cdot \nabla\mathcal{A}\right]\right\rangle, \end{aligned} \tag{63}$$

holds for any function $\mathcal{A}(\mathbf{X},t)$. The relation in Eq. (63) is used in Sec. V B to derive Eq. (101).

V. CONSERVATION LAWS

In this section, conservation laws for energy and toroidal angular momentum are derived from Noether's theorem in the way similar to that in Ref. 9. First, we consider general infinitesimal transformations of the Eulerian field variables given as function of $(\mathbf{x}, t)$

$$\begin{aligned} t' &= t + \delta t_E(\mathbf{x},t), \\ \mathbf{x}' &= \mathbf{x} + \delta\mathbf{x}_E(\mathbf{x},t), \\ \phi'(\mathbf{x}',t') &= \phi(\mathbf{x},t) + \delta\phi(\mathbf{x},t), \\ \mathbf{A}_1'(\mathbf{x}',t') &= \mathbf{A}_1(\mathbf{x},t) + \delta\mathbf{A}_1(\mathbf{x},t), \\ \mathbf{A}_0'(\mathbf{x}',t') &= \mathbf{A}_0(\mathbf{x},t) + \delta\mathbf{A}_0(\mathbf{x},t), \\ \lambda'(\mathbf{x}',t') &= \lambda(\mathbf{x},t) + \delta\lambda(\mathbf{x},t), \\ \alpha'(\mathbf{x}',t') &= \alpha(\mathbf{x},t) + \delta\alpha(\mathbf{x},t), \\ \boldsymbol{\Lambda}'(\mathbf{x}',t') &= \boldsymbol{\Lambda}(\mathbf{x},t) + \delta\boldsymbol{\Lambda}(\mathbf{x},t). \end{aligned} \tag{64}$$

Here, δt_E and $\delta\ \mathbf{x}_E$ are generally functions of $(\mathbf{x},\ t)$ while $\delta\phi$, $\delta\mathbf{A}_1$, $\delta\mathbf{A}_0$, $\delta\lambda$, $\delta\alpha$, and $\delta\boldsymbol{\Lambda}$ are produced by the variations in their functional forms and those in the variables $(\mathbf{x},\ t)$

$$\begin{aligned} \delta\phi(\mathbf{x},t) &= \bar{\delta}\phi(\mathbf{x},t) + \delta t_E\,\partial_t\phi + \delta\mathbf{x}_E \cdot \nabla\phi, \\ \delta\mathbf{A}_1(\mathbf{x},t) &= \bar{\delta}\mathbf{A}_1(\mathbf{x},t) + \delta t_E\,\partial_t\mathbf{A}_1 + \delta\mathbf{x}_E \cdot \nabla\mathbf{A}_1, \\ \delta\mathbf{A}_0(\mathbf{x},t) &= \bar{\delta}\mathbf{A}_0(\mathbf{x},t) + \delta t_E\,\partial_t\mathbf{A}_0 + \delta\mathbf{x}_E \cdot \nabla\mathbf{A}_0, \\ \delta\lambda(\mathbf{x},t) &= \bar{\delta}\lambda(\mathbf{x},t) + \delta t_E\,\partial_t\lambda + \delta\mathbf{x}_E \cdot \nabla\lambda, \\ \delta\alpha(\mathbf{x},t) &= \bar{\delta}\alpha(\mathbf{x},t) + \delta t_E\,\partial_t\alpha + \delta\mathbf{x}_E \cdot \nabla\alpha, \\ \delta\boldsymbol{\Lambda}(\mathbf{x},t) &= \bar{\delta}\boldsymbol{\Lambda}(\mathbf{x},t) + \delta t_E\,\partial_t\boldsymbol{\Lambda} + \delta\mathbf{x}_E \cdot \nabla\boldsymbol{\Lambda}, \end{aligned} \tag{65}$$

where $\bar{\delta}\phi(\mathbf{x},t) = \phi'(\mathbf{x},t) - \phi(\mathbf{x},t)$, $\bar{\delta}\mathbf{A}_1(\mathbf{x},t) = \mathbf{A}_1'(\mathbf{x},t) - \mathbf{A}_1(\mathbf{x},t)$, $\bar{\delta}\mathbf{A}_0(\mathbf{x},t) = \mathbf{A}_0'(\mathbf{x},t) - \mathbf{A}_0(\mathbf{x},t)$, $\bar{\delta}\lambda(\mathbf{x},t) = \lambda'(\mathbf{x},t) - \lambda(\mathbf{x},t)$, $\bar{\delta}\alpha(\mathbf{x},t) = \alpha'(\mathbf{x},t) - \alpha(\mathbf{x},t)$, $\bar{\delta}\boldsymbol{\Lambda}(\mathbf{x},t) = \boldsymbol{\Lambda}'(\mathbf{x},t) - \boldsymbol{\Lambda}(\mathbf{x},t)$, and the second-order variation terms are neglected. Infinitesimal transformations of the axisymmetric functions χ and I associated with the equilibrium magnetic field are given by

$$\begin{aligned} \chi'(R',z',t') &= \chi(R,z,t) + \delta\chi(R,z,t), \\ I'(\chi'(R',z',t'),t') &= I(\chi(R,z,t),t) + \delta I(R,z,t), \end{aligned} \tag{66}$$

where $R' = R(\mathbf{x}')$ and $z' = z(\mathbf{x}')$ represent the R and z coordinates of the position $\mathbf{x}'$. Then, $\delta\chi(R, z, t)$ and $\delta I(R, z, t)$ are written as

$$\begin{aligned} \delta\chi(R, z, t) &= \bar{\delta}\chi(R, z, t) + \delta t_E\, \partial_t\chi + \delta R\, \partial_R\chi + \delta z\, \partial_z\chi, \\ \delta I(R, z, t) &= \bar{\delta} I(R, z, t) + \delta t_E\, \partial_t I(\chi, t) + \delta\chi\, \partial_\chi I(\chi, t), \end{aligned} \tag{67}$$

where $\bar{\delta}\chi(R, z, t) = \chi'(R, z, t) - \chi(R, z, t)$, $\bar{\delta} I(R, z, t) = I'(\chi(R, z, t), t) - I(\chi(R, z, t), t)$, $\delta R = R' - R$, and $\delta z = z' - z$.

We also consider the following infinitesimal transformations based on the Lagrangian description using $(\mathbf{Z}_0, t_0; t)$ as independent variables

$$\begin{aligned} t' &= t + \delta t_a(\mathbf{Z}_0, t_0; t), \\ \mathbf{Z}'_a(\mathbf{Z}_0, t_0; t') &= \mathbf{Z}_a(\mathbf{Z}_0, t_0; t) + \delta\mathbf{Z}_a(\mathbf{Z}_0, t_0; t), \end{aligned} \tag{68}$$

where the Lagrangian variations δt_a and $\delta\mathbf{X}_a$ are related to the Eulerian variations δt_E and $\delta\mathbf{x}_E$ by

$$\begin{aligned} \delta t_a(\mathbf{Z}_0, t_0; t) &= \delta t_E(\mathbf{X}_a(\mathbf{Z}_0, t_0; t), t), \\ \delta\mathbf{X}_a(\mathbf{Z}_0, t_0; t) &= \delta\mathbf{x}_E(\mathbf{X}_a(\mathbf{Z}_0, t_0; t), t). \end{aligned} \tag{69}$$

Similar to Eqs. (65) and (67), $\delta\mathbf{Z}_a$ is caused by the variations in their functional forms and the variation δt_a

$$\delta\mathbf{Z}_a(\mathbf{Z}_0, t_0; t) = \bar{\delta}\mathbf{Z}_a(\mathbf{Z}_0, t_0; t) + \delta t_a\, \partial_t\mathbf{Z}_a(\mathbf{Z}_0, t_0; t), \tag{70}$$

where $\bar{\delta}\mathbf{Z}_a(\mathbf{Z}_0, t_0; t) = \mathbf{Z}'_a(\mathbf{Z}_0, t_0; t) - \mathbf{Z}_a(\mathbf{Z}_0, t_0; t)$. Recall that, in Sec. III, the governing equations for $\mathbf{Z}_a$, ϕ, $\mathbf{A}_1$, $\mathbf{A}_0$, λ, α, and $\boldsymbol{\Lambda}$ are derived as Euler-Lagrange equations from $\delta\mathcal{I} = 0$ by considering only the variations in the functional forms which are assumed to vanish on the integral boundaries. On the other hand, when the variations given by Eqs. (64)–(70) are taken about the solutions of the Euler-Lagrange equations, the variation $\delta\mathcal{I}$ of the action integral does not generally vanish but it is written as

$$\delta\mathcal{I} = \delta\mathcal{I}_p + \delta\mathcal{I}_{pf} + \delta\mathcal{I}_f, \tag{71}$$

where

$$\delta\mathcal{I}_p = \sum_a \int dt \int d^6\mathbf{Z}_{a0} D_a(\mathbf{Z}_{a0}, t_0) F_a(\mathbf{Z}_{a0}, t_0) \times \frac{\partial}{\partial t}\left[\left(L_a - \frac{\partial L_a}{\partial(\partial_t\mathbf{Z}_a)}\cdot\partial_t\mathbf{Z}_a\right)\delta t_a + \frac{\partial L_a}{\partial(\partial_t\mathbf{Z}_a)}\cdot\delta\mathbf{Z}_a\right], \tag{72}$$

$$\delta\mathcal{I}_{pf} = \sum_a \int dt \int d^6\mathbf{Z}\, \nabla\cdot\left[D_a(\mathbf{Z}, t) F_a(\mathbf{Z}, t)\bar{\delta}\mathbf{A}_0 \times\left(-\mu\mathbf{b} + \frac{m_a U}{B_0}(\mathbf{v}_a^{(\mathrm{gc})})_\perp - \mathbf{N}_a\right) - \delta\mathbf{R}_a\right], \tag{73}$$

and

$$\begin{aligned} \delta\mathcal{I}_f = \int dt \int d^3\mathbf{x} \Bigg[&\frac{\partial}{\partial t}(\mathcal{L}_f \delta t_E) + \nabla\cdot\Bigg\{ -\left(\frac{\partial\mathcal{L}_f}{\partial(\nabla\phi)}\partial_t\phi + \sum_{n=0}^{1}\sum_{k=1}^{3}\frac{\partial\mathcal{L}_f}{\partial(\nabla A_{nk})}\partial_t A_{nk} + \frac{\partial\mathcal{L}_f}{\partial(\nabla\chi)}\partial_t\chi\right)\delta t_E + \mathcal{L}_f\delta\mathbf{x}_E \\ &- \left(\frac{\partial\mathcal{L}_f}{\partial(\nabla\phi)}\nabla\phi + \sum_{n=0}^{1}\sum_{k=1}^{3}\frac{\partial\mathcal{L}_f}{\partial(\nabla A_{nk})}\nabla A_{nk} + \frac{\partial\mathcal{L}_f}{\partial(\nabla\chi)}\nabla\chi\right)\cdot\delta\mathbf{x}_E \\ &+ \frac{\partial\mathcal{L}_f}{\partial(\nabla\phi)}\delta\phi + \sum_{n=0}^{1}\sum_{k=1}^{3}\frac{\partial\mathcal{L}_f}{\partial(\nabla A_{nk})}\delta A_{nk}\Bigg\} + \frac{\partial\mathcal{L}_f}{\partial(\nabla\chi)}\delta\chi\Bigg]. \end{aligned} \tag{74}$$

On the right-hand side of Eq. (73), $\delta\mathbf{R}_a$ is associated with effects of finite gyroradii [see Eq. (52)] on electromagnetic fields and defined by

$$\begin{aligned} \delta\mathbf{R}_a = e_a\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{i_1=1}^{\infty}\cdots\sum_{i_{n-1}=1}^{\infty}\Bigg[&D_a F_a^*\boldsymbol{\rho}_a\rho_{ai_1}\cdots\rho_{ai_{n-1}}\frac{\partial^{n-1}\bar{\delta}\psi_a}{\partial X_{i_1}\cdots\partial X_{i_{n-1}}} - \frac{\partial(D_a F_a^*\boldsymbol{\rho}_a\rho_{ai_1}\cdots\rho_{ai_{n-1}})}{\partial X_{i1}}\frac{\partial^{n-2}\bar{\delta}\psi_a}{\partial X_{i_2}\cdots\partial X_{i_{n-1}}} \\ &+\cdots+(-1)^{n-1}\frac{\partial^{n-1}(D_a F_a^*\boldsymbol{\rho}_a\rho_{ai_1}\cdots\rho_{ai_{n-1}})}{\partial X_{i_1}\cdots\partial X_{i_{n-1}}}\bar{\delta}\psi_a + \frac{e_a}{m_a c^2}\Bigg(D_a F_a\boldsymbol{\rho}_a\rho_{ai_1}\cdots\rho_{ai_{n-1}}\frac{\partial^{n-1}(\mathbf{A}_1\cdot\bar{\delta}\mathbf{A}_1)}{\partial X_{i_1}\cdots\partial X_{i_{n-1}}} \\ &-\frac{\partial(D_a F_a\boldsymbol{\rho}_a\rho_{ai_1}\cdots\rho_{ai_{n-1}})}{\partial X_{i1}}\frac{\partial^{n-2}(\mathbf{A}_1\cdot\bar{\delta}\mathbf{A}_1)}{\partial X_{i_2}\cdots\partial X_{i_{n-1}}} + \cdots + (-1)^{n-1}\frac{\partial^{n-1}(D_a F_a\boldsymbol{\rho}_a\rho_{ai_1}\cdots\rho_{ai_{n-1}})}{\partial X_{i_1}\cdots\partial X_{i_{n-1}}}(\mathbf{A}_1\cdot\bar{\delta}\mathbf{A}_1)\Bigg)\Bigg], \end{aligned} \tag{75}$$

where $\bar{\delta}\psi_a$ is given by

$$\bar{\delta}\psi_a = \bar{\delta}\phi(\mathbf{X}_a, t) - \frac{1}{c}\mathbf{v}_{a0}(\mathbf{Z}_a, t)\cdot\bar{\delta}\mathbf{A}_1(\mathbf{X}_a, t). \tag{76}$$

We recall that the conservation of the magnetic moment μ results from the invariance under the variation of the gyrophase ξ_a although, in order to prepare for deriving the conservation laws of energy and toroidal momentum in the following subsections, we hereafter consider the case in which $\delta\xi_a = 0$ and accordingly $\partial L_a/\partial(\partial_t\mathbf{Z}_a)\cdot\delta\mathbf{Z}_a = \partial L_a/\partial(\partial_t\mathbf{X}_a)\cdot\delta\mathbf{X}_a$ in Eq. (72). Then, using Eqs. (72)–(74), we can rewrite Eq. (71) as

$$\delta\mathcal{I} = -\int dt \int d^3\mathbf{X} \left[\frac{\partial}{\partial t}\delta G_0(\mathbf{X},t) + \nabla\cdot\delta\mathbf{G}(\mathbf{X},t) \right], \qquad (77)$$

with the functions δG_0 and $\delta\mathbf{G}$ defined by

$$\begin{aligned} \delta G_0(\mathbf{X},t) &= \mathcal{E}_c\,\delta t_E - \mathbf{P}_c\cdot\delta\mathbf{x}_E, \\ \delta\mathbf{G}(\mathbf{X},t) &= \mathbf{Q}_c\,\delta t_E - \mathbf{\Pi}_c\cdot\delta\mathbf{x}_E + \mathbf{S}_\phi\,\delta\phi - \mathbf{\Sigma}_{A1}\cdot\delta\mathbf{A}_1 \\ &\quad -\mathbf{\Sigma}_{A0}\cdot\delta\mathbf{A}_0 + \mathbf{S}_\chi\delta\chi + \delta\mathbf{T}, \end{aligned} \qquad (78)$$

where

$$\begin{aligned} \mathcal{E}_c &= \sum_a \int dU \int d\mu \int d\xi\, D_a F_a H_a \\ &\quad + \frac{1}{8\pi}\left(-|\nabla\phi|^2 + |\mathbf{B}_0+\mathbf{B}_1|^2\right), \\ \mathbf{P}_c &= \sum_a \int dU \int d\mu \int d\xi\, D_a F_a \left(m_a U\mathbf{b} + \frac{e_a}{c}\mathbf{A}_0 \right), \\ \mathbf{Q}_c &= \sum_a \int dU \int d\mu \int d\xi\, D_a F_a \left[H_a \mathbf{v}_a^{(\mathrm{gc})} + \frac{\partial \mathbf{A}_0}{\partial t} \right. \\ &\quad \left. \times\left(-\mu\mathbf{b} + \frac{m_a U}{B_0}(\mathbf{v}_a^{(\mathrm{gc})})_\perp - \mathbf{N}_a \right) \right] + \frac{1}{4\pi}\frac{\partial\phi}{\partial t}\nabla\phi \\ &\quad - \frac{1}{4\pi}\frac{\partial(\mathbf{A}_0+\mathbf{A}_1)}{\partial t}\times(\mathbf{B}_0+\mathbf{B}_1) \\ &\quad + \frac{1}{4\pi c}\left(\lambda\frac{\partial\mathbf{A}_1}{\partial t} + \alpha\frac{\partial\mathbf{A}_0}{\partial t} \right) - \frac{1}{4\pi}\mathbf{\Lambda}\times\left(\frac{\partial\mathbf{A}_0}{\partial t} + \frac{\partial\chi}{\partial t}\nabla\zeta \right), \\ \mathbf{\Pi}_c &= \sum_a \int dU \int d\mu \int d\xi\, D_a F_a \left[\mathbf{v}_a^{(\mathrm{gc})}\left(m_a U\mathbf{b} + \frac{e_a}{c}\mathbf{A}_0 \right) \right. \\ &\quad \left. + \left(-\mu\mathbf{b} + \frac{m_a U}{B_0}(\mathbf{v}_a^{(\mathrm{gc})})_\perp - \mathbf{N}_a \right)\times(\nabla\mathbf{A}_0)^T \right] \\ &\quad + \frac{1}{8\pi}\left(|\nabla\phi|^2 - B^2\right)\mathbf{I} + \frac{1}{4\pi}\left[-(\nabla\phi)(\nabla\phi) \right. \\ &\quad + ((\nabla\mathbf{A}) - (\nabla\mathbf{A})^T)\cdot(\nabla\mathbf{A})^T - \frac{\lambda}{c}(\nabla\mathbf{A}_1)^T \\ &\quad \left. - \frac{\alpha}{c}(\nabla\mathbf{A}_0)^T + \mathbf{\Lambda}\times((\nabla\mathbf{A}_0)^T + (\nabla\zeta)(\nabla\chi)) \right], \\ \mathbf{S}_\phi &= -\frac{1}{4\pi}\nabla\phi, \\ \mathbf{\Sigma}_{A1} &= \frac{1}{4\pi}\left(\mathbf{B}\times\mathbf{I} + \frac{\lambda}{c}\mathbf{I} \right), \\ \mathbf{\Sigma}_{A0} &= \frac{1}{4\pi}\left((\mathbf{B}-\mathbf{\Lambda})\times\mathbf{I} + \frac{\alpha}{c}\mathbf{I} \right) + \sum_a \int dU \int d\mu \int d\xi \\ &\quad \times D_a F_a \left(\mu\mathbf{b} - \frac{m_a U}{B_0}(\mathbf{v}_a^{(\mathrm{gc})})_\perp + \mathbf{N}_a \right)\times\mathbf{I}, \\ \mathbf{S}_\chi &= \frac{1}{4\pi}\mathbf{\Lambda}\times\nabla\zeta, \\ \delta\mathbf{T} &= \sum_a \int dU \int d\mu \int d\xi\, \delta\mathbf{R}_a. \end{aligned} \qquad (79)$$

Here, the superscript T represents the transpose of the tensor, and $\mathbf{I}$ denotes the unit tensor. Comparing the variation $\delta\mathcal{I}$ of the gyrokinetic action integral shown in Eqs. (77)–(79) and the similar expression of $\delta\mathcal{I}$ given in Ref. 9 for the Vlasov-Poisson-Ampère system, we find that more complicated terms appear in δG_0 and $\delta\mathbf{G}$ in the present system due to effects of the finite gyroradii and the new variational fields included for separately determining the turbulent and background fields.

It should be noted that Eq. (77) is derived by using the Euler-Lagrange equations shown in Sec. III, of which Eq. (35) requires the integral over the flux surface. Thus, given any spatial point in the integral domain of Eq. (77), the flux surface including the point should be wholly contained in the integral domain in order for Eq. (77) to be valid. If the variations in the variables are such that $\delta\mathcal{I} = 0$ holds for an arbitrary spatiotemporal integral domain represented by $[t_1, t_2]\times[s_1, s_2]$ where $[s_1, s_2]$ represents the spatial volume region sandwiched between two flux surfaces labeled by s_1 and s_2, then the conservation law is derived as

$$\begin{aligned} &\left\langle \frac{\partial}{\partial t}\delta G_0(\mathbf{X},t) + \nabla\cdot\delta\mathbf{G}(\mathbf{X},t) \right\rangle \\ &\quad = \left\langle \frac{\partial}{\partial t}\delta G_0(\mathbf{X},t) \right\rangle + \frac{1}{V'}\frac{\partial}{\partial s}\left(V'\langle\delta\mathbf{G}\cdot\nabla s\rangle \right) = 0. \end{aligned} \qquad (80)$$

This is Noether's theorem for the present gyrokinetic system. Here, we use flux coordinates (s, θ, ζ), where s denotes an arbitrary radial coordinate to label flux surfaces, so that χ is written as a function $\chi = \chi(s, t)$. The volume enclosed by the flux surface with the label s at the time t is denoted by $V(s, t)$ and its radial derivative is represented by $V' \equiv \partial V/\partial s$. Under the nonstationary background field $\mathbf{B}_0$, flux surfaces may change their shapes and the grid of the flux coordinates moves. Then, the grid velocity[30] is given by

$$\mathbf{u}_s = \frac{\partial\mathbf{x}(s,\theta,\zeta,t)}{\partial t}, \qquad (81)$$

and we obtain the following formula

$$\left\langle \frac{\partial}{\partial t}\delta G_0 \right\rangle = \frac{1}{V'}\left[\frac{\partial}{\partial t}\left(V'\langle\delta G_0\rangle\right) - \frac{\partial}{\partial s}\left(V'\langle\delta G_0\mathbf{u}_s\cdot\nabla s\rangle\right) \right], \qquad (82)$$

where on the right-hand side, the partial derivatives $\partial/\partial t$ and $\partial/\partial s$ act on functions of (s, t) obtained after taking the flux-surface average while, on the left-hand side, the partial time derivative $\partial/\partial t$ is taken with fixed $\mathbf{X}$ before the flux-surface average [see Eq. (2.35) in Ref. 30]. On the right-hand side of Eq. (82), $\mathbf{u}_s\cdot\nabla s$ represents the radial velocity of the flux surface and the last term gives the correction due to the radial surface motion for evaluating the surface-averaged rate of change. Substituting Eq. (82) into Eq. (80), we obtain

$$\frac{\partial}{\partial t}\left(V'\langle\delta G_0\rangle\right) + \frac{\partial}{\partial s}\left(V'\langle(\delta\mathbf{G} - \delta G_0\mathbf{u}_s)\cdot\nabla s\rangle\right) = 0. \qquad (83)$$

A. Energy conservation

In order to derive the energy conservation law, we consider the infinitesimal translation in time represented by $\delta t_E = \epsilon$ where ϵ is an infinitesimally small constant. Here, all other variations $\delta\mathbf{x}_E$, $\delta\phi$, $\delta\mathbf{A}_1$, $\delta\mathbf{A}_0$, $\delta\chi$, δI, $\delta\lambda$, $\delta\alpha$, $\delta\mathbf{\Lambda}$, and $\delta\mathbf{Z}_a$ are regarded as zero. Under this infinitesimal time translation, $\delta\mathcal{I} = 0$ is satisfied for an arbitrary integral domain in the form of $[t_1, t_2] \times [s_1, s_2]$ [see the remark before Eq. (80)] because the integrands in the action integral I given by Eq. (2) have no explicit time dependence while they implicitly depend on t through ϕ, $\mathbf{A}_1$, $\mathbf{A}_0$, λ, α, $\mathbf{\Lambda}$, and $\mathbf{Z}_a$. Then, using the time translational symmetry and Eqs. (78)–(80), we obtain

$$\left\langle \frac{\partial \mathcal{E}_c}{\partial t} \right\rangle + \langle \nabla \cdot (\mathbf{Q}_c + \mathbf{Q}_R) \rangle = 0, \tag{84}$$

where $\mathbf{Q}_R$ arises from Eq. (75) and is defined by

$$\begin{aligned}
\mathbf{Q}_R = &\sum_a e_a \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1=1}^{\infty} \cdots \sum_{i_{n-1}=1}^{\infty} \int dU \int d\mu \int d\xi \\
&\times \Bigg[-D_a F_a^* \boldsymbol{\rho}_a \rho_{ai_1} \cdots \rho_{ai_{n-1}} \frac{\partial^{n-1} \partial_t \psi_a}{\partial X_{i_1} \cdots \partial X_{i_{n-1}}} \\
&+ \frac{\partial (D_a F_a^* \boldsymbol{\rho}_a \rho_{ai_1} \cdots \rho_{ai_{n-1}})}{\partial X_{i1}} \frac{\partial^{n-2} \partial_t \psi_a}{\partial X_{i_2} \cdots \partial X_{i_{n-1}}} \\
&+ \cdots + (-1)^n \frac{\partial^{n-1} (D_a F_a^* \boldsymbol{\rho}_a \rho_{ai_1} \cdots \rho_{ai_{n-1}})}{\partial X_{i_1} \cdots \partial X_{i_{n-1}}} \partial_t \psi_a \\
&+ \frac{e_a}{m_a c^2} \Bigg(-D_a F_a^* \boldsymbol{\rho}_a \rho_{ai_1} \cdots \rho_{ai_{n-1}} \frac{\partial^{n-1} (\mathbf{A}_1 \cdot \partial_t \mathbf{A}_1)}{\partial X_{i_1} \cdots \partial X_{i_{n-1}}} \\
&+ \frac{\partial (D_a F_a^* \boldsymbol{\rho}_a \rho_{ai_1} \cdots \rho_{ai_{n-1}})}{\partial X_{i1}} \frac{\partial^{n-2} (\mathbf{A}_1 \cdot \partial_t \mathbf{A}_1)}{\partial X_{i_2} \cdots \partial X_{i_{n-1}}} + \cdots \\
&+ (-1)^n \frac{\partial^{n-1} (D_a F_a^* \boldsymbol{\rho}_a \rho_{ai_1} \cdots \rho_{ai_{n-1}})}{\partial X_{i_1} \cdots \partial X_{i_{n-1}}} (\mathbf{A}_1 \cdot \partial_t \mathbf{A}_1) \Bigg) \Bigg].
\end{aligned} \tag{85}$$

With the help of the gyrokinetic Poisson equation in Eq. (25), the canonical energy density $\mathcal{E}_c$ is rewritten as

$$\mathcal{E}_c = \mathcal{E} + \nabla \cdot \left(-\frac{1}{4\pi} \phi \nabla \phi + \mathbf{\Phi}_R \right), \tag{86}$$

where $\mathcal{E}$ and $\mathbf{\Phi}_R$ are defined by

$$\begin{aligned}
\mathcal{E} = &\sum_a \int dU \int d\mu \int d\xi \, D_a F_a \left(\frac{m_a}{2} \left| \mathbf{v}_{a0} - \frac{e_a}{m_a c} \mathbf{A}_1 \right|^2 \right. \\
&\left. + \frac{e_a}{2B_0} \frac{\partial}{\partial \mu} \langle \tilde{\psi}_a (2\tilde{\phi} - \tilde{\psi}_a) \rangle_\xi \right) \\
&+ \frac{1}{8\pi} \left(|\nabla \phi|^2 + |\mathbf{B}_0 + \mathbf{B}_1|^2 \right)
\end{aligned} \tag{87}$$

and

$$\begin{aligned}
\mathbf{\Phi}_R = &\sum_a e_a \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1=1}^{\infty} \cdots \sum_{i_{n-1}=1}^{\infty} \int dU \int d\mu \int d\xi \\
&\times \Bigg[D_a F_a^* \boldsymbol{\rho}_a \rho_{ai_1} \cdots \rho_{ai_{n-1}} \frac{\partial^{n-1} \phi}{\partial X_{i_1} \cdots \partial X_{i_{n-1}}} \\
&- \frac{\partial (D_a F_a^* \boldsymbol{\rho}_a \rho_{ai_1} \cdots \rho_{ai_{n-1}})}{\partial X_{i1}} \frac{\partial^{n-2} \phi}{\partial X_{i_2} \cdots \partial X_{i_{n-1}}} \\
&+ \cdots + (-1)^{n-1} \frac{\partial^{n-1} (D_a F_a^* \boldsymbol{\rho}_a \rho_{ai_1} \cdots \rho_{ai_{n-1}})}{\partial X_{i_1} \cdots \partial X_{i_{n-1}}} \phi \Bigg],
\end{aligned} \tag{88}$$

respectively. Substituting Eq. (86) into Eq. (84), we obtain the energy conservation written as

$$\left\langle \frac{\partial \mathcal{E}}{\partial t} \right\rangle + \langle \nabla \cdot \mathbf{Q} \rangle = 0, \tag{89}$$

which is rewritten by using Eq. (83) as

$$\frac{\partial}{\partial t} \left(V' \langle \mathcal{E} \rangle \right) + \frac{\partial}{\partial s} \left(V' \langle (\mathbf{Q} - \mathcal{E} \mathbf{u}_s) \cdot \nabla s \rangle \right) = 0. \tag{90}$$

Here, $\mathbf{Q}$ is defined by

$$\mathbf{Q} = \mathbf{Q}_c^* + \mathbf{Q}_R^*, \tag{91}$$

where

$$\begin{aligned}
\mathbf{Q}_c^* &= \mathbf{Q}_c - \frac{1}{4\pi} \frac{\partial}{\partial t} (\phi \nabla \phi) \\
&= \sum_a \int dU \int d\mu \int d\xi \, D_a F_a \left[H_a \mathbf{v}_a^{(\mathrm{gc})} + \frac{\partial \mathbf{A}_0}{\partial t} \right. \\
&\quad \left. \times \left(-\mu \mathbf{b} + \frac{m_a U}{B_0} (\mathbf{v}_a^{(\mathrm{gc})})_\perp - \mathbf{N}_a \right) \right] - \frac{1}{4\pi} \phi \nabla \frac{\partial \phi}{\partial t} \\
&\quad - \frac{1}{4\pi} \frac{\partial (\mathbf{A}_0 + \mathbf{A}_1)}{\partial t} \times (\mathbf{B}_0 + \mathbf{B}_1) \\
&\quad + \frac{1}{4\pi c} \left(\lambda \frac{\partial \mathbf{A}_1}{\partial t} + \alpha \frac{\partial \mathbf{A}_0}{\partial t} \right) - \frac{1}{4\pi} \mathbf{\Lambda} \times \left(\frac{\partial \mathbf{A}_0}{\partial t} + \frac{\partial \chi}{\partial t} \nabla \zeta \right)
\end{aligned} \tag{92}$$

and

$$\mathbf{Q}_R^* = \mathbf{Q}_R + \frac{\partial \mathbf{\Phi}_R}{\partial t}. \tag{93}$$

In the energy conservation law given by Eq. (89) [or Eq. (90)], the energy density $\mathcal{E}$ is defined by Eq. (87) and its volume integral gives the same energy integral as shown by Eq. (59) in Ref. 7. The energy flux $\mathbf{Q}$ defined by Eq. (91) with Eqs. (92) and (93) has a complicated form although it is shown later in Sec. VI B that the ensemble average of $\mathbf{Q}$ coincides with the well-known expression of the radial energy transport to the lowest order in the δ-expansion.

B. Conservation of toroidal angular momentum

In Subsection V B, the energy conservation law is derived from the invariance of the system under the time translation. It should be noted that, since the Lagrangian explicitly contains $\nabla\zeta$ through $\mathcal{L}_{B0}$ defined in Eq. (15) to derive the axisymmetric equilibrium field, the present gyrokinetic system is not invariant under the spatial translation but it is still invariant under the toroidal rotation. Therefore, the toroidal angular momentum conservation is derived from the fact that $\delta\mathcal{I} = 0$ under the infinitesimal toroidal rotation represented by $\delta \mathbf{x}_E = \epsilon \mathbf{e}_\zeta(\mathbf{X})$. Here, ϵ is again an infinitesimally small constant, and $\mathbf{e}_\zeta(\mathbf{X})$ is defined by

$$\mathbf{e}_\zeta(\mathbf{X}) = \partial \mathbf{X} / \partial \zeta = R^2 \nabla \zeta, \tag{94}$$

where the right-handed cylindrical spatial coordinates (R, z, ζ) are used. We also define $\hat{\mathbf{z}}$ by

$$\hat{\mathbf{z}} = R\nabla\zeta \times \nabla R, \tag{95}$$

which represents the unit vector in the z-direction. Then, if putting the origin of the position vector $\mathbf{X}$ at $(R, z) = (0, 0)$, we have $\mathbf{e}_\zeta(\mathbf{X}) = \mathbf{X} \times \hat{\mathbf{z}}$. Under the infinitesimal toroidal rotation, the variations δt_E, $\delta\phi$, $\delta\chi$, δI, $\delta\lambda$, and $\delta\alpha$ are all regarded as zero while the variations of the vector variables are given by

$$\delta\mathbf{A}_1 = \epsilon\mathbf{A}_1 \times \hat{\mathbf{z}}, \quad \delta\mathbf{A}_0 = \epsilon\mathbf{A}_0 \times \hat{\mathbf{z}}, \quad \delta\mathbf{\Lambda} = \epsilon\mathbf{\Lambda} \times \hat{\mathbf{z}}. \tag{96}$$

Then, using $\delta\mathcal{I} = 0$ under the infinitesimal toroidal rotation and Eqs. (78)–(80), we obtain

$$\left\langle \frac{\partial(\mathbf{P}_c \cdot \mathbf{e}_\zeta)}{\partial t} \right\rangle + \frac{1}{V'}\frac{\partial}{\partial s}\left[V'\langle\nabla s \cdot (\mathbf{\Pi}_c \cdot \mathbf{e}_\zeta + (\mathbf{\Sigma}_{A1} \times \mathbf{A}_1 + \mathbf{\Sigma}_{A0} \times \mathbf{A}_0) \cdot \hat{\mathbf{z}} + \mathbf{P}_{R\zeta}]\rangle\right) = 0, \tag{97}$$

where

$$\mathbf{P}_c \cdot \mathbf{e}_\zeta = \sum_a \int dU \int d\mu \int d\xi \, D_a F_a \left(m_a U b_\zeta + \frac{e_a}{c} A_{0\zeta} \right), \tag{98}$$

$$\begin{aligned} &\langle \nabla s \cdot [\mathbf{\Pi}_c \cdot \mathbf{e}_\zeta + (\mathbf{\Sigma}_{A1} \times \mathbf{A}_1 + \mathbf{\Sigma}_{A0} \times \mathbf{A}_0) \cdot \hat{\mathbf{z}}] \rangle \\ &= \sum_a \int dU \int d\mu \int d\xi \, D_a F_a \mathbf{v}_a^{(\mathrm{gc})} \left(m_a U b_\zeta + \frac{e_a}{c} A_{0\zeta} \right) \\ &\quad + \frac{1}{4\pi}\left[-\left\langle (\nabla\phi \cdot \nabla s) \frac{\partial\phi}{\partial\zeta} \right\rangle - \langle B_1^s B_{1\zeta} \rangle \right. \\ &\quad \left. - \langle [(\nabla \times \mathbf{B}_1) \cdot \nabla s] A_{1\zeta} \rangle + \frac{1}{c}\left\langle (\mathbf{A}_1 \cdot \nabla s) \frac{\partial\lambda}{\partial\zeta} \right\rangle \right], \end{aligned} \tag{99}$$

and

$$\begin{aligned} \mathbf{P}_{R\zeta} = &\sum_a e_a \sum_{n=1}^\infty \frac{1}{n!} \sum_{i_1=1}^\infty \cdots \sum_{i_{n-1}=1}^\infty \int dU \int d\mu \int d\xi \\ &\times \left[D_a F_a^* \boldsymbol{\rho}_a \rho_{ai_1} \cdots \rho_{ai_{n-1}} \frac{\partial^{n-1}\partial_\zeta \psi_a}{\partial X_{i_1} \cdots \partial X_{i_{n-1}}} \right. \\ &- \frac{\partial(D_a F_a^* \boldsymbol{\rho}_a \rho_{ai_1} \cdots \rho_{ai_{n-1}})}{\partial X_{i1}} \frac{\partial^{n-2}\partial_\zeta \psi_a}{\partial X_{i_2} \cdots \partial X_{i_{n-1}}} \\ &+ \cdots + (-1)^{n-1} \frac{\partial^{n-1}(D_a F_a^* \boldsymbol{\rho}_a \rho_{ai_1} \cdots \rho_{ai_{n-1}})}{\partial X_{i_1} \cdots \partial X_{i_{n-1}}} \partial_\zeta \psi_a \\ &+ \frac{e_a}{m_a c^2}\left(D_a F_a^* \boldsymbol{\rho}_a \rho_{ai_1} \cdots \rho_{ai_{n-1}} \frac{\partial^{n-1}(\mathbf{A}_1 \cdot \partial_\zeta \mathbf{A}_1)}{\partial X_{i_1} \cdots \partial X_{i_{n-1}}} \right. \\ &- \frac{\partial(D_a F_a^* \boldsymbol{\rho}_a \rho_{ai_1} \cdots \rho_{ai_{n-1}})}{\partial X_{i1}} \frac{\partial^{n-2}(\mathbf{A}_1 \cdot \partial_\zeta \mathbf{A}_1)}{\partial X_{i_2} \cdots \partial X_{i_{n-1}}} + \cdots \\ &\left.\left. + (-1)^{n-1} \frac{\partial^{n-1}(D_a F_a^* \boldsymbol{\rho}_a \rho_{ai_1} \cdots \rho_{ai_{n-1}})}{\partial X_{i_1} \cdots \partial X_{i_{n-1}}} (\mathbf{A}_1 \cdot \partial_\zeta \mathbf{A}_1) \right) \right]. \end{aligned} \tag{100}$$

Using Eqs. (97)–(100) and Eq. (63) with $\mathcal{A} = A_{0\zeta}$, the toroidal angular momentum conservation law is written as

$$\begin{aligned} &\left\langle \frac{\partial}{\partial t}\left[P_{\|\zeta} - \frac{1}{c}\left(\mathbf{P}_L^{(\mathrm{pol})} + \frac{\mathbf{E}_L}{4\pi} \right) \cdot \nabla A_{0\zeta} \right] \right\rangle \\ &+ \frac{1}{V'}\frac{\partial}{\partial s}\left[V'\left\{ \Pi_{\|\zeta}^s + \Pi_{R\zeta}^s - \frac{1}{4\pi}\langle A_{1\zeta}(\nabla \times \mathbf{B}_1) \cdot \nabla s \rangle \right.\right. \\ &- \frac{1}{4\pi}\langle E_{L\zeta} E_L^s + B_{1\zeta} B_1^s \rangle + \frac{1}{4\pi c}\left\langle \frac{\partial\lambda}{\partial\zeta} A_1^s \right\rangle \\ &\left.\left. + \frac{1}{c}\left\langle \frac{\partial A_{0\zeta}}{\partial t}\left(\mathbf{P}_L^{(\mathrm{pol})} + \frac{\mathbf{E}_L}{4\pi} \right) \cdot \nabla s \right\rangle \right\} \right] = 0, \end{aligned} \tag{101}$$

where

$$P_{\|\zeta} = \sum_a \int dU \int d\mu \int d\xi \, D_a F_a m_a U b_\zeta, \tag{102}$$

$$\Pi_{\|\zeta}^s = \sum_a \int dU \int d\mu \int d\xi \, D_a F_a m_a U b_\zeta \mathbf{v}_a^{(\mathrm{gc})} \cdot \nabla s, \tag{103}$$

and

$$\Pi_{R\zeta}^s = \mathbf{P}_{R\zeta} \cdot \nabla s. \tag{104}$$

In Sec. VI C, we derive the ensemble-averaged toroidal angular momentum conservation from Eq. (101) in order to confirm that it is consistent with the conventional result up to the second order in δ.

VI. ENSEMBLE-AVERAGED CONSERVATION LAWS

In this section, the conservation laws derived in Sec. V are ensemble-averaged for the purpose of verifying their consistency with those obtained by previous works.[14,16,18]

First, we divide the vector potential $\mathbf{A}$ and the magnetic field $\mathbf{B}$ into the average and fluctuation parts as

$$\begin{aligned} \mathbf{A}(\mathbf{x}, t) &= \langle \mathbf{A}(\mathbf{x}, t) \rangle_{\mathrm{ens}} + \hat{\mathbf{A}}(\mathbf{x}, t), \\ \mathbf{B}(\mathbf{x}, t) &= \langle \mathbf{B}(\mathbf{x}, t) \rangle_{\mathrm{ens}} + \hat{\mathbf{B}}(\mathbf{x}, t), \end{aligned} \tag{105}$$

where $\langle \cdots \rangle_{\mathrm{ens}}$ represents the ensemble average, and we immediately find $\langle \hat{\mathbf{A}} \rangle_{\mathrm{ens}} = \langle \hat{\mathbf{B}} \rangle_{\mathrm{ens}} = 0$. We also identify the zeroth fields $\mathbf{A}_0$ and $\mathbf{B}_0$ with the ensemble-averaged parts to write

$$\begin{aligned} \mathbf{A}_0 &= \langle \mathbf{A} \rangle_{\mathrm{ens}}, \quad \mathbf{A}_1 = \hat{\mathbf{A}}, \\ \mathbf{B}_0 &= \langle \mathbf{B} \rangle_{\mathrm{ens}}, \quad \mathbf{B}_1 = \hat{\mathbf{B}}. \end{aligned} \tag{106}$$

Regarding the electrostatic potential ϕ, it is written as the sum of the average and fluctuation parts

$$\phi(\mathbf{x}, t) = \langle \phi(\mathbf{x}, t) \rangle_{\mathrm{ens}} + \hat{\phi}(\mathbf{x}, t). \tag{107}$$

Here, assuming that $\langle \phi(\mathbf{x}, t) \rangle_{\mathrm{ens}} \neq 0$, the background the $\mathbf{E} \times \mathbf{B}$ flow is retained and its velocity is regarded as $\mathcal{O}(\delta v_T)$, where δ and v_T represent the drift ordering parameter and the thermal velocity, respectively. Combining Eqs. (8), (106), and (107), we have

$$\psi_a = \langle \psi_a \rangle_{\mathrm{ens}} + \hat{\psi}_a, \tag{108}$$

where

$$\langle \psi_a \rangle_{\text{ens}} = \langle \phi \rangle_{\text{ens}}, \quad \hat{\psi}_a = \hat{\phi} - \frac{\mathbf{v}_0}{c} \cdot \hat{\mathbf{A}}. \tag{109}$$

We assume that the ensemble average $\langle \mathcal{Q} \rangle_{\text{ens}}$ of any variable $\mathcal{Q}$ considered here has a slow temporal variation subject to the so-called the transport ordering and that it has a gradient scale length L which is on the same order as gradient scale lengths of the equilibrium field and pressure profiles. These assumptions are expressed by

$$\begin{aligned} &\frac{\partial}{\partial t} \ln \langle \mathcal{Q} \rangle_{\text{ens}} = \mathcal{O}(\delta^2 v_T / L), \\ &|\nabla \ln \langle \mathcal{Q} \rangle_{\text{ens}}| = \mathcal{O}(1/L). \end{aligned} \tag{110}$$

We also impose the constraint of axisymmetry on $\langle \mathcal{Q} \rangle_{\text{ens}}$ that is written as

$$\frac{\partial \langle \mathcal{Q} \rangle_{\text{ens}}}{\partial \zeta} = 0, \tag{111}$$

even though $\mathcal{Q}$ itself is not axisymmetric. The spatiotemporal variations of the fluctuation part $\hat{\mathcal{Q}}$ are assumed to be subject to the conventional gyrokinetic orderings

$$\begin{aligned} &\frac{\partial}{\partial t} \ln \hat{\mathcal{Q}} = \mathcal{O}(v_T/L), \\ &|\mathbf{b} \cdot \nabla \ln \hat{\mathcal{Q}}| = \mathcal{O}(1/L), \\ &|\mathbf{b} \times \nabla \ln \hat{\mathcal{Q}}| = \mathcal{O}(1/\rho_T), \end{aligned} \tag{112}$$

where ρ_T stands for the thermal gyroradius. Besides, we use the WKB representation[19] of $\hat{\mathcal{Q}}$

$$\hat{\mathcal{Q}}(\mathbf{x}, t) = \sum_{\mathbf{k}_\perp} \hat{\mathcal{Q}}_{\mathbf{k}_\perp}(\mathbf{x}, t) \exp\left[i S_{\mathbf{k}_\perp}(\mathbf{x}, t) \right]. \tag{113}$$

Here, $\hat{\mathcal{Q}}_{\mathbf{k}_\perp}(\mathbf{x}, t)$ has a gradient scale length L while the eikonal $S_{\mathbf{k}_\perp}(\mathbf{x}, t)$ represents the rapid perpendicular variation with the wave number vector $\mathbf{k}_\perp \equiv \nabla S_{\mathbf{k}_\perp} (\sim 1/\rho)$ that is perpendicular to the background field $\mathbf{B}_0$. It is found from Eqs. (110), (112), and (113) that $\langle \hat{\mathcal{Q}}^*_{\mathbf{k}_\perp} \hat{\mathcal{Q}}_{\mathbf{k}'_\perp} \rangle_{\text{ens}} = \langle |\hat{\mathcal{Q}}_{\mathbf{k}_\perp}|^2 \rangle_{\text{ens}} \delta_{\mathbf{k}_\perp \mathbf{k}'_\perp}$, where $\delta_{\mathbf{k}_\perp \mathbf{k}'_\perp} = 1$ for $\mathbf{k}_\perp = \mathbf{k}'_\perp$ and $= 0$ for $\mathbf{k}_\perp \neq \mathbf{k}'_\perp$.

The distribution function F_a for species a is also divided into the average and fluctuation parts as

$$F_a = \langle F_a \rangle_{\text{ens}} + \hat{F}_a, \tag{114}$$

where the ensemble-averaged part $\langle F_a \rangle_{\text{ens}}$ consists of the local Maxwellian part and the deviation from it

$$\langle F_a \rangle_{\text{ens}} = F_{aM} + \langle F_{a1} \rangle_{\text{ens}}. \tag{115}$$

The local Maxwellian distribution function F_{aM} is written as

$$F_{aM} = n_{a0} \left(\frac{m_a}{2\pi T_{a0}} \right)^{3/2} \exp\left[-\frac{1}{T_{a0}} \left(\frac{1}{2} m_a U^2 + \mu B_0 \right) \right], \tag{116}$$

where the equilibrium density n_{a0} and temperature T_{a0} are regarded as uniform on flux surfaces. The first-order ensemble-averaged distribution function $\langle F_{a1} \rangle_{\text{ens}}$ is determined by the drift kinetic equation, which can be derived by substituting Eq. (115) into the ensemble average of Eq. (24). The derived equation agrees, to $\mathcal{O}(\delta)$, with the well-known linearized drift kinetic equation, on which the neoclassical transport theory is based.[30]

We write the fluctuation part $\hat{F}_a$ as

$$\hat{F}_a = -F_{aM} \frac{e_a \langle \hat{\psi}_a \rangle_\xi}{T_a} + \hat{h}_a. \tag{117}$$

Then, from the fluctuation part of the gyrokinetic equation in Eq. (24), we can derive

$$\begin{aligned} &\frac{\partial \hat{h}_a}{\partial t} + \{ \hat{h}_a, H_a \} \\ &= F_{aM} \left[\frac{e_a}{T_{a0}} \frac{\partial \langle \hat{\psi}_a \rangle_\xi}{\partial t} - \hat{\mathbf{v}}_a^{(\text{gc})} \cdot \left(\nabla \ln p_{a0} + \frac{e_a}{T_{a0}} \nabla \langle \phi \rangle_{\text{ens}} \right.\right. \\ &\quad \left.\left. + \left(\frac{\frac{1}{2} m_a U^2 + \mu B_0}{T_{a0}} - \frac{5}{2} \right) \nabla \ln T_{a0} \right) \right], \end{aligned} \tag{118}$$

which is valid to the lowest order in δ. Equation (118) agrees with the conventional gyrokinetic equation for the nonadiabatic part $\hat{h}_a$ of the perturbed distribution function derived from using the WKB representation.[18,31] On the right-hand side of Eq. (118), the turbulent part $\hat{\mathbf{v}}_a^{(\text{gc})}$ of the gyrocenter drift velocity $\mathbf{v}_a^{(\text{gc})} = d\mathbf{X}_a/dt = \{ \mathbf{X}_a, H_a \}$ is written as

$$\hat{\mathbf{v}}_a^{(\text{gc})} = \frac{c}{B_0} \mathbf{b} \times \nabla \langle \hat{\psi}_a (\mathbf{X} + \boldsymbol{\rho}_a, t) \rangle_\xi + \mathcal{O}(\delta^2). \tag{119}$$

Here, the gyrophase-averaged turbulent field is given in terms of the WKB representation by $\langle \hat{\psi}_a (\mathbf{X} + \boldsymbol{\rho}_a, t) \rangle_\xi = \sum_{\mathbf{k}_\perp} \hat{\psi}_{a\mathbf{k}_\perp} \exp[i S_{\mathbf{k}_\perp}(\mathbf{X}, t)]$ and

$$\hat{\psi}_{a\mathbf{k}_\perp} = J_0 \left(\frac{k_\perp v_\perp}{\Omega_a} \right) \left(\phi_{\mathbf{k}_\perp} - \frac{U}{c} A_{\parallel \mathbf{k}_\perp} \right) + J_1 \left(\frac{k_\perp v_\perp}{\Omega_a} \right) \frac{v_\perp}{c} \frac{B_{\parallel \mathbf{k}_\perp}}{k_\perp}, \tag{120}$$

where J_0 and J_1 are the zeroth- and first-order Bessel functions, respectively, and $v_\perp \equiv |(\mathbf{b} \times \mathbf{v}_{a0}) \times \mathbf{b}|$.

We find from Eqs. (56)–(58) that the ensemble average of $\mathbf{P}^{(\text{pol})}$ are written as

$$\langle \mathbf{P}^{(\text{pol})} \rangle_{\text{ens}} = \langle \mathbf{P}_g \rangle_{\text{ens}} + \langle \mathbf{P}_\psi \rangle_{\text{ens}}, \tag{121}$$

where

$$\begin{aligned} \langle \mathbf{P}_g \rangle_{\text{ens}} &= -\frac{1}{2} \sum_a e_a \nabla \cdot \left(\int dU \int d\mu \int d\xi \, D_a F_{aM} \boldsymbol{\rho}_a \boldsymbol{\rho}_a \right) + \mathcal{O}(\delta^2) \\ &= -\sum_a \frac{m_a c^2}{2 e_a} \nabla \cdot \left[\frac{n_{a0} T_{a0}}{B_0^2} (\mathbf{I} - \mathbf{bb}) \right] + \mathcal{O}(\delta^2), \end{aligned} \tag{122}$$

and

$$\begin{aligned} \langle \mathbf{P}_\psi \rangle_{\text{ens}} &= \sum_a \frac{e_a^2}{B_0} \int dU \int d\mu \int d\xi \, D_a \langle \tilde{\psi} \rangle_{\text{ens}} \frac{\partial F_{aM}}{\partial \mu} \boldsymbol{\rho}_a + \mathcal{O}(\delta^2) \\ &= \sum_a \frac{n_{a0} m_a c^2}{B_0^2} \langle \mathbf{E}_L \rangle_{\text{ens}} + \mathcal{O}(\delta^2). \end{aligned} \tag{123}$$

Here, $\langle\tilde{\psi}\rangle_{\rm ens} = \langle\tilde{\phi}\rangle_{\rm ens} \simeq \boldsymbol{\rho}_a \cdot \nabla\langle\phi\rangle_{\rm ens} = -\boldsymbol{\rho}_a \cdot \langle\mathbf{E}_L\rangle_{\rm ens}$ is used.

Taking the ensemble average of the gyrokinetic Poisson equation, Eq. (53), and noting that $-\nabla\cdot\langle\mathbf{P}^{\rm (pol)}\rangle_{\rm ens}$ and $\nabla\cdot\langle\mathbf{E}_L\rangle_{\rm ens}$ are of $\mathcal{O}(\delta)$, we obtain $\sum_a e_a\langle n_a^{\rm (gc)}\rangle_{\rm ens} = \mathcal{O}(\delta)$. On the other hand, it is shown by substituting Eq. (117) into Eq. (25) and using the Debye length $\lambda_D \equiv (4\pi\sum_a n_{a0}e_a^2/T_{a0})^{-2}$ that the turbulent part of the gyrokinetic Poisson equation is written in the WKB representation as

$$\left(k_\perp^2 + \lambda_D^{-2}\right)\hat{\phi}_{\mathbf{k}_\perp} = 4\pi\sum_a e_a\int dU\int d\mu\int d\xi\, D_a \times \hat{h}_{a\mathbf{k}_\perp} J_0\left(\frac{k_\perp v_\perp}{\Omega}\right), \tag{124}$$

which is valid to the lowest order in δ. Equation (124) coincides with the gyrokinetic Poisson equation derived by the conventional recursive formulation.[18,32,33] It can also be shown from Eqs. (27) and (117) that, to the lowest order in δ, the WKB representation of the turbulent part of Eq. (29) agrees with the conventional expression of Ampère's law.[18,32,33]

A. Ensemble-averaged particle transport equation

Before deriving the ensemble-averaged conservation laws of energy and toroidal angular momentum in the next subsections, we here consider the ensemble-averaged particle transport equation. Taking the ensemble average of Eq. (51) immediately yields

$$\left\langle\frac{\partial\langle n_a^{\rm (gc)}\rangle_{\rm ens}}{\partial t}\right\rangle + \frac{1}{V'}\frac{\partial}{\partial s}\left(V'\langle\langle n_a^{\rm (gc)}\mathbf{u}_a^{\rm (gc)}\cdot\nabla s\rangle\rangle\right) = 0, \tag{125}$$

where

$$\langle n_a^{\rm (gc)}\rangle_{\rm ens} = n_{a0} + \mathcal{O}(\delta) \tag{126}$$

and $\langle\langle\cdots\rangle\rangle$ represents a double average over the flux surface and the ensemble. Here, n_{a0} is the equilibrium density which is a flux-surface function and characterizes the Maxwellian distribution function F_{aM} in Eq. (116). The radial particle flux is written as

$$(\Gamma_a)^s = \langle\langle n_a^{\rm (gc)}\mathbf{u}_a^{\rm (gc)}\cdot\nabla s\rangle\rangle = (\Gamma_a^{\rm NA})^s + (\Gamma_a^{\rm A})^s, \tag{127}$$

which consists of the nonturbulent part

$$(\Gamma_a^{\rm NA})^s = \left\langle\int dU\int d\mu\int d\xi\, D_a\langle F_a\rangle_{\rm ens}\langle\mathbf{v}_a^{\rm (gc)}\rangle_{\rm ens}\cdot\nabla s\right\rangle \tag{128}$$

and the turbulence-driven part

$$(\Gamma_a^{\rm A})^s = \left\langle\int dU\int d\mu\int d\xi\, D_a\langle\hat{F}_a\hat{\mathbf{v}}_a^{\rm (gc)}\rangle_{\rm ens}\cdot\nabla s\right\rangle. \tag{129}$$

Here, the gyrocenter drift velocity is written as the sum of the ensemble-averaged and turbulent parts

$$\mathbf{v}_a^{\rm (gc)} = \langle\mathbf{v}_a^{\rm (gc)}\rangle_{\rm ens} + \hat{\mathbf{v}}_a^{\rm (gc)}, \tag{130}$$

where $\langle\mathbf{v}_a^{\rm (gc)}\rangle_{\rm ens}$ is obtained by taking the ensemble average of the right-hand side of Eq. (17) and the turbulent part of the gyrocenter drift velocity is given by Eq. (119). Using Eqs. (82), (125), and (127), the ensemble-averaged particle transport equation is written as

$$\frac{\partial}{\partial t}(V'n_{a0}) + \frac{\partial}{\partial s}\left(V'\left[(\Gamma_a^{\rm NA})^s + (\Gamma_a^{\rm A})^s - n_{a0}\langle\mathbf{u}_s\cdot\nabla s\rangle\right]\right) = 0. \tag{131}$$

Substituting Eq. (17) into Eq. (128), the nonturbulent radial particle flux is expressed by

$$(\Gamma_a^{\rm NA})^s = \left\langle\frac{c}{e_aB_0}\left[\mathbf{b}\times\left(\nabla\cdot\mathbf{P}_{a1}^{\rm CGL}\right)\right]\cdot\nabla s\right\rangle + n_{a0}\left\langle\frac{c}{B_0}\left(\langle\mathbf{E}\rangle_{\rm ens}\times\mathbf{b}\right)\cdot\nabla s\right\rangle + \mathcal{O}(\delta^3), \tag{132}$$

where $\mathbf{P}_{a1}^{\rm CGL}$ represents the first-order part of the pressure tensor in the Chew-Goldberger-Low (CGL) form[30] defined by

$$\mathbf{P}_{a1}^{\rm CGL} = \int dU\int d\mu\int d\xi\, D_a\langle F_{a1}\rangle_{\rm ens} \times\left[m_aU^2\mathbf{bb} + \mu B_0(\mathbf{I}-\mathbf{bb})\right], \tag{133}$$

and the ensemble-averaged electric field is given by

$$\langle\mathbf{E}\rangle_{\rm ens} = -\nabla\langle\phi\rangle_{\rm ens} - \frac{1}{c}\frac{\partial\mathbf{A}_0}{\partial t}.$$

The right-hand side of Eq. (132) expresses the neoclassical radial particle flux and the radial $\mathbf{E}\times\mathbf{B}$ drift which are well-known by the conventional neoclassical transport theory[30] although the collisional effects are not included in the present formulation based on the Lagrangian shown in Eq. (2). Substituting Eqs. (119) into Eq. (129) yields the turbulent radial particle flux given by

$$(\Gamma_a^{\rm A})^s = -\left\langle\left\langle\frac{c}{B_0}\int dU\int d\mu\int d\xi\, D_a\hat{h}_a(\nabla\hat{\psi}_a\times\mathbf{b})\cdot\nabla s\right\rangle\right\rangle + \mathcal{O}(\delta^3), \tag{134}$$

which is equivalent to the expression obtained by the conventional gyrokinetic theory based on the WKB formalism.[18]

As shown above, the well-known expressions of the neoclassical and turbulent particle fluxes are included in $(\Gamma_a^{\rm NA})^s$ and $(\Gamma_a^{\rm A})^s$. However, the classical particle flux does not appear in $(\Gamma_a^{\rm NA})^s$, respectively. This is because collisional processes are disregarded by the Lagrangian. The neoclassical particle flux included in $(\Gamma_a^{\rm NA})^s$ can also be shown to vanish after all by using the fact that $\langle F_{a1}\rangle_{\rm ens}$ should satisfy the drift kinetic equation with no collision term.

B. Ensemble-averaged energy conservation law

In the subsequent subsections, the ensemble-averaged energy and toroidal angular momentum conservation laws are derived from the results obtained by Noether's theorem in Secs. V A and V B. Taking the ensemble average of the energy density defined by Eq. (87) and expanding it in δ, we have

$$\langle \mathcal{E} \rangle_{\rm ens} = \frac{3}{2}\sum_a n_{a0}T_{a0} + \frac{B_0^2}{8\pi} + \mathcal{O}(\delta), \quad (135)$$

where the energy density of the electric field is neglected as a small quantity of $\mathcal{O}(\delta^2)$. The radial components of the first two terms on the right-hand side of Eq. (92) are double-averaged over the ensemble and the flux surface to obtain

$$\sum_a \left\langle \left\langle \int dU \int d\mu \int d\xi\, D_a F_a \left(H_a \mathbf{v}_a^{\rm (gc)} - \mu \frac{\partial \mathbf{A}_0}{\partial t} \times \mathbf{b} \right) \cdot \nabla s \right\rangle \right\rangle = \sum_a \left[(q_a)^s + \frac{5}{2} T_{a0} (\Gamma_a)^s \right] + \mathcal{O}(\delta^3), \quad (136)$$

Here, the radial particle flux $(\Gamma_a)^s$ is given by Eqs. (127) and the radial heat flux $(q_a)^s$ is written as

$$(q_a)^s = (q_a^{\rm NA})^s + (q_a^{\rm A})^s, \quad (137)$$

which consists of the nonturbulent part

$$(q_a^{\rm NA})^s = \left\langle \int dU \int d\mu \int d\xi\, D_a \langle F_{a1} \rangle_{\rm ens} \langle \mathbf{v}_a^{\rm (gc)} \rangle_{\rm ens} \cdot \nabla s \times \left(\frac{1}{2} m_a U^2 + \mu B_0 - \frac{5}{2} T_{a0} \right) \right\rangle, \quad (138)$$

and the turbulence-driven part

$$(q_a^{\rm A})^s = \left\langle \int dU \int d\mu \int d\xi\, D_a \langle \hat{F}_a \hat{\mathbf{v}}_a^{\rm (gc)} \rangle_{\rm ens} \cdot \nabla s \times \left(\frac{1}{2} m_a U^2 + \mu B_0 - \frac{5}{2} T_{a0} \right) \right\rangle. \quad (139)$$

In the similar manner to Eq. (132), the nonturbulent heat flux is written as

$$(q_a^{\rm NA})^s = T_{a0} \left\langle \frac{c}{e_a B_0} \left[\mathbf{b} \times \left(\nabla \cdot \boldsymbol{\Theta}_a^{\rm CGL} \right) \right] \cdot \nabla s \right\rangle + \mathcal{O}(\delta^3), \quad (140)$$

where the heat stress tensor $\boldsymbol{\Theta}_a^{\rm CGL}$ is defined by

$$T_{a0} \boldsymbol{\Theta}_a^{\rm CGL} = \int dU \int d\mu \int d\xi\, D_a \langle F_{a1} \rangle_{\rm ens} \times \left(\frac{1}{2} m_a U^2 + \mu B_0 - \frac{5}{2} T_{a0} \right) \times \left[m_a U^2 \mathbf{bb} + \mu B_0 (\mathbf{I} - \mathbf{bb}) \right]. \quad (141)$$

The expression of Eq. (140) coincides with that of the neoclassical radial heat flux in terms of the heat stress tensor.[30] The turbulent heat flux in Eq. (139) is also written in the same form as used in the conventional gyrokinetic theory[18]

$$(q_a^{\rm A})^s = -\left\langle \left\langle \frac{c}{B_0} \int dU \int d\mu \int d\xi\, D_a \hat{h}_a (\nabla \hat{\psi}_a \times \mathbf{b}) \cdot \nabla s \times \left(\frac{1}{2} m_a U^2 + \mu B_0 - \frac{5}{2} T_{a0} \right) \right\rangle \right\rangle + \mathcal{O}(\delta^3). \quad (142)$$

Now, using Eqs. (82), (90)–(93), (135), and (136), we find

$$\frac{\partial}{\partial t} \left(V' \left[\frac{3}{2} \sum_a n_{a0} T_{a0} + \frac{B_0^2}{8\pi} \right] \right) = -\frac{\partial}{\partial s} \left(V' \left[\sum_a \left((q_a)^s + \frac{5}{2} T_{a0} (\Gamma_a)^s \right) + \langle \mathbf{S}^{\rm (Poynting)} \cdot \nabla s \rangle - \left(\frac{3}{2} \sum_a n_{a0} T_{a0} + \frac{B_0^2}{8\pi} \right) \langle \mathbf{u}_s \cdot \nabla s \rangle \right] \right) + \mathcal{O}(\delta^3), \quad (143)$$

where $\mathbf{S}^{\rm (Poynting)} \equiv (c/4\pi) \langle \mathbf{E} \rangle_{\rm ens} \times \mathbf{B}_0$ represents the nonturbulent part of the Poynting vector. It is shown in Ref. 18 that the turbulent Poynting energy flux $(c/4\pi)\langle\langle(\hat{\mathbf{E}} \times \hat{\mathbf{B}}) \cdot \nabla s\rangle\rangle$ of $\mathcal{O}(\delta^2)$ is contained in $\sum_a (q_a^{\rm A})^s$. Here, using $(\nabla \times \mathbf{B}_0) \cdot \nabla s = (4\pi/c) \mathbf{J}_0 \cdot \nabla s = 0$, we have $\langle \mathbf{S}^{\rm (Poynting)} \cdot \nabla s \rangle = -(1/4\pi) \langle (\partial \mathbf{A}_0/\partial t \times \mathbf{B}_0) \cdot \nabla s \rangle$, which leads to

$$\frac{\partial}{\partial s} \left(V' \langle \mathbf{S}^{\rm (Poynting)} \cdot \nabla s \rangle \right) = -\frac{1}{4\pi} \frac{\partial}{\partial s} \left[V' \left\langle \left(\frac{\partial \mathbf{A}_0}{\partial t} \times \mathbf{B}_0 \right) \cdot \nabla s \right\rangle \right], = -\frac{V'}{4\pi} \left\langle \nabla \cdot \left(\frac{\partial \mathbf{A}_0}{\partial t} \times \mathbf{B}_0 \right) \right\rangle, \quad (144)$$

and

$$\left\langle \frac{\partial}{\partial t} \left(\frac{B_0^2}{8\pi} \right) \right\rangle = \frac{1}{4\pi} \left\langle \mathbf{B}_0 \cdot \left(\nabla \times \frac{\partial \mathbf{A}_0}{\partial t} \right) \right\rangle, = \frac{1}{4\pi} \left\langle \nabla \cdot \left(\frac{\partial \mathbf{A}_0}{\partial t} \times \mathbf{B}_0 \right) + (\nabla \times \mathbf{B}_0) \cdot \frac{\partial \mathbf{A}_0}{\partial t} \right\rangle, = -\frac{1}{V'} \frac{\partial}{\partial s} \left(V' \langle \mathbf{S}^{\rm (Poynting)} \cdot \nabla s \rangle \right) - \langle \mathbf{J}_0 \cdot \langle \mathbf{E} \rangle_{\rm ens} \rangle. \quad (145)$$

Combining Eqs. (143)–(145), we obtain

$$\frac{\partial}{\partial t} \left(V' \frac{3}{2} \sum_a n_{a0} T_{a0} \right) + \frac{\partial}{\partial s} \left(V' \left[\sum_a \left((q_a)^s + \frac{5}{2} T_{a0} (\Gamma_a)^s \right) - \frac{3}{2} \sum_a n_{a0} T_{a0} \langle \mathbf{u}_s \cdot \nabla s \rangle \right] \right) = V' \langle \mathbf{J}_0 \cdot \langle \mathbf{E} \rangle_{\rm ens} \rangle + \mathcal{O}(\delta^3). \quad (146)$$

Equations (143) and (146) take the well-known forms of the energy balance equations[34] except that the terms associated with the electric field energy and the kinetic energies due to the fluid velocities are neglected here as small quantities of higher order in δ. As explained in the end of Sec. VI A, since collisions are disregarded, the radial particle and heat fluxes, $(\Gamma_a)^s$ and $(q_a)^s$, which appear in Eq. (146), do not contain the contributions of the classical fluxes. Besides, it is shown from the drift kinetic equation without the collision term that the neoclassical parts included in $(\Gamma_a^{\rm NA})^s$ and $(q_a^{\rm NA})^s$ become zero.

C. Ensemble-averaged conservation law of toroidal angular momentum

In this subsection, the ensemble-averaged toroidal angular momentum conservation law is derived from using the results in Sec. V B. Using Eqs. (30), (121)–(123), and $A_{0\zeta} = -\chi$, we obtain

$$\left\langle -\frac{1}{c}\left(\mathbf{P}_L^{(\mathrm{pol})} + \frac{\mathbf{E}_L}{4\pi}\right)\cdot\nabla A_{0\zeta}\right\rangle_{\mathrm{ens}} = \left\langle \sum_a \frac{n_{a0}m_a c}{B_0}\mathbf{E}_L\times\mathbf{b} + \frac{1}{4\pi c}\mathbf{E}_L\times\mathbf{B}_0\right\rangle_{\mathrm{ens}}\cdot\mathbf{e}_\zeta + \frac{1}{c}\langle\mathbf{P}_g\rangle_{\mathrm{ens}}\cdot\nabla\chi + \mathcal{O}(\delta^2), = \left(\sum_a n_{a0}m_a\mathbf{u}_E + \frac{\mathbf{S}^{(\mathrm{Poynting})}}{c^2}\right)\cdot\mathbf{e}_\zeta + \frac{1}{c}\langle\mathbf{P}_g\rangle_{\mathrm{ens}}\cdot\nabla\chi + \mathcal{O}(\delta^2), \tag{147}$$

where $\mathbf{u}_E \equiv c\langle\mathbf{E}\rangle_{\mathrm{ens}}\times\mathbf{b}/B_0$ represents the nonturbulent part of the $\mathbf{E}\times\mathbf{B}$ drift velocity [note that the contributions of $\mathbf{E}_T \equiv -c^{-1}\partial\mathbf{A}/\partial t$ to $\mathbf{u}_E$ and $\mathbf{S}^{(\mathrm{Poynting})}$ are smaller by the factor of δ than those of $\mathbf{E}_L \equiv -\nabla\phi$]. We find that the term $\frac{1}{c}\langle\mathbf{P}_g\rangle_{\mathrm{ens}}\cdot\nabla\chi$ in Eq. (147) cannot be written in the form of the toroidal component of the momentum $\sum_a n_{a0}m_a\mathbf{u}_a^{(\mathrm{dia})}$ due to the diamagnetic drift velocity $\mathbf{u}_a^{(\mathrm{dia})} \equiv (c/e_a n_{a0}B_0)\mathbf{b}\times\nabla(n_{a0}T_{a0})$ although the magnitude of $\frac{1}{c}\langle\mathbf{P}_g\rangle_{\mathrm{ens}}\cdot\nabla\chi$ is on the same order of $\sum_a n_{a0}m_a\mathbf{u}_a^{(\mathrm{dia})}\cdot\mathbf{e}_\zeta$. Then, using Eqs. (10) and (47) and (110) and (147), we have

$$\frac{\partial}{\partial t}\left\langle P_{\parallel\zeta} - \frac{1}{c}\left(\mathbf{P}_L^{(\mathrm{pol})} + \frac{\mathbf{E}_L}{4\pi}\right)\cdot\nabla A_{0\zeta}\right\rangle_{\mathrm{ens}} = \frac{\partial}{\partial t}\left[\left(\sum_a n_{a0}m_a\left(u_{a\parallel}\mathbf{b} + \mathbf{u}_E\right) + \frac{\mathbf{S}^{(\mathrm{Poynting})}}{c^2}\right)\cdot\mathbf{e}_\zeta + \frac{1}{c}\langle\mathbf{P}_g\rangle_{\mathrm{ens}}\cdot\nabla\chi\right] + \mathcal{O}(\delta^4), \tag{148}$$

where $u_{a\parallel}$ represents the nonturbulent part of the parallel fluid velocity for particle species a defined by $n_{a0}u_{a\parallel} \equiv \int dU\int d\mu\int d\xi\langle F_{a1}\rangle_{\mathrm{ens}}U$. It should be noted that, on the right-hand side of Eqs. (148), each of the temporal variation terms including $u_{a\parallel}$, $\mathbf{u}_E$, $\mathbf{S}^{(\mathrm{Poynting})}$, $\langle\mathbf{P}_g\rangle_{\mathrm{ens}}$ is of $\mathcal{O}(\delta^3)$ while $\mathbf{S}^{(\mathrm{Poynting})}/c^2 = (v_A^2/c^2)\sum_a n_{a0}m_a\mathbf{u}_E$ is obtained from using the Alfvén velocity $v_A \equiv B_0/(4\pi\sum_a n_{a0}m_a)^{1/2}$.

Using Eqs. (8), (100), (103), (104), (113), (117), and (119), we have

$$\langle\langle\Pi_{R\zeta}^s\rangle\rangle = \sum_a e_a\sum_{\mathbf{k}_\perp}\sum_{n=1}^{\infty}\frac{i^n}{(n-1)!}\left\langle\left\langle\int dU\int d\mu\int d\xi\, D_a\hat{h}_{a\mathbf{k}}^* \times(\nabla s\cdot\boldsymbol{\rho}_a)(\mathbf{k}_\perp\cdot\mathbf{e}_\zeta)(\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a)^{n-1} \times\left(\hat{\phi}_{\mathbf{k}_\perp} - \frac{\mathbf{v}_{a0}}{c}\cdot\hat{\mathbf{A}}_{\mathbf{k}_\perp}\right)\right\rangle\right\rangle + \mathcal{O}(\delta^3), = \sum_a e_a\left\langle\left\langle\int dU\int d\mu\int d\xi\, D_a\hat{h}_a(\mathbf{X})(\nabla s\cdot\boldsymbol{\rho}_a) \times(\mathbf{e}_\zeta\cdot\nabla_\perp)\hat{\psi}_a(\mathbf{X}+\boldsymbol{\rho}_a)\right\rangle\right\rangle + \mathcal{O}(\delta^3), = \sum_a\left\langle\left\langle\int dU\int d\mu\int d\xi\, D_a\hat{h}_a m_a(\mathbf{v}_{a0\perp}\cdot\mathbf{e}_\zeta) \times(\hat{\mathbf{v}}_a^{(\mathrm{gc})}\cdot\nabla s)\right\rangle\right\rangle + \mathcal{O}(\delta^3) \tag{149}$$

and

$$\langle\langle\Pi_{\parallel\zeta}^s + \Pi_{R\zeta}^s\rangle\rangle = \sum_a\left[(\Pi_a^{\mathrm{NA}})^s + (\Pi_a^{\mathrm{A}})^s\right] + \mathcal{O}(\delta^3), \tag{150}$$

where the nonturbulent and turbulence-driven parts of the radial flux of the toroidal angular momentum are defined by

$$(\Pi_a^{\mathrm{NA}})^s = \left\langle\int dU\int d\mu\int d\xi\, D_a\langle F_{a1}\rangle_{\mathrm{ens}} \times m_a U b_\zeta\langle\mathbf{v}_a^{(\mathrm{gc})}\rangle_{\mathrm{ens}}\cdot\nabla s\right\rangle, \tag{151}$$

and

$$(\Pi_a^{\mathrm{A}})^s = \left\langle\left\langle\int dU\int d\mu\int d\xi\, D_a\hat{h}_a \times m_a(U\mathbf{b} + \mathbf{v}_{a0\perp})\cdot\mathbf{e}_\zeta(\hat{\mathbf{v}}_a^{(\mathrm{gc})}\cdot\nabla s)\right\rangle\right\rangle, \tag{152}$$

respectively. Finally, the ensemble-averaged toroidal angular momentum conservation law is rewritten by using Eqs. (82), (101), (148), and (150) as

$$\frac{\partial}{\partial t}\left(V'\left\langle\left[\sum_a n_{a0}m_a(u_{a\parallel}\mathbf{b} + \mathbf{u}_E) + \frac{\mathbf{S}^{(\mathrm{Poynting})}}{c^2}\right]\cdot\mathbf{e}_\zeta\right\rangle\right) + \frac{\partial}{\partial s}\left(V'\left[\sum_a\left\{(\Pi_a^{\mathrm{NA}})^s + (\Pi_a^{\mathrm{A}})^s - \left\langle\left[\sum_a n_{a0}m_a(u_{a\parallel}\mathbf{b} + \mathbf{u}_E) + \frac{\mathbf{S}^{(\mathrm{Poynting})}}{c^2}\right]\cdot\mathbf{e}_\zeta(\mathbf{u}_s\cdot\nabla s)\right\rangle\right\} - \frac{1}{4\pi}\langle\langle\nabla s\cdot\left[\hat{\mathbf{E}}_L\hat{\mathbf{E}}_L + \hat{\mathbf{B}}\hat{\mathbf{B}} + (\nabla\times\hat{\mathbf{B}})\hat{\mathbf{A}}\right]\cdot\mathbf{e}_\zeta\rangle\rangle\right]\right) = \mathcal{O}(\delta^3), \tag{153}$$

where the terms including $(u_{a\parallel}\mathbf{b} + \mathbf{u}_E)$ and $\mathbf{S}^{(\mathrm{Poynting})}$ are of $\mathcal{O}(\delta^3)$ although they are explicitly written down for comparison with the toroidal momentum balance equation in Ref. 16.

In Ref. 16, the toroidal flow velocity $\mathbf{V}_0$ on the order of the sonic speed (high-flow ordering) is assumed to be given by the sum of the parallel flow velocity and the $\mathbf{E}\times\mathbf{B}$ drift velocity of $\mathcal{O}(v_T)$ although the toroidal momentum balance equation, Eq. (57), in Ref. 16 is still valid to $\mathcal{O}(\delta^2)$ even if we use it for the present case of $\mathbf{V}_0 = \mathcal{O}(\delta v_T)$ (low-flow ordering). Keeping this in mind, we see that, up to $\mathcal{O}(\delta^2)$, Eq. (153) is consistent with Eq. (57) in Ref. 16. The classical toroidal momentum flux is not included in the nonturbulent toroidal momentum flux $(\Pi_a^{\mathrm{NA}})^s$ because collisions are not taken into account in the present formulation. It is also shown from the drift kinetic equation with no collision term that $(\Pi_a^{\mathrm{NA}})^s$ vanishes. The expression of the turbulent toroidal momentum flux $(\Pi_a^{\mathrm{A}})^s$ in Eq. (152) coincides with the one derived from the WKB formalism [see Eq. (53) in Ref. 16]. The terms associated with the turbulent Maxwell stress tensor $(\hat{\mathbf{E}}_L\hat{\mathbf{E}}_L + \hat{\mathbf{B}}\hat{\mathbf{B}})/(4\pi)$ in Eq. (153) corresponds to the anomalous toroidal momentum production term $\sum_a\langle\int d^3v\, m_a v_\zeta\mathcal{D}_a\rangle$ [see Eq. (63) in Ref. 16]. The last stress term including $(\nabla \times \hat{\mathbf{B}})\hat{\mathbf{A}}$ on the left-hand side is combined with $(\Pi_a^{\rm A})^s$ to generate the anomalous stress term $(\Pi_a^{\rm anom})^s$ [see Eq. (60) in Ref. 16]. The inertia terms $\partial(\cdots)/\partial t$ in Eq. (153) agree with $(\partial/\partial t)\langle(\sum_a m_a n_a)(1 + v_{PA}^2/c^2)R^2 V^\zeta\rangle$ in Ref. 16 although they are of $\mathcal{O}(\delta^3)$ due to the low-flow ordering. The correction term due to the radial motion $\mathbf{u}_s \cdot \nabla s$ of the flux surface is retained here but it is also of $\mathcal{O}(\delta^3)$. Besides, it is shown in Ref. 14 that, when there exists the up-down symmetry of the background magnetic field, $(\Pi_a^{\rm A})^s$ and all other stress terms due to $\hat{E}_L = -\nabla\hat{\phi}$ and $\hat{\mathbf{B}}$ in Eq. (153) become zero. Therefore, for that case, the nontrivial toroidal momentum balance equation is of $\mathcal{O}(\delta^3)$ as argued in Ref. 14.

VII. CONCLUSIONS

In this work, a gyrokinetic system of equations for turbulent toroidal plasmas in time-dependent axisymmetric background magnetic fields are derived from the variational principle using the Lagrangian which includes the constraint on the background fields. From these equations, the background fields, which vary on the transport time scale, can be determined self-consistently with the relaxation of the pressure profile due to the turbulent particle and heat transport.

Conservation laws of energy and toroidal angular momentum are derived from applying Noether's theorem to the action integral of the Lagrangian. Besides, assuming separate spatiotemporal scales for the average and fluctuation parts of physical variables, ensemble averages of particle, energy, and toroidal angular momentum conservation laws are taken. The resultant ensemble-averaged conservation laws are consistent to the lowest order in the gyrokinetic ordering parameter δ, namely $\mathcal{O}(\delta^2)$, with those obtained by the conventional gyrokinetic theory based on the WKB formalism. We should note that the present and conventional gyrokinetic equations are both accurate up to $\mathcal{O}(\delta)$ and that the classical and neoclassical transport fluxes vanish in the present work because collisional processes are ignored here.

As shown in Ref. 14, in the case of the low-flow ordering, all terms in the ensemble-averaged toroidal momentum conservation law vanish to $\mathcal{O}(\delta^2)$ in the axisymmetric background magnetic field with the up-down symmetry, for which the background radial electric field E_s cannot be determined by the $\mathcal{O}(\delta^2)$ toroidal momentum balance equation although the $\mathcal{O}(\delta^2)$ transport equations for particles and energy are not influenced by E_s either. It is known that, for rotating plasmas with large toroidal flows on the order of the ion thermal speed, E_s can be determined from the $\mathcal{O}(\delta^2)$ toroidal momentum transport equation.[16] It should be noted here that the above-mentioned remarks on the momentum transport strongly depend on the ordering argument combined with the scale separation assumptions using the WKB representation as described in Sec. VI. The scale separation assumptions may not be satisfied by some solutions of the gyrokinetic equations, which may show significantly nonlocal turbulent momentum transport different from the prediction by the above ordering argument.

In the present work, collisions are neglected so that the resistive diffusion of the background magnetic field is not treated here. In order to describe the resistive diffusion process, we need to add the collision term into the gyrokinetic equation, which yields the resistivity relating the electric current to the inductive electric field. As future tasks, we plan to extend the present work to include effects of collisions, external sources, and large toroidal flows.

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