Entropy production and Onsager symmetry in neoclassical transport processes of toroidal plasmas

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Entropy production and Onsager symmetry in neoclassical transport processes of magnetically confined plasmas are studied in detail for general toroidal systems, including nonaxisymmetric configurations. It is found that the flux surface average of the entropy production defined from the linearized collision operator and the gyroangle-averaged distribution function coincides with the sum of the inner products of the thermodynamic forces and the conjugate fluxes consisting of the Pfirsch–Schlüter, banana-plateau, nonaxisymmetric parts of the neoclassical radial fluxes and the parallel current. It is proved from the self-adjointness of the linearized collision operator that the Onsager symmetry is robustly valid for the neoclassical transport equations in the cases of general toroidal plasmas consisting of electrons and multi-species ions with arbitrary collision frequencies. It is shown that the Onsager symmetry holds whether or not the ambipolarity condition is used to reduce the number of the conjugate pairs of the transport fluxes and the thermodynamic forces. The full transport coefficients for the banana-plateau and nonaxisymmetric parts are separately derived, and their symmetry properties are investigated. The nonaxisymmetric transport equations are obtained for arbitrary collision frequencies in the Pfirsch–Schlüter and plateau regimes, and it is directly confirmed that the total banana-plateau and nonaxisymmetric transport equations satisfy the Onsager symmetry. © *1996 American Institute of Physics*. [S1070-664X(96)02001-3]

I. INTRODUCTION

Transport processes caused by binary Coulomb collisions between charged particles in toroidal magnetic configurations are described by the neoclassical transport theory.[1–3] Particle and heat transport fluxes observed in most fusion devices exceed the predictions of the neoclassical transport theory, and thus are called anomalous transport.[4] The anomalous transport is considered to result from the turbulent fluctuations driven by various plasma instabilities, which are not taken into account by the neoclassical theory. However, the neoclassical theory is regarded as a standard model with a well-established framework, which gives lower limits of the transport fluxes in quiescent plasmas close to thermal equilibria, and it also has practical use for predicting the parallel transport such as a bootstrap current with more accuracy than the predictions for the radial transport. The characteristics of the neoclassical theory lie in the inclusion of the effects of the global magnetic field geometry due to the long mean free path of the hot particles. In less collisional regimes such as plateau and banana regimes, the processes in deriving the neoclassical transport equations, which relate the neoclassical transport fluxes to the thermodynamic forces, are more complicated than the derivation of the classical transport equations in more collisional regimes. Since the neoclassical transport coefficients contain the parameters relating to both the collisionality and the magnetic geometry, it is less trivial than in the case of the classical transport to show the validity of the Onsager symmetry[5] for the neoclassical transport. In the cases of axisymmetric systems, the analytical expressions of the full neoclassical transport coefficients for all collisionality regimes have been obtained, and it is well-known that the Onsager symmetry holds for the axisymmetric neoclassical transport matrix.

The neoclassical transport equations for nonaxisymmetric systems[6–11] are more complicated due to the geometrical complexities, and they involve the nonambipolar parts. The absence of symmetry may also cause the breakup of magnetic surfaces into islands and ergordic regions,[12] although such problems are beyond the scope of this work. Here we assume the existence of toroidal nested magnetic surfaces as in other previous works.[6–11] Balescu and Fantechi derive the full neoclassical transport coefficients for the nonaxisymmetric plasma in the plateau regime, and claim that the Onsager symmetry partly breaks down in that case.[11] Contrary to Ref. 11, we show in the present work that the Onsager symmetry is robustly valid for the neoclassical transport equations in the cases of general toroidal plasmas consisting of electrons and multi-species ions with arbitrary collision frequencies.

Concerning the Onsager relation, it is important to discuss the entropy production resulting from the transport processes since the Onsager relation, if it holds, is satisfied only by the transport matrix connecting the conjugate pairs of transport fluxes and thermodynamic forces, which should be specified through the entropy production.[5] According to the terminology in Ref. 3, we expect that the kinetic form of the entropy production defined from the collision operator should coincide with its thermodynamic form, in which the entropy production is expressed as the sum of the products of the conjugate pairs of the fluxes and forces. In Chap. 17 of Ref. 3, Balescu presented detailed analyses on the kinetic and thermodynamic forms of the entropy production in the classical and neoclassical transport processes for the axisymmetric case. Using the Hermitian moment representation, he confirmed that the kinetic form of the entropy production includes the thermodynamic form given by the products of the thermodynamic forces and their conjugate classical and

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neoclassical Pfirsch–Schlüter fluxes, although he did not identify the thermodynamic form corresponding to the neoclassical banana-plateau transport. In this work, before the proof of the Onsager symmetry, we show for general toroidal geometry the complete coincidence between the kinetic form of the entropy production and its full thermodynamic form, including all contributions from the classical and neoclassical (Pfirsch–Schlüter, banana-plateau, nonaxisymmetric) fluxes. The main difference between our treatment and that in Ref. 3 is that we use directly the distribution function and the drift kinetic equations instead of the Hermitian moment expansion.

The proof of the Onsager symmetry for the neoclassical transport equations in general toroidal configurations is given in the similar manner to that in the Appendix of Ref. 2. The proof uses the self-adjointness of the linearized collision operator and the formal solution of the linearized drift kinetic equation, although neither axisymmetry of the magnetic configuration nor any condition for collisionality is required. We also derive the full neoclassical transport coefficients in the nonaxisymmetric system for collision frequencies in the Pfirsch–Schlüter and plateau regimes, from which the Onsager symmetry of the full neoclassical transport matrix is directly confirmed.

In the axisymmetric configurations, the ambipolarity is automatically satisfied by the neoclassical transport and the radial electric field does not affect the transport fluxes. On the other hand, in the nonaxisymmetric configurations, the radial electric field is determined through the particle transport equations if the ambipolarity condition is imposed. In both cases with and without the ambipolarity condition, we give the neoclassical transport equations and check their Onsager symmetry.

This work is organized as follows. In Sec. II, the entropy production defined from the collision operator is divided into the two parts, which are derived from the gyroangle-averaged and gyroangle-dependent parts of the distribution function. The entropy production from the gyroangle-dependent distribution function is given by the sum of the inner products of the classical radial particle and heat fluxes and the radial gradient thermodynamic forces. The transport matrix relating these classical fluxes and forces is shown to satisfy the Onsager symmetry. We find that the entropy production from the gyroangle-averaged distribution function is written as the sum of the inner products of the thermodynamic forces and the corresponding conjugate fluxes which consist of the Pfirsch–Schlüter, banana-plateau, nonaxisymmetric parts of the neoclassical radial fluxes and the parallel current. In Sec. III, using the formal solution of the linearized drift kinetic equation and the self-adjointness of the linearized collision operator, we prove that the Onsager symmetry is satisfied by the neoclassical transport equations for arbitrary collision frequencies in general toroidal systems, including nonaxisymmetric cases. The effects of the ambipolarity on the neoclassical transport coefficients are examined for both axisymmetric and nonaxisymmetric cases to show the robust validity of the Onsager symmetry independent of the use of the ambipolarity condition. In Sec. IV, the full transport coefficients are derived for the banana-plateau and nonaxisymmetric parts, separately, and their symmetry properties are investigated. We derive the nonaxisymmetric transport coefficients for arbitrary collision frequencies in the Pfirsch–Schlüter and plateau regimes, and directly confirm that the total banana-plateau and nonaxisymmetric transport equations satisfy the Onsager symmetry. Finally, conclusions and discussions are given in Sec. V. There, we discuss the reason why our two main results, i.e., the complete correspondence between the kinetic and thermodynamic forms of the entropy production, and the Onsager symmetry in the nonaxisymmetric case, were not confirmed in Ref. 3 and in Ref. 11, respectively.

II. ENTROPY PRODUCTION IN CLASSICAL AND NEOCLASSICAL TRANSPORT PROCESSES

Here, we show that the thermodynamic form of the entropy production is equivalent to its kinetic form defined from the collision operator for the classical and neoclassical transport in the general case of magnetically confined plasma with arbitrary toroidal geometry. For that purpose, we first describe several properties of the collision operator which is denoted for species a by

$$C_a=\sum_b C_{ab}(f_a,f_b), \tag{1}$$

where f_a (f_b) is the distribution function of the species a (b) and C_{ab} represents the contribution from the collision between the particles a and b.

The collision operator conserves the particles' number, momentum, and kinetic energy, which is written as

$$\begin{aligned} &\int d^3v\, C_{ab}=0,\\ &\int d^3v\, m_a\mathbf{v}C_{ab}+\int d^3v\, m_b\mathbf{v}C_{ba}=0,\\ &\int d^3v \frac{1}{2}m_a v^2 C_{ab}+\int d^3v \frac{1}{2}m_b v^2 C_{ba}=0. \end{aligned} \tag{2}$$

Furthermore, the collision operator is invariant under arbitrary translational and rotational transform of the velocity variable $\mathbf{v}$ of distribution functions, which is expressed by

$$\begin{aligned} &\mathscr{T}C_{ab}(f_a,f_b)=C_{ab}(\mathscr{T}f_a,\mathscr{T}f_b),\\ &\mathscr{R}C_{ab}(f_a,f_b)=C_{ab}(\mathscr{R}f_a,\mathscr{R}f_b), \end{aligned} \tag{3}$$

where $\mathscr{T}f$ and $\mathscr{R}f$ denote functions f with arbitrary translational and rotational transform of the velocity variable $\mathbf{v}$, respectively.

The entropy production for the species a is defined from the collision operator by

$$\dot{S}_a\equiv\sum_b \dot{S}_{ab}\equiv-\sum_b \int d^3v(\ln f_a)C_{ab}(f_a,f_b). \tag{4}$$

The second law of the thermodynamics or the positive definiteness of the entropy production is given by

$$\dot{S}_{ab}+\dot{S}_{ba}\geqslant 0, \quad \sum_a \dot{S}_a\geqslant 0, \tag{5}$$

where the total entropy production $\Sigma_a \dot{S}_a$ vanishes if and only if the distribution functions for all species are the Maxwellian with the same temperature and the same mean velocity. However, even if two particle species with much different masses (such as for electrons and ions) have the Maxwellian distributions with different temperatures, the collisional heat exchange between the two species are negligibly slow and the entropy production due to their collisions is so small that we hereafter neglect it based on the small mass ratio ordering.

For magnetically confined plasmas, the lowest-order distribution function for the species a with respect to the drift ordering is given by the Maxwellian with no mean velocity as

$$f_{a0} = \pi^{-3/2} n_a v_{Ta}^{-3} \exp(-x_a^2) \equiv f_{aM}, \tag{6}$$

where $v_{Ta}^2 \equiv 2T_a/m_a$ and $x_a \equiv v/v_{Ta}$ are defined from the temperature T_a. The distribution functions and the collision operator are perturbatively expanded in the drift ordering parameter $\delta \equiv \rho/L$ (ρ: the thermal gyroradius, L: the equilibrium scale length) as

$$\begin{aligned} f_a &= f_{a0} + f_{a1} + \mathcal{O}(\delta^2), \\ C_{ab} &= C_{ab}(f_{a0}, f_{b0}) + C_{ab}^L(f_{a1}, f_{b1}) + \mathcal{O}(\delta^2), \end{aligned} \tag{7}$$

where the linearized collision operator C_{ab}^L is defined by

$$C_{ab}^L(f_{a1}, f_{b1}) \equiv C_{ab}(f_{a1}, f_{b0}) + C_{ab}(f_{a0}, f_{b1}). \tag{8}$$

The linearized collision operator also has the conservation and symmetry properties, which are expressed by Eqs. (2) and (3) with C_{ab} replaced by C_{ab}^L. The self-adjointness of the linearized collision operator[13] is written as

$$\begin{aligned} &T_a \int d^3v \frac{g_{a1}}{f_{a0}} C_{ab}^L(h_{a1}, h_{b1}) + T_b \int d^3v \frac{g_{b1}}{f_{b0}} C_{ba}^L(h_{b1}, h_{a1}) \\ &= T_a \int d^3v \frac{h_{a1}}{f_{a0}} C_{ab}^L(g_{a1}, g_{b1}) \\ &\quad + T_b \int d^3v \frac{h_{b1}}{f_{b0}} C_{ba}^L(g_{b1}, g_{a1}), \end{aligned} \tag{9}$$

which is exactly valid for $T_a = T_b$ and is approximately satisfied for $T_a \neq T_b$ when $m_a/m_b \ll 1$ or $m_b/m_a \ll 1$. For example, in the case of collisions between ions ($a=i$) and electrons ($b=e$) with $T_i \neq T_e$, C_{ie}^L contains a part which breaks the complete self-adjointness although it is neglected to the lowest order in $(m_e/m_i)^{1/2}$ (see Sec. IV of Ref. 13). Concerning the positive definiteness of the entropy production described in Eq. (5), we find the positive definiteness of the quadratic form associated with the linearized collision operator as

$$\begin{aligned} &-T_a \int d^3v \frac{g_{a1}}{f_{a0}} C_{ab}^L(g_{a1}, g_{b1}) \\ &\quad - T_b \int d^3v \frac{g_{b1}}{f_{b0}} C_{ba}^L(g_{b1}, g_{a1}) \geq 0, \end{aligned} \tag{10}$$

which is valid for $T_a \neq T_b$ to the lowest order of the small mass ratio $m_a/m_b \ll 1$ (or $m_b/m_a \ll 1$) as Eq. (9).

Using in Eq. (4)

$$\ln f_a = \ln f_{a0} + \frac{f_{a1}}{f_{a0}} + \mathcal{O}(\delta^2),$$

the entropy production $\dot{S}_a$ up to $\mathcal{O}(\delta^2)$ is given by

$$\sigma_a \equiv \sum_b \sigma_{ab} \equiv -\sum_b \int d^3v \frac{f_{a1}}{f_{a0}} C_{ab}^L(f_{a1}, f_{b1}). \tag{11}$$

Instead of Eq. (5), we have from Eq. (10)

$$T_a \sigma_{ab} + T_b \sigma_{ba} \geq 0, \quad \sum_a T_a \sigma_a \geq 0. \tag{12}$$

Now, let us divide the first-order distribution function f_{a1} into the gyroangle-averaged part $\bar{f}_{a1}$ and the gyroangle-dependent part $\tilde{f}_{a1}$ as

$$f_{a1} = \bar{f}_{a1} + \tilde{f}_{a1}. \tag{13}$$

Due to the rotational symmetry of the collision operator in the velocity space shown in Eq. (3), the entropy production σ_a separates into the corresponding parts as

$$\sigma_a = \bar{\sigma}_a + \tilde{\sigma}_a, \tag{14}$$

where

$$\bar{\sigma}_a = -\sum_b \int d^3v \frac{\bar{f}_{a1}}{f_{a0}} C_{ab}^L(\bar{f}_{a1}, \bar{f}_{b1}), \tag{15}$$

$$\tilde{\sigma}_a = -\sum_b \int d^3v \frac{\tilde{f}_{a1}}{f_{a0}} C_{ab}^L(\tilde{f}_{a1}, \tilde{f}_{b1}). \tag{16}$$

First, we consider the entropy production $\tilde{\sigma}_a$ due to the gyroangle-dependent part of the distribution function. The gyroangle-dependent part $\tilde{f}_{a1}$ is given from the lowest-order distribution function f_{a0} by

$$\begin{aligned} \tilde{f}_{a1} &= \frac{\mathbf{v}\times\mathbf{n}}{\Omega_a} \cdot \nabla f_{a0} = -f_{a0} \frac{\mathbf{v}\times\mathbf{n}\cdot\nabla V}{\Omega_a T_a} \left[X_{a1} + X_{a2} \left(x_a^2 - \frac{5}{2} \right) \right] \\ &= \frac{m_a}{T_a} \mathbf{v} \cdot \left[\mathbf{u}_{\perp a} + \frac{2}{5} \frac{\mathbf{q}_{\perp a}}{p_a} \left(x_a^2 - \frac{5}{2} \right) \right] f_{a0}, \end{aligned} \tag{17}$$

where $\mathbf{n} = \mathbf{B}/B$ is the unit vector along the magnetic field $\mathbf{B}$ and $\Omega_a = e_a B / m_a c$ is the gyrofrequency of the particle with the mass m_a and the charge e_a. In Eq. (17), f_{a0} is regarded as a function of (V, E, μ) (V: the volume inside the flux surface, $E \equiv \frac{1}{2} m_a v^2 + e_a \Phi$: the particle's energy, and $\mu \equiv m_a v_\perp^2 / 2B$: the magnetic moment), and we have used

$$\frac{\partial \ln f_{a0}}{\partial V} = -\frac{1}{T_a} \left[X_{a1} + X_{a2} \left(x_a^2 - \frac{5}{2} \right) \right], \tag{18}$$

where the thermodynamic forces X_{a1} and X_{a2} are defined from the radial gradients of the pressure p_a, the electrostatic potential Φ, and the temperature T_a as

$$X_{a1} \equiv -\frac{p_a'}{n_a} - e_a \Phi', \quad X_{a2} \equiv -T_a' \quad \left(' \equiv \frac{\partial}{\partial V} \right), \tag{19}$$

in terms of which the perpendicular components of the fluid velocity and the heat flow are

$$
\mathbf{u}_{\perp a} \equiv \frac{1}{n_a} \int d^3v \tilde{f}_{a1} \mathbf{v}_{\perp} = \frac{cX_{a1}}{e_aB} \nabla V \times \mathbf{n},
$$

$$
\frac{\mathbf{q}_{\perp a}}{p_a} \equiv \frac{1}{n_a} \int d^3v \tilde{f}_{a1} \mathbf{v}_{\perp} \left(x_a^2 - \frac{5}{2} \right) = \frac{5}{2} \frac{cX_{a2}}{e_aB} \nabla V \times \mathbf{n}. \tag{20}
$$

Substituting Eq. (17) into Eq. (16), we obtain

$$
T_a \tilde{\sigma}_a = J^{\mathrm{cl}}_{a1} X_{a1} + J^{\mathrm{cl}}_{a2} X_{a2}, \tag{21}
$$

where

$$
J^{\mathrm{cl}}_{a1} \equiv \boldsymbol{\Gamma}^{\mathrm{cl}}_a \cdot \nabla V, \quad J^{\mathrm{cl}}_{a2} \equiv \frac{1}{T_a} \mathbf{q}^{\mathrm{cl}}_a \cdot \nabla V \tag{22}
$$

are the radial components of the classical particle and heat fluxes defined by

$$
\boldsymbol{\Gamma}^{\mathrm{cl}}_a = \frac{c}{e_aB} \mathbf{F}_{a1} \times \mathbf{n}, \quad \frac{1}{T_a} \mathbf{q}^{\mathrm{cl}}_a = \frac{c}{e_aB} \mathbf{F}_{a2} \times \mathbf{n}. \tag{23}
$$

Here the friction forces $\mathbf{F}_{a1}$ and $\mathbf{F}_{a2}$ are given by

$$
\mathbf{F}_{a1} = \sum_b \int d^3v m_a \mathbf{v} C^L_{ab}(f_{a1}, f_{b1}),
$$

$$
\mathbf{F}_{a2} = \sum_b \int d^3v m_a \mathbf{v} \left(x_a^2 - \frac{5}{2} \right) C^L_{ab}(f_{a1}, f_{b1}). \tag{24}
$$

It should be noted that, from the rotational symmetry of the collision operator, that $\bar{f}_{a1}$ does not contribute to the perpendicular friction forces $\mathbf{F}_{\perp aj}$ while $\tilde{f}_{a1}$ does not contribute to the parallel friction forces $\mathbf{F}_{\parallel aj}$. Equation (21) shows that the entropy production $\tilde{\sigma}_a$ defined from the gyroangle-dependent part of the distribution function is caused by the classical particle and heat transport, and that the classical fluxes J^{cl}_{a1} and J^{cl}_{a2} are conjugate to the thermodynamic forces X_{a1} and X_{a2}, respectively. The momentum conservation due to the collision operator gives

$$
\sum_a \mathbf{F}_{a1} = 0, \tag{25}
$$

which in turn causes the classical particle fluxes to satisfy the ambipolarity as

$$
\sum_a e_a \boldsymbol{\Gamma}^{\mathrm{cl}}_a = 0. \tag{26}
$$

We have the relations between the perpendicular friction forces and flows from Eqs. (17) and (24) as

$$
\begin{bmatrix} \mathbf{F}_{\perp a1} \\ -\mathbf{F}_{\perp a2} \end{bmatrix} = \sum_b \begin{bmatrix} l^{ab}_{11} & l^{ab}_{12} \\ l^{ab}_{21} & l^{ab}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\perp b} \\ -\dfrac{2}{5p_b} \mathbf{q}_{\perp b} \end{bmatrix}, \tag{27}
$$

where the coefficients l^{ab}_{jk} are the same ones defined in Ref. 2 and are given by

$$
l^{ab}_{jk} = \delta_{ab} \frac{m_a^2}{T_a} \sum_{a'} \int d^3v v_{\parallel} L^{(3/2)}_{j-1}(x_a^2) C_{aa'}[v_{\parallel} \times L^{(3/2)}_{k-1}(x_a^2) f_{a0}, f_{a'0}] + \frac{m_a m_b}{T_b} \int d^3v v_{\parallel} \times L^{(3/2)}_{j-1}(x_a^2) C_{ab}[f_{a0}, v_{\parallel} L^{(3/2)}_{k-1}(x_b^2) f_{b0}]. \tag{28}
$$

Here, $L^{(3/2)}_0(x^2) = 1$, $L^{(3/2)}_1(x^2) = \frac{5}{2} - x^2$, $\cdots$, are the Laguerre polynomials of order $\frac{3}{2}$. In Eqs. (27) and (28), the rotational symmetry of the linearized collision operator is used. From the self-adjointness of the linearized collision operator shown in Eq. (9), the coefficients l^{ab}_{jk} have the following symmetry :

$$
l^{ab}_{jk} = l^{ba}_{kj}. \tag{29}
$$

The momentum conservation property described in Eq. (25) imposes another constraint on the coefficients l^{ab}_{jk}:

$$
\sum_a l^{ab}_{1k} = 0. \tag{30}
$$

From Eq. (21) and the ambipolarity condition $\Sigma_a e_a J^{\mathrm{cl}}_{a1} = 0$ given by Eq. (26), we obtain

$$
\sum_a T_a \tilde{\sigma}_a = \sum_a [J^{\mathrm{cl}}_{a1} X_{a1} + J^{\mathrm{cl}}_{a2} X_{a2}] = \sum_{a \neq I} J^{\mathrm{cl}}_{a1} X^*_{a1} + \sum_a J^{\mathrm{cl}}_{a2} X_{a2}, \tag{31}
$$

where X^*_{a1} ($a \neq I$) is defined by

$$
X^*_{a1} = X_{a1} - \frac{e_a}{e_I} X_{I1} = -\frac{p'_a}{n_a} + \frac{e_a}{e_I} \frac{p'_I}{n_I} \quad (a \neq I). \tag{32}
$$

Here we have chosen a certain particle species denoted by I. We hereafter regard I as the ion species with the smallest particle number density. If a plasma consists of electrons and a single ion species i, we take $I = i$.

Equation (31) shows that the number of the conjugated pairs of the classical fluxes and thermodynamic forces is reduced by employing the new pairs $(J_{a1}, X^*_{a1})_{a \neq I}, (J_{a2}, X_{a2})$ instead of $(J_{a1}, X_{a1}), (J_{a2}, X_{a2})$. We also find that the radial electric field does not appear in the new set of the thermodynamic forces. The transport equations which relate the classical fluxes (J_{a1}, J_{a2}) to the thermodynamic forces (X_{a1}, X_{a2}) are obtained from Eqs. (20), (22), (23), and (27) as

$$
\begin{bmatrix} J^{\mathrm{cl}}_{a1} \\ J^{\mathrm{cl}}_{a2} \end{bmatrix} = \sum_b \begin{bmatrix} (L^{\mathrm{cl}})^{ab}_{11} & (L^{\mathrm{cl}})^{ab}_{12} \\ (L^{\mathrm{cl}})^{ab}_{21} & (L^{\mathrm{cl}})^{ab}_{22} \end{bmatrix} \begin{bmatrix} X_{b1} \\ X_{b2} \end{bmatrix}, \tag{33}
$$

where the classical transport coefficients $(L^{\mathrm{cl}})^{ab}_{jk}$ are given by

$$
(L^{\mathrm{cl}})^{ab}_{jk} = (-1)^{j+k-1} \frac{c^2 \langle |\nabla V|^2 \rangle}{e_a e_b B^2} l^{ab}_{jk} \quad (j,k = 1,2). \tag{34}
$$

From Eq. (29), we have the Onsager symmetry for the classical transport as

$$
(L^{\mathrm{cl}})^{ab}_{jk} = (L^{\mathrm{cl}})^{ba}_{kj} \quad (j,k = 1,2). \tag{35}
$$

Equation (30) yields

$$
\sum_a e_a (L^{\mathrm{cl}})^{ab}_{1k} = 0 \quad (k = 1,2). \tag{36}
$$

We easily find from Eqs. (32), (33), (35), and (36) that the classical transport equations for the pairs $(J^{\mathrm{cl}}_{a1}, X^*_{a1})_{a \neq I}, (J^{\mathrm{cl}}_{a2}, X_{a2})$ are given by

$$J^{\rm cl}_{a1(a\neq I)}=\sum_{b\neq I}(L^{\rm cl})^{ab}_{11}X^*_{b1}+\sum_b (L^{\rm cl})^{ab}_{12}X_{b2},$$
$$J^{\rm cl}_{a2}=\sum_{b\neq I}(L^{\rm cl})^{ab}_{21}X^*_{b1}+\sum_b (L^{\rm cl})^{ab}_{22}X_{b2}, \qquad (37)$$

which shows that the transport coefficients $(L^{\rm cl})^{ab}_{jk}$ are the same as in Eq. (33), except for the limitation $(a,j),(b,k) \neq (I,1)$, and that the Onsager symmetry is valid for both of the conjugate pairs. We should note that the ambipolarity condition (36) reduces the number of the thermodynamic forces required for determining the classical fluxes by one, and that the radial electric field does not enter the reduced set of the thermodynamic forces $(X^*_{a1(a\neq I)},X_{a2})$.

Next, let us consider the entropy production $\bar{\sigma}_a$ due to the first-order gyroangle-averaged distribution function $\bar{f}_{a1}$, which satisfies the linearized drift kinetic equation:[1–3,7–11,14,15]

$$v_\parallel \mathbf{n}\cdot\nabla\bar{f}_{a1}+\mathbf{v}_{da}\cdot\nabla f_{a0}-\frac{e_a}{T_a}v_\parallel E_\parallel f_{a0}=C^L_a(\bar{f}_{a1}), \qquad (38)$$

where $\mathbf{v}_{da}$ is the sum of the $\mathbf{E}\times\mathbf{B}$, ∇B and curvature drift velocities, and $C^L_a(\bar{f}_{a1})\equiv\Sigma_b C^L_{ab}(\bar{f}_{a1},\bar{f}_{b1})$. Here, it should be noted that the electric drift term $\mathbf{v}_E\cdot\nabla\bar{f}_{a1}$ is sometimes retained in the linearized drift kinetic equation[16] which causes the nonlinear radial electric field dependence of the neoclassical transport coefficients and of the ambipolarity condition for a nonaxisymmetric system. There are two main reasons why the electric drift term $\mathbf{v}_E\cdot\nabla\bar{f}_{a1}$ is not included in Eq. (38). One is due to the drift ordering $\mathbf{v}_{da}/v_\parallel\sim\delta$, from which $\mathbf{v}_{da}\cdot\nabla\bar{f}_{a1}$ including $\mathbf{v}_E\cdot\nabla\bar{f}_{a1}$ should be neglected compared with $v_\parallel\mathbf{n}\cdot\nabla\bar{f}_{a1}$ in the linearized drift kinetic equation. Equation (38) is the standard linearized drift kinetic equation widely accepted in literatures (see Refs. 1–3 and Refs. 7–11). (However, if the radial electric field much larger than assumed by the drift ordering is generated by some techniques such as neutral beam injections, the electric drift term $\mathbf{v}_E\cdot\nabla\bar{f}_{a1}$ should be retained in the drift kinetic equation as treated in Ref. 16.) Another important reason is as follows. The radial electric field is regarded as one of the thermodynamic forces as seen from Eq. (19). (When the ambipolarity condition is imposed for nonaxisymmetric systems, the radial electric field is regarded as a function of the other thermodynamic forces as shown later.) If the electric drift term is added in the left-hand side of the linearized drift kinetic equation, the radial electric field dependence appears in the neoclassical transport coefficients as mentioned before, and accordingly we obtain the nonlinear transport equations of the form $\mathbf{J}=\mathsf{L}(\mathbf{X})\cdot\mathbf{X}$ in which the transport fluxes $\mathbf{J}$ are nonlinear functions of the thermodynamic forces $\mathbf{X}$. On the other hand, as shown in detail in Ref. 5, the Onsager symmetry is relevant to linear transport equations of the form $\mathbf{J}=\mathsf{L}\cdot\mathbf{X}$ with the transport matrix L independent of the thermodynamic forces $\mathbf{X}$. Since, in this work, we are concerned with the Onsager symmetry for the transport matrix in the linear neoclassical transport equations, the electric drift term causing the nonlinear dependence on the thermodynamic forces should be neglected.

We obtain from Eqs. (15) and (38),

$$\bar{\sigma}_a\equiv-\left\langle\int d^3v\frac{\bar{f}_{a1}}{f_{a0}}C^L_a(\bar{f}_{a1})\right\rangle=-\nabla\cdot\left\langle\mathbf{n}\int d^3v\, v_\parallel\frac{(\bar{f}_{a1})^2}{2f_{a0}}\right\rangle$$
$$-\left\langle\int d^3v\bar{f}_{a1}\mathbf{v}_{da}\cdot\nabla V\right\rangle\frac{\partial\ln f_{a0}}{\partial V}+\frac{1}{T_a}n_a e_a u_{\parallel a}E_\parallel, \qquad (39)$$

where the parallel flow velocity $u_{\parallel a}$ is defined by $n_a u_{\parallel a}\equiv\int d^3v\bar{f}_{a1}v_\parallel$. Here, the flux surface average of the first term in the right-hand side vanishes. In taking the flux surface average of the second term, we use the following two equations for the radial particle and heat fluxes:

$$\left\langle\int d^3v\bar{f}_{a1}\mathbf{v}_{da}\cdot\nabla V\right\rangle=\left\langle\frac{\nabla V}{m_a\Omega_a}\cdot\mathbf{n}\times(\nabla p_{a1}+\nabla\cdot\boldsymbol{\pi}_a)\right\rangle$$
$$=-\frac{c}{e_a B^\theta}\left\langle\frac{B_\zeta}{B}\mathbf{n}\cdot(\nabla p_{a1}+\nabla\cdot\boldsymbol{\pi}_a)\right\rangle+\frac{c}{e_a B^\theta B^\zeta}\langle\mathbf{B}_t\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle$$
$$=-\frac{c}{e_a B^\theta}\frac{\langle B_\zeta\rangle}{\langle B^2\rangle}\langle\mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle+\frac{c}{e_a B^\theta B^\zeta}\langle\mathbf{B}_t\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle-\frac{c}{e_a B^\theta}\left\langle\frac{1}{B}\underbrace{\mathbf{n}\cdot(\nabla p_{a1}+\nabla\cdot\boldsymbol{\pi}_a)}_{(F_{\parallel a1}+n_a e_a E_\parallel)}\left(B_\zeta-\langle B_\zeta\rangle\frac{B^2}{\langle B^2\rangle}\right)\right\rangle$$
$$=J^{\rm ncl}_{a1}-\frac{n_a c}{B^\theta}\left\langle\frac{E_\parallel}{B}\left(B_\zeta-\langle B_\zeta\rangle\frac{B^2}{\langle B^2\rangle}\right)\right\rangle, \qquad (40)$$

$$\left\langle\int d^3v\bar{f}_{a1}\left(x_a^2-\frac{5}{2}\right)\mathbf{v}_{da}\cdot\nabla V\right\rangle=\left\langle\frac{\nabla V}{m_a\Omega_a}\cdot\mathbf{n}\times(\nabla\theta_{a1}+\nabla\cdot\boldsymbol{\Theta}_a)\right\rangle$$
$$=-\frac{c}{e_a B^\theta}\frac{\langle B_\zeta\rangle}{\langle B^2\rangle}\langle\mathbf{B}\cdot\nabla\cdot\boldsymbol{\Theta}_a\rangle+\frac{c}{e_a B^\theta B^\zeta}\langle\mathbf{B}_t\cdot\nabla\cdot\boldsymbol{\Theta}_a\rangle$$
$$-\frac{c}{e_a B^\theta}\left\langle\frac{1}{B}\underbrace{\mathbf{n}\cdot(\nabla\theta_{a1}+\nabla\cdot\boldsymbol{\Theta}_a)}_{F_{\parallel a2}}\left(B_\zeta-\langle B_\zeta\rangle\frac{B^2}{\langle B^2\rangle}\right)\right\rangle=J^{\rm ncl}_{a2}, \qquad (41)$$

where the Hamada coordinates (V,θ,ζ) are employed to represent the poloidal and toroidal components of the magnetic fields(see Appendix A) and Eq. (A2) is used. Here, we have used the definitions $p_{a1}\equiv\int d^3v\frac{1}{3}m_a v^2\bar{f}_{a1}$, $\theta_{a1}\equiv\int d^3v\frac{1}{3}\times m_a v^2(x_a^2-\frac{5}{2})\bar{f}_{a1}$, $\boldsymbol{\pi}_a\equiv(p_{\parallel a}-p_{\perp a})(\mathbf{nn}-\frac{1}{3}\mathbf{I})\equiv\int d^3v m_a(v_\parallel^2-\frac{1}{2}v_\perp^2)\bar{f}_{a1}(\mathbf{nn}-\frac{1}{3}\mathbf{I})$, $\boldsymbol{\Theta}_a\equiv(\Theta_{\parallel a}-\Theta_{\perp a})(\mathbf{nn}-\frac{1}{3}\mathbf{I})\equiv\int d^3v m_a\times(v_\parallel^2-\frac{1}{2}v_\perp^2)(x_a^2-\frac{5}{2})\bar{f}_{a1}(\mathbf{nn}-\frac{1}{3}\mathbf{I})$. The neoclassical particle and heat fluxes are given by

$$J_{a1}^{\rm ncl}\equiv\langle\boldsymbol{\Gamma}_a\cdot\nabla V\rangle^{\rm ncl}\equiv J_{a1}^{\rm PS}+J_{a1}^{\rm bp}+J_{a1}^{\rm na},$$
$$J_{a2}^{\rm ncl}\equiv\frac{1}{T_a}\langle\mathbf{q}_a\cdot\nabla V\rangle^{\rm ncl}\equiv J_{a2}^{\rm PS}+J_{a2}^{\rm bp}+J_{a2}^{\rm na}, \tag{42}$$

which consist of the Pfirsch–Schlüter ($J_{aj}^{\rm PS}$), the banana-plateau ($J_{aj}^{\rm bp}$), and nonaxisymmetric ($J_{aj}^{\rm na}$) parts defined by

$$J_{a1}^{\rm PS}\equiv\langle\boldsymbol{\Gamma}_a\cdot\nabla V\rangle^{\rm PS}\equiv-\frac{c}{e_aB^\theta}\left\langle\frac{F_{\parallel a1}}{B}\left(B_\zeta-\langle B_\zeta\rangle\frac{B^2}{\langle B^2\rangle}\right)\right\rangle,$$

$$J_{a1}^{\rm bp}\equiv\langle\boldsymbol{\Gamma}_a\cdot\nabla V\rangle^{\rm bp}\equiv-\frac{c}{e_aB^\theta}\frac{\langle B_\zeta\rangle}{\langle B^2\rangle}\langle\mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle,$$

$$J_{a1}^{\rm na}\equiv\langle\boldsymbol{\Gamma}_a\cdot\nabla V\rangle^{\rm na}\equiv\frac{c}{e_aB^\theta B^\zeta}\langle\mathbf{B}_t\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle, \tag{43}$$

$$J_{a2}^{\rm PS}\equiv\frac{1}{T_a}\langle\mathbf{q}_a\cdot\nabla V\rangle^{\rm PS}\equiv-\frac{c}{e_aB^\theta}\left\langle\frac{F_{\parallel a2}}{B}\left(B_\zeta-\langle B_\zeta\rangle\frac{B^2}{\langle B^2\rangle}\right)\right\rangle,$$

$$J_{a2}^{\rm bp}\equiv\frac{1}{T_a}\langle\mathbf{q}_a\cdot\nabla V\rangle^{\rm bp}\equiv-\frac{c}{e_aB^\theta}\frac{\langle B_\zeta\rangle}{\langle B^2\rangle}\langle\mathbf{B}\cdot\nabla\cdot\boldsymbol{\Theta}_a\rangle,$$

$$J_{a2}^{\rm na}\equiv\frac{1}{T_a}\langle\mathbf{q}_a\cdot\nabla V\rangle^{\rm na}\equiv\frac{c}{e_aB^\theta B^\zeta}\langle\mathbf{B}_t\cdot\nabla\cdot\boldsymbol{\Theta}_a\rangle.$$

Here the Pfirsch–Schlüter fluxes are written as parts of the neoclassical fluxes as in Refs. 1–3 where the term "neoclassical" is used for the transport due to guiding center motions in a toroidal magnetic configuration affected by collisions, which is a contrast to the "classical" transport caused by particle gyro-motions with collisions. (However, the term "neoclassical" is sometimes used in a narrower sense for referring to the transport fluxes due to particles with long mean free paths such as the banana-plateau fluxes, which exclude the Pfirsch–Schlüter fluxes.)

Then, we find that the flux surface average of the second term in the right-hand side of Eq. (39) is given by the products of the neoclassical radial fluxes and the thermodynamic forces as

$$-\left\langle\int d^3v\bar{f}_{a1}\mathbf{v}_{da}\cdot\nabla V\frac{\partial\ln f_{a0}}{\partial V}\right\rangle=\frac{1}{T_a}\left[J_{a1}^{\rm ncl}-\frac{n_ac}{B^\theta}\left\langle\frac{E_\parallel}{B}\left(B_\zeta-\langle B_\zeta\rangle\frac{B^2}{\langle B^2\rangle}\right)\right\rangle\right]\times X_{a1}+\frac{1}{T_a}J_{a2}^{\rm ncl}X_{a2}. \tag{44}$$

The flux surface average of the third term in the right-hand side of Eq. (39) is given by

$$\frac{n_ae_a\langle u_{\parallel a}E_\parallel\rangle}{T_a}=\frac{n_ae_a\langle Bu_{\parallel a}\rangle\langle BE_\parallel\rangle}{T_a\langle B^2\rangle}+\frac{n_ac}{T_aB^\theta}\times\left\langle\frac{E_\parallel}{B}\left(B_\zeta-\langle B_\zeta\rangle\frac{B^2}{\langle B^2\rangle}\right)\right\rangle X_{a1}, \tag{45}$$

where Eq. (A7) is used. Then, we finally obtain from Eqs. (39), (44), and (45) the thermodynamic form of the flux-surface-averaged entropy production $\langle\bar{\sigma}_a\rangle$ as

$$T_a\langle\bar{\sigma}_a\rangle=J_{a1}^{\rm ncl}X_{a1}+J_{a2}^{\rm ncl}X_{a2}+J_{a3}X_{a3}, \tag{46}$$

where the parallel flux J_{a3} and the parallel force X_{a3} are defined by

$$J_{a3}\equiv\frac{n_a\langle Bu_{\parallel a}\rangle}{\langle B^2\rangle^{1/2}},\quad X_{a3}\equiv e_a\frac{\langle BE_\parallel\rangle}{\langle B^2\rangle^{1/2}}. \tag{47}$$

Taking the species summation of Eq. (46), we have

$$\sum_a T_a\langle\bar{\sigma}_a\rangle=\sum_a(J_{a1}^{\rm ncl}X_{a1}+J_{a2}^{\rm ncl}X_{a2})+J_EX_E, \tag{48}$$

where J_E and X_E are defined from the total parallel current $J_\parallel$ and the parallel electric field $E_\parallel$ as

$$J_E\equiv\frac{\langle BJ_\parallel\rangle}{\langle B^2\rangle^{1/2}}\equiv\sum_a e_aJ_{a3},\quad X_E\equiv\frac{\langle BE_\parallel\rangle}{\langle B^2\rangle^{1/2}}. \tag{49}$$

Thus the flux surface average of the entropy production due to the gyroangle-averaged distribution functions is given in the thermodynamic form, in which the neoclassical radial fluxes $J_{a1}^{\rm ncl}$, $J_{a2}^{\rm ncl}$ and the parallel current J_E are conjugate to the radial gradient forces X_{a1}, X_{a2} and the parallel electric field X_E, respectively. It should be noted that the neoclassical thermodynamic form of the entropy production can be obtained only through the magnetic surface average as in Eq. (48), which is a remarkable contrast to the classical thermodynamic form (21) defined locally in the configuration space.

Now, let us consider the ambipolarity condition for the neoclassical particle fluxes. Using the momentum conservation (25) by collisions and the charge neutrality condition $\Sigma_a n_ae_a=0$, we obtain the flux surface average of the total parallel momentum balance equation as $\Sigma_a\langle\mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle=0$. Then, we find from the definitions in Eq. (43) that the intrinsic ambipolarity holds for both of the Pfirsch–Schlüter and banana-plateau particle fluxes in the same way as for the classical particle fluxes, which implies that the ambipolar conditions,

$$\sum_a e_aJ_{a1}^{\rm PS}=\sum_a e_aJ_{a1}^{\rm bp}=0, \tag{50}$$

are valid for arbitrary values of the thermodynamic forces (X_{a1},X_{a2},X_E). On the other hand, the nonaxisymmetric particle fluxes $J_{a1}^{\rm na}$ and accordingly the total neoclassical particle fluxes $J_{a1}^{\rm ncl}$ are not ambipolar generally. If the ambipolarity condition for the total neoclassical particle fluxes,

$$\sum_a e_aJ_{a1}^{\rm ncl}=\sum_a e_aJ_{a1}^{\rm na}=0, \tag{51}$$

is imposed, we find in the similar way as in Eq. (31) that Eq. (48) is rewritten as

$$\sum_a T_a\langle\bar{\sigma}_a\rangle=\sum_{a\neq I} J_{a1}^{\rm ncl}X_{a1}^*+\sum_a J_{a2}^{\rm ncl}X_{a2}+J_EX_E\,. \tag{52}$$

In axisymmetric toroidal systems, the nonaxisymmetric fluxes $J_{aj}^{\rm na}$ $(j=1,2)$ vanish and therefore the neoclassical particle fluxes $J_{a1}^{\rm ncl}$ are intrinsically ambipolar. As discussed in the next section, in nonaxisymmetric systems, the ambipolarity condition (51) combined with the neoclassical transport equations gives a constraint on the thermodynamic forces (X_{a1},X_{a2},X_E) from which the radial electric field $-\Phi'$ is expressed by a linear form in the pressure and temperature gradients and the parallel electric field. Then, independent thermodynamic forces for nonaxisymmetric systems are given not by the set (X_{a1},X_{a2},X_E) but by the reduced one $(X_{a1(a\neq I)}^*,X_{a2},X_E)$. In the present work, we show the neoclassical transport equations for both cases with (X_{a1},X_{a2},X_E) and with $(X_{a1(a\neq I)}^*,X_{a2},X_E)$ used as the forces, in order to elucidate the relation of the ambipolarity to the axisymmetry and to the Onsager symmetry.

III. ONSAGER SYMMETRY OF NEOCLASSICAL TRANSPORT EQUATIONS FOR GENERAL TOROIDAL SYSTEMS

In this section, it is proved that the Onsager symmetry is satisfied by the neoclassical transport equations for general toroidal systems, including nonaxisymmetric cases. For that, it is convenient to define the distribution function $\bar{g}_a$ by

$$\bar{g}_a=\bar{f}_{a1}-\frac{e_a}{T_a}f_{a0}\int^l\frac{dl}{B}\left(BE_\parallel-\frac{B^2}{\langle B^2\rangle}\langle BE_\parallel\rangle\right), \tag{53}$$

where $\int^l dl$ denotes the integral along the magnetic field line. Then, Eq. (38) is rewritten as

$$v_\parallel\mathbf{n}\cdot\nabla\bar{g}_a-C_a^L(\bar{g}_a)=\frac{1}{T_a}f_{a0}(S_{a1}X_{a1}+S_{a2}X_{a2}+S_{a3}X_{a3}), \tag{54}$$

where the functions S_{aj} $(j=1,2,3)$ are defined by

$$S_{aj}=(x_a^2-\tfrac{5}{2})^{j-1}\mathbf{v}_{da}\cdot\nabla V\quad(j=1,2),$$
$$S_{a3}=v_\parallel B/\langle B^2\rangle^{1/2}. \tag{55}$$

Here, it is worthwhile making some remarks on the case where, just as in Ref. 16, the electric drift term $\mathscr{A}_E\bar{g}_a\equiv\mathbf{v}_E\cdot\nabla\bar{g}_a$ is added in the left-hand side of Eq. (54). Here, in $\mathbf{v}_E\cdot\nabla\bar{g}_a$, the spatial derivative ∇ is taken with $(\mathbf{x},v,\alpha)$ as independent phase space variables defined in Ref. 16, and the electric drift velocity is given by $\mathbf{v}_E=c\mathbf{E}\times\mathbf{B}/\langle B^2\rangle$ to satisfy an important phase space conservative property (see Ref. 16). In that case, nonlinear neoclassical transport equations are derived due to the nonlinear dependence on the radial electric field as mentioned after Eq. (38). Here, let us artificially regard the radial electric field $E_V\equiv-\partial\Phi/\partial V$ added in the left-hand side of Eq. (54) as an independent parameter, although E_V is already contained as a part of the thermodynamic forces in the right-hand side of Eq. (54). By doing this, the resulting transport equations are written in the apparently linear form $\mathbf{J}=\mathsf{L}(E_V)\cdot\mathbf{X}$ with E_V as a parameter in the transport matrix L. Then, it is shown that the proof of the Onsager symmetry for $\mathsf{L}(E_V)$ in this section is still valid even if $\mathscr{A}_E\bar{g}_a$ is retained. This follows from the fact that the operator $\mathscr{A}_E$ has the same properties as $v_\parallel\mathbf{n}\cdot\nabla$ which are given by $\mathscr{A}_Ef_{a0}=0$ and $\langle\int d^3v\,\mathscr{A}_EF\rangle=0$ for an arbitrary function F on the phase space.

The neoclassical radial fluxes $J_{aj}^{\rm ncl}$ $(j=1,2)$ and the parallel flux J_{a3} are expressed in terms of $\bar{g}_a$ and S_{aj} $(j=1,2,3)$ as

$$J_{aj}^{\rm ncl}\equiv\left\langle\int d^3v\,\bar{g}_aS_{aj}\right\rangle\quad(j=1,2),$$
$$J_{a3}\equiv\left\langle\int d^3v\,\bar{g}_aS_{a3}\right\rangle, \tag{56}$$

where we have used the following formula for an arbitrary function $F(\mathbf{x})$:

$$\left\langle F\int d^3v\left(x_a^2-\frac{5}{2}\right)^{j-1}(\mathbf{v}_{da}\cdot\nabla V)f_{a0}\right\rangle$$
$$=-\delta_{j1}\frac{cp_a}{e_aB^\theta}\left\langle\frac{B_\zeta}{B}\mathbf{n}\cdot\nabla F\right\rangle\quad(j=1,2). \tag{57}$$

Noting in Eq. (54) that the left-hand side is linear with respect to $\bar{g}_a$ and that X_{aj} $(j=1,2,3)$ occur in the right-hand side as parameters, we find that the solution $\bar{g}_a$ of Eq. (54) is given by

$$\bar{g}_a=\sum_b(G_{ab1}X_{b1}+G_{ab2}X_{b2}+G_{ab3}X_{b3}), \tag{58}$$

where G_{abj} $(j=1,2,3)$ are defined as the solutions of

$$v_\parallel\mathbf{n}\cdot\nabla G_{abj}-\sum_{a'}C_{aa'}^L(G_{abj},G_{a'bj})$$
$$=\delta_{ab}\frac{1}{T_b}f_{b0}S_{bj}\quad(j=1,2,3). \tag{59}$$

Then, using Eqs. (56) and (58), we obtain the transport equations relating $J_{aj}^{\rm ncl}$ $(j=1,2)$ and J_{a3} to X_{aj} $(j=1,2,3)$ as

$$\begin{pmatrix}J_{a1}^{\rm ncl}\\ J_{a2}^{\rm ncl}\\ J_{a3}\end{pmatrix}=\sum_b\begin{pmatrix}L_{11}^{ab} & L_{12}^{ab} & L_{13}^{ab}\\ L_{21}^{ab} & L_{22}^{ab} & L_{32}^{ab}\\ L_{31}^{ab} & L_{32}^{ab} & L_{33}^{ab}\end{pmatrix}\begin{pmatrix}X_{b1}\\ X_{b2}\\ X_{b3}\end{pmatrix}, \tag{60}$$

where the transport coefficients L_{jk}^{ab} are given by

$$L_{jk}^{ab}=\left\langle\int d^3v\,S_{aj}G_{abk}\right\rangle\quad(j,k=1,2,3). \tag{61}$$

Equation (60) is rewritten as the transport equations relating $J_{aj}^{\rm ncl}$ $(j=1,2)$ and J_E to X_{aj} $(j=1,2)$ and X_E:

$$J_{aj}^{\rm ncl}=\sum_b\sum_{k=1,2}L_{jk}^{ab}X_{bk}+L_{jE}^aX_E\quad(j=1,2),$$
$$J_E=\sum_b\sum_{k=1,2}L_{Ek}^bX_{bk}+L_{EE}X_E, \tag{62}$$

where the coefficients L_{jE}^a, L_{Ej}^a $(j=1,2)$ and L_{EE} are given by

$$L^a_{jE} \equiv \sum_b e_b L^{ab}_{j3}, \quad L^a_{Ej} \equiv \sum_b e_b L^{ba}_{3j} \quad (j=1,2),$$
$$L_{EE} \equiv \sum_{a,b} e_a e_b L^{ab}_{33}. \tag{63}$$

In order to show the Onsager symmetry of the transport coefficients, it is useful to separate Eq. (59) into even and odd parts with respect to $v_{\parallel}$ as

$$v_{\parallel}\mathbf{n}\cdot\nabla G^-_{abj} - \sum_{a'} C^L_{aa'}(G^+_{abj}, G^+_{a'bj}) = \delta_{ab}\frac{1}{T_b} f_{b0} S^+_{bj},$$
$$v_{\parallel}\mathbf{n}\cdot\nabla G^+_{abj} - \sum_{a'} C^L_{aa'}(G^-_{abj}, G^-_{a'bj}) = \delta_{ab}\frac{1}{T_b} f_{b0} S^-_{bj}, \tag{64}$$

where the superscripts + and − denote the even and odd parts, respectively. Noting that S_{aj} are even for $j=1,2$ and odd for $j=3$, and using Eqs. (61) and (64), we obtain

$$\sum_A T_A \left\langle \int d^3v \frac{1}{f_{A0}} G^+_{Abk} v_{\parallel}\mathbf{n}\cdot\nabla G^-_{Aaj} \right\rangle - \sum_{A,B} T_A \left\langle \int d^3v \frac{1}{f_{A0}} G^+_{Abk} C^L_{AB}(G^+_{Aaj}, G^+_{Baj}) \right\rangle = (\delta_{j1}+\delta_{j2}) L^{ab}_{jk},$$
$$\sum_A T_A \left\langle \int d^3v \frac{1}{f_{A0}} G^-_{Abk} v_{\parallel}\mathbf{n}\cdot\nabla G^+_{Aaj} \right\rangle - \sum_{A,B} T_A \left\langle \int d^3v \frac{1}{f_{A0}} G^-_{Abk} C^L_{AB}(G^-_{Aaj}, G^-_{Baj}) \right\rangle = \delta_{j3} L^{ab}_{jk}, \tag{65}$$

from which we have

$$(\delta_{j1}+\delta_{j2}) L^{ab}_{jk} + \delta_{k3} L^{ba}_{kj} = -\sum_{A,B} T_A \left\langle \int d^3v \frac{1}{f_{A0}} [G^+_{Abk} C^L_{AB}(G^+_{Aaj}, G^+_{Baj}) + G^-_{Aaj} C^L_{AB}(G^-_{Abk}, G^-_{Bbk})] \right\rangle. \tag{66}$$

We find from the self-adjointness of the linearized collision operator given by Eq. (9) that the right-hand side of Eq. (66) is invariant under the permutation of the subscripts $(a,j)\leftrightarrow(b,k)$ and that

$$(\delta_{j1}+\delta_{j2}) L^{ab}_{jk} + \delta_{k3} L^{ba}_{kj} = (\delta_{k1}+\delta_{k2}) L^{ba}_{kj} + \delta_{j3} L^{ab}_{jk}.$$

Thus, we obtain

$$(\delta_{j1}+\delta_{j2}-\delta_{j3}) L^{ab}_{jk} = (\delta_{k1}+\delta_{k2}-\delta_{k3}) L^{ba}_{kj},$$

which is rewritten as the well-known Onsager relations :

$$L^{ab}_{jk} = L^{ba}_{kj} \quad (j,k=1,2),$$
$$L^{ab}_{j3} = -L^{ba}_{3j} \quad (j=1,2), \tag{67}$$
$$L^{ab}_{33} = L^{ba}_{33}.$$

We see from Eqs. (63) and (67) that

$$L^a_{jE} = -L^a_{Ej} \quad (j=1,2). \tag{68}$$

Equations (67) and (68) show that the Onsager symmetry is satisfied by the transport matrix which combines the conjugate pairs of the fluxes $(J^{\rm ncl}_{a1}, J^{\rm ncl}_{a2}, J_E)$ and the forces (X_{a1}, X_{a3}, X_E) for general toroidal systems.

Here, let us discuss the relation between the transport equations and the ambipolarity condition. In axisymmetric systems, intrinsic ambipolarity holds for the neoclassical particle fluxes and is expressed in terms of the relation between the transport coefficients as

$$\sum_a e_a L^{ab}_{1j} = \sum_a e_a L^a_{1E} = 0 \quad (j=1,2,3). \tag{69}$$

Then, the number of the conjugate pairs of fluxes and forces is reduced by one as shown in Eq. (52) using $X^*_{a1(a\neq I)}$ instead of X_{a1}. We find from Eqs. (67), (68), and (69) that the transport equations relating the fluxes $(J^{\rm ncl}_{a1(a\neq I)}, J^{\rm ncl}_{a2}, J_E)$ to $(X^*_{a1(a\neq I)}, X_{a2}, X_E)$ in the axisymmetric case are given by

$$J^{\rm ncl}_{a1(a\neq I)} = \sum_{b\neq I} L^{ab}_{11} X^*_{b1} + \sum_b L^{ab}_{12} X_{b2} + L^a_{1E} X_E,$$
$$J^{\rm ncl}_{a2} = \sum_{b\neq I} L^{ab}_{21} X^*_{b1} + \sum_b L^{ab}_{22} X_{b2} + L^a_{2E} X_E, \tag{70}$$
$$J_E = \sum_{b\neq I} L^b_{E1} X^*_{b1} + \sum_b L^b_{E2} X_{b2} + L_{EE} X_E.$$

In the transport equations (70), the transport coefficients L^{ab}_{jk} are the same as in Eq. (62) except for the limitation $(a,j),(b,k) \neq (I,1)$, and the Onsager symmetry is still valid.

In nonaxisymmetric systems, the ambipolarity condition (51) gives a relation between the thermodynamic forces which is used to express one thermodynamic force X_I in terms of the other thermodynamic forces $(X^*_{a1(a\neq I)}, X_{a2}, X_E)$ as

$$X_{I1} = -e_I \left(\sum_{a,b} e_a e_b L^{ab}_{11} \right)^{-1} \sum_a e_a \times \left(\sum_{b\neq I} L^{ab}_{11} X^*_{b1} + \sum_b L^{ab}_{12} X_{b2} + L^a_{1E} X_E \right). \tag{71}$$

Equation (71) determines the radial electric field $-\Phi' = X_{I1}/e_I + p'_I/(n_I e_I)$ as a linear combination of the pressure and temperature gradients and the parallel electric field.

Then, in the nonaxisymmetric case with the ambipolarity condition (71), the transport equations relating $(J^{\rm ncl}_{a1(a\neq I)}, J^{\rm ncl}_{a2}, J_E)$ to $(X^*_{a1(a\neq I)}, X_{a2}, X_E)$ are given by

$$J^{\rm ncl}_{a1(a\neq I)} = \sum_{b\neq I} \mathscr{L}^{ab}_{11} X^*_{b1} + \sum_b \mathscr{L}^{ab}_{12} X_{b2} + \mathscr{L}^a_{1E} X_E,$$
$$J^{\rm ncl}_{a2} = \sum_{b\neq I} \mathscr{L}^{ab}_{21} X^*_{b1} + \sum_b \mathscr{L}^{ab}_{22} X_{b2} + \mathscr{L}^a_{2E} X_E, \tag{72}$$
$$J_E = \sum_{b\neq I} \mathscr{L}^b_{E1} X^*_{b1} + \sum_b \mathscr{L}^b_{E2} X_{b2} + \mathscr{L}_{EE} X_E,$$

where the transport coefficients $\mathscr{L}^{ab}_{jk}$, $\mathscr{L}^a_{jE}$, $\mathscr{L}^a_{Ej}$, and $\mathscr{L}_{EE}$ are

$$\mathscr{L}^{ab}_{jk} = L^{ab}_{jk} - \left(\sum_A e_A L^{aA}_{j1}\right)\left(\sum_A e_A L^{Ab}_{1k}\right) \times \left(\sum_{A,B} e_A e_B L^{AB}_{11}\right)^{-1},$$

$$\mathscr{L}^{a}_{jE} = L^{a}_{jE} - \left(\sum_A e_A L^{aA}_{j1}\right)\left(\sum_A e_A L^{A}_{1E}\right) \times \left(\sum_{A,B} e_A e_B L^{AB}_{11}\right)^{-1},$$

$$\mathscr{L}^{a}_{Ej} = L^{a}_{Ej} - \left(\sum_A e_A L^{A}_{E1}\right)\left(\sum_A e_A L^{Aa}_{1j}\right) \times \left(\sum_{A,B} e_A e_B L^{AB}_{11}\right)^{-1},$$

$$\mathscr{L}_{EE} = L_{EE} - \left(\sum_A e_A L^{A}_{E1}\right)\left(\sum_A e_A L^{A}_{1E}\right) \times \left(\sum_{A,B} e_A e_B L^{AB}_{11}\right)^{-1}. \quad (73)$$

We see from Eqs. (67) and (68) that the Onsager symmetry still holds for the transport coefficients given by Eq. (73):

$$\mathscr{L}^{ab}_{jk} = \mathscr{L}^{ba}_{kj}, \quad \mathscr{L}^{a}_{jE} = -\mathscr{L}^{a}_{Ej}. \quad (74)$$

Thus, we have established for nonaxisymmetric systems with no net radial current that the radial electric field is determined by the pressure, temperature gradients, and the parallel electric field, and that transport satisfies the Onsager symmetry.

IV. BANANA-PLATEAU AND NONAXISYMMETRIC TRANSPORT COEFFICIENTS

In the previous section, we have shown the Onsager symmetry for the transport coefficients relating the neoclassical radial fluxes and the parallel current $(J^{\rm ncl}_{a1}, J^{\rm ncl}_{a2}, J_E)$ to the radial gradient forces and the parallel electric field (X_{a1}, X_{a2}, X_E) in general toroidal systems. The neoclassical radial particle and heat fluxes $(J^{\rm ncl}_{a1}, J^{\rm ncl}_{a2})$ consist of the Pfirsch–Schlüter, banana-plateau, and nonaxisymmetric parts while, as shown in Appendix B, it is well known that the transport equations for the Pfirsch–Schlüter fluxes $(J^{\rm PS}_{a1}, J^{\rm PS}_{a2})$ and the radial gradient forces (X_{a1}, X_{a2}) satisfy the Onsager symmetry. Thus, it is clear that the sum of the banana-plateau and nonaxisymmetric radial fluxes, and the parallel current $(J^{\rm bp}_{a1}+J^{\rm na}_{a1}, J^{\rm bp}_{a2}+J^{\rm na}_{a2}, J_E)$ are related to the forces (X_{a1}, X_{a2}, X_E) by the transport coefficients with the Onsager symmetry. In this section, using the 13 moment (13M) approximation,[3] we derive the transport equations for the banana-plateau fluxes and those for the nonaxisymmetric fluxes separately, in the case of general toroidal plasma consisting of electrons and one ion species. Then, the symmetry properties are investigated for each of the transport equations, and the Onsager symmetry for the total transport is directly confirmed.

The parallel momentum balance equations combined with the friction-flow relations are given in the 13M approximation by

$$\begin{bmatrix} \langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_e\rangle + n_e e \langle E_\parallel B\rangle \\ \langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\Theta}_e\rangle \end{bmatrix} = \begin{bmatrix} \langle BF_{\parallel e1}\rangle \\ \langle BF_{\parallel e2}\rangle \end{bmatrix} = -\frac{n_e m_e}{\tau_{ee}} \begin{bmatrix} \hat{l}^e_{11} & -\hat{l}^e_{12} \\ -\hat{l}^e_{12} & \hat{l}^e_{22} \end{bmatrix} \begin{bmatrix} \langle B(u_{\parallel e} - u_{\parallel i})\rangle \\ \dfrac{2}{5p_a}\langle Bq_{\parallel e}\rangle \end{bmatrix}, \quad (75)$$

$$\langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\Theta}_i\rangle = \langle BF_{\parallel i2}\rangle = -\frac{n_i m_i}{\tau_{ii}} \hat{l}^i_{22} \frac{2}{5p_i}\langle Bq_{\parallel i}\rangle, \quad (76)$$

where the dimensionless friction coefficients $\hat{l}^a_{ij} \equiv -(\tau_{aa}/n_a m_a) l^{aa}_{ij}$ are given by $\hat{l}^e_{11} = Z_i$, $\hat{l}^e_{12} = \frac{3}{2} Z_i$, $\hat{l}^e_{22} = \sqrt{2} + \frac{13}{4} Z_i$, and $\hat{l}^i_{22} = \sqrt{2}$ with the ion charge number Z_i. The species summation of the parallel momentum balances reduces to

$$\langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_e\rangle + \langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_i\rangle = 0, \quad (77)$$

where the momentum conservation (25) in collisions and the charge neutrality condition $\Sigma_a e_a n_a = 0$ are used.

Solving the linearized drift kinetic equation gives the equations for the parallel viscosities, which have the following form for all collision frequencies in the Pfirsch–Schlüter, plateau, and banana regimes:

$$\begin{bmatrix} \langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle \\ \langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\Theta}_a\rangle \end{bmatrix} = \frac{n_a m_a}{\tau_{aa}} c_a \begin{bmatrix} \hat{\mu}_{a1} & \hat{\mu}_{a2} \\ \hat{\mu}_{a2} & \hat{\mu}_{a3} \end{bmatrix} \times \left(\begin{bmatrix} \langle u_{\parallel a} B\rangle \\ \dfrac{2}{5p_a}\langle q_{\parallel a} B\rangle \end{bmatrix} - \langle G_a\rangle \frac{c}{e_a} \begin{bmatrix} X_{a1} \\ X_{a2} \end{bmatrix} \right), \quad (78)$$

where c_a and $\hat{\mu}_{aj}$ $(j=1,2,3)$ are the dimensionless parameters for the viscosity coefficients, and $\langle G_a\rangle$ represents the geometrical factor which measures the deviation from the axisymmetric configuration. These parameters c_a, $\hat{\mu}_{aj}$ $(j=1,2,3)$ and $\langle G_a\rangle$ are given in Appendix C and all of them are generally dependent on the collision frequencies although, in the axisymmetric case, the geometrical factor is given by $\langle G_a\rangle = \langle B_\zeta\rangle / B^\theta$ for both species and for all the collision frequencies.

Using Eqs. (76)–(78), the ion parallel flows and viscosity are written as

$$\langle Bu_{\parallel i}\rangle = \langle G_i\rangle \frac{c}{e_i}\left(X_{i1} + \frac{\hat{\mu}'_{i2}}{\hat{\mu}'_{i1}} X_{i2} \right) - \frac{1}{c_i \hat{\mu}'_{i1}} \frac{\tau_{ii}}{n_i m_i} \langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_e\rangle, \quad (79)$$

$$\frac{2}{5p_i}\langle Bq_{\parallel i}\rangle = \frac{c_i(\hat{\mu}_{i1}\hat{\mu}_{i3} - (\hat{\mu}_{i2})^2)}{\hat{\mu}'_{i1}(c_i\hat{\mu}_{i3} + \hat{l}^i_{22})} \langle G_i\rangle \frac{c}{e_i} X_{i2} + \frac{\hat{\mu}'_{i2}}{\hat{\mu}'_{i1}\hat{l}^i_{22}} \frac{\tau_{ii}}{n_i m_i} \langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_e\rangle, \quad (80)$$

$$\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Theta}_i\rangle = -\frac{n_i m_i}{\tau_{ii}}\,\frac{c_i\hat{l}^i_{22}(\hat{\mu}_{i1}\hat{\mu}_{i3}-(\hat{\mu}_{i2})^2)}{\hat{\mu}'_{i1}(c_i\hat{\mu}_{i3}+\hat{l}^i_{22})}\langle G_i\rangle\frac{c}{e_i}X_{i2} - \frac{\hat{\mu}'_{i2}}{\hat{\mu}'_{i1}}\langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_e\rangle, \qquad (81)$$

where we have defined $\hat{\mu}'_{i1}\equiv\hat{\mu}_{i1}-\gamma\hat{\mu}_{i2}$ and $\hat{\mu}'_{i2}\equiv\hat{\mu}_{i2}-\gamma\hat{\mu}_{i3}$ with $\gamma\equiv c_i\hat{\mu}_{i2}/(c_i\hat{\mu}_{i3}+\hat{l}^i_{22})$. Equations relating the electron parallel viscosities $\langle\mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_e\rangle$ and $\langle\mathbf{B}\cdot\nabla\cdot\mathbf{\Theta}_e\rangle$ to the thermodynamic forces $(X_{e1},X_{e2},X_{i1},X_{i2},X_E)$ are obtained from Eqs. (75), (78), and (79) as

$$\left\{\begin{bmatrix}\hat{l}^e_{11} & -\hat{l}^e_{12}\\ -\hat{l}^e_{12} & \hat{l}^e_{22}\end{bmatrix}^{-1} + \frac{1}{c_e}\begin{bmatrix}\hat{\mu}_{e1} & \hat{\mu}_{e2}\\ \hat{\mu}_{e2} & \hat{\mu}_{e3}\end{bmatrix}^{-1} + \frac{1}{c_i\hat{\mu}'_{i1}}\frac{n_e m_e\tau_{ii}}{n_i m_i\tau_{ee}}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right\}\begin{bmatrix}\langle\mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_e\rangle\\ \langle\mathbf{B}\cdot\nabla\cdot\mathbf{\Theta}_e\rangle\end{bmatrix}$$
$$= -\begin{bmatrix}\hat{l}^e_{11} & -\hat{l}^e_{12}\\ -\hat{l}^e_{12} & \hat{l}^e_{22}\end{bmatrix}^{-1}\begin{bmatrix}n_e e\langle B^2\rangle^{1/2}X_E\\ 0\end{bmatrix} + \frac{n_e m_e}{\tau_{ee}}\begin{bmatrix}\langle G_e\rangle\frac{c}{e}X_{e1}+\langle G_i\rangle\frac{c}{e_i}\left(X_{i1}+\frac{\hat{\mu}'_{i2}}{\hat{\mu}'_{i1}}X_{i2}\right)\\ \langle G_e\rangle\frac{c}{e}X_{e2}\end{bmatrix}. \qquad (82)$$

We also find from Eq. (75) that the parallel current is divided into the classical part $J_E^{\rm cl}$ and the neoclassical part $J_E^{\rm bp}$ due to the electron parallel viscosities as

$$J_E\equiv\langle BJ_{\parallel}\rangle/\langle B^2\rangle^{1/2}=J_E^{\rm cl}+J_E^{\rm bp},$$
$$J_E^{\rm cl}\equiv\sigma_s X_E, \qquad (83)$$
$$J_E^{\rm bp}\equiv\frac{\sigma_s}{n_e e}\langle B^2\rangle^{-1/2}\left(\langle\mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_e\rangle+\frac{\hat{l}^e_{12}}{\hat{l}^e_{22}}\langle\mathbf{B}\cdot\nabla\cdot\mathbf{\Theta}_e\rangle\right),$$

where $\sigma_s\equiv(n_e e^2\tau_{ee}/m_e)\hat{l}^e_{22}/[\hat{l}^e_{11}\hat{l}^e_{22}-(\hat{l}^e_{12})^2]$ is the classical Spitzer conductivity.

Finally, we obtain from Eqs. (77) and (81)–(83) the banana-plateau transport equations which relate $(J^{\rm bp}_{e1},J^{\rm bp}_{e2},J^{\rm bp}_{i1},J^{\rm bp}_{i2},J^{\rm bp}_E)$ to $(X_{e1},X_{e2},X_{i1},X_{i2},X_E)$ as

$$\begin{bmatrix}J^{\rm bp}_{e1}\\ J^{\rm bp}_{e2}\\ J^{\rm bp}_{i1}\\ J^{\rm bp}_{i2}\\ J^{\rm bp}_E\end{bmatrix} = \begin{bmatrix}(L^{\rm bp})^{ee}_{11} & (L^{\rm bp})^{ee}_{12} & (L^{\rm bp})^{ei}_{11} & (L^{\rm bp})^{ei}_{12} & (L^{\rm bp})^{e}_{1E}\\ (L^{\rm bp})^{ee}_{21} & (L^{\rm bp})^{ee}_{22} & (L^{\rm bp})^{ei}_{21} & (L^{\rm bp})^{ei}_{22} & (L^{\rm bp})^{e}_{2E}\\ (L^{\rm bp})^{ie}_{11} & (L^{\rm bp})^{ie}_{12} & (L^{\rm bp})^{ii}_{11} & (L^{\rm bp})^{ii}_{12} & (L^{\rm bp})^{i}_{1E}\\ (L^{\rm bp})^{ie}_{21} & (L^{\rm bp})^{ie}_{22} & (L^{\rm bp})^{ii}_{21} & (L^{\rm bp})^{ii}_{22} & (L^{\rm bp})^{i}_{2E}\\ (L^{\rm bp})^{e}_{E1} & (L^{\rm bp})^{e}_{E2} & (L^{\rm bp})^{i}_{E1} & (L^{\rm bp})^{i}_{E2} & (L^{\rm bp})_{EE}\end{bmatrix} \times\begin{bmatrix}X_{e1}\\ X_{e2}\\ X_{i1}\\ X_{i2}\\ X_E\end{bmatrix}, \qquad (84)$$

where the banana-plateau transport coefficients are given by

$$\begin{bmatrix}(L^{\rm bp})^{ee}_{11} & (L^{\rm bp})^{ee}_{12}\\ (L^{\rm bp})^{ee}_{21} & (L^{\rm bp})^{ee}_{22}\end{bmatrix} = \frac{n_e m_e}{\tau_{ee}}\frac{c^2}{e^2}\frac{\langle B_\zeta\rangle\langle G_e\rangle}{B^\theta\langle B^2\rangle}\begin{bmatrix}M_1 & M_2\\ M_2 & M_3\end{bmatrix}, \qquad (85)$$

$$\begin{bmatrix}M_1 & M_2\\ M_2 & M_3\end{bmatrix}\equiv\left(\begin{bmatrix}\hat{l}^e_{11} & -\hat{l}^e_{12}\\ -\hat{l}^e_{12} & \hat{l}^e_{22}\end{bmatrix}^{-1}+\frac{1}{c_e}\begin{bmatrix}\hat{\mu}_{e1} & \hat{\mu}_{e2}\\ \hat{\mu}_{e2} & \hat{\mu}_{e3}\end{bmatrix}^{-1} + \frac{1}{c_i\hat{\mu}'_{i1}}\frac{n_e m_e\tau_{ii}}{n_i m_i\tau_{ee}}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right)^{-1}, \qquad (86)$$

$$\begin{bmatrix}(L^{\rm bp})^{ei}_{11}\\ (L^{\rm bp})^{ei}_{21}\end{bmatrix}=\frac{1}{Z_i}\frac{\langle G_i\rangle}{\langle G_e\rangle}\begin{bmatrix}(L^{\rm bp})^{ee}_{11}\\ (L^{\rm bp})^{ee}_{21}\end{bmatrix},$$
$$\begin{bmatrix}(L^{\rm bp})^{ei}_{12}\\ (L^{\rm bp})^{ei}_{22}\end{bmatrix}=\frac{1}{Z_i}\frac{\hat{\mu}'_{i2}}{\hat{\mu}'_{i1}}\frac{\langle G_i\rangle}{\langle G_e\rangle}\begin{bmatrix}(L^{\rm bp})^{ee}_{11}\\ (L^{\rm bp})^{ee}_{21}\end{bmatrix}, \qquad (87)$$

$$\begin{bmatrix}(L^{\rm bp})^{e}_{1E}\\ (L^{\rm bp})^{e}_{2E}\end{bmatrix}=-\frac{n_e c}{[\hat{l}^e_{11}\hat{l}^e_{22}-(\hat{l}^e_{12})^2]}\frac{\langle B_\zeta\rangle}{B^\theta\langle B^2\rangle^{1/2}}\begin{bmatrix}M_1 & M_2\\ M_2 & M_3\end{bmatrix}\begin{bmatrix}\hat{l}^e_{22}\\ \hat{l}^e_{12}\end{bmatrix}, \qquad (88)$$

$$[(L^{\rm bp})^{ie}_{11},(L^{\rm bp})^{ie}_{12},(L^{\rm bp})^{ii}_{11},(L^{\rm bp})^{ii}_{12},(L^{\rm bp})^{i}_{1E}]$$
$$=\frac{1}{Z_i}[(L^{\rm bp})^{ee}_{11},(L^{\rm bp})^{ee}_{12},(L^{\rm bp})^{ei}_{11},(L^{\rm bp})^{ei}_{12},(L^{\rm bp})^{e}_{1E}], \qquad (89)$$

$$[(L^{\rm bp})^{ie}_{21},(L^{\rm bp})^{ie}_{22},(L^{\rm bp})^{ii}_{21},(L^{\rm bp})^{ii}_{22},(L^{\rm bp})^{i}_{2E}]$$
$$=\frac{\hat{\mu}'_{i2}}{\hat{\mu}'_{i1}Z_i}[(L^{\rm bp})^{ee}_{11},(L^{\rm bp})^{ee}_{12},(L^{\rm bp})^{ei}_{11},(L^{\rm bp})^{ei}_{12},(L^{\rm bp})^{e}_{1E}]$$
$$+\frac{n_i m_i}{\tau_{ii}}\frac{c^2}{e_i^2}\frac{c_i\hat{l}^i_{22}(\hat{\mu}_{i1}\hat{\mu}_{i3}-(\hat{\mu}_{i2})^2)}{\hat{\mu}'_{i1}(c_i\hat{\mu}_{i3}+\hat{l}^i_{22})}\frac{\langle B_\zeta\rangle\langle G_i\rangle}{B^\theta\langle B^2\rangle}$$
$$\times[0,0,0,1,0], \qquad (90)$$

$$[(L^{\rm bp})^{e}_{E1},(L^{\rm bp})^{e}_{E2}]=-\frac{B^\theta\langle G_e\rangle}{\langle B_\zeta\rangle}[(L^{\rm bp})^{e}_{1E},(L^{\rm bp})^{e}_{2E}], \qquad (91)$$

$$[(L^{\rm bp})^{i}_{E1},(L^{\rm bp})^{i}_{E2}]=-\frac{B^\theta\langle G_i\rangle}{\langle B_\zeta\rangle}[(L^{\rm bp})^{i}_{1E},(L^{\rm bp})^{i}_{2E}], \qquad (92)$$

$$(L^{\rm bp})_{EE}=-\frac{\sigma_s}{\hat{l}^e_{22}[\hat{l}^e_{11}\hat{l}^e_{22}-(\hat{l}^e_{12})^2]}[\hat{l}^e_{22}\ \hat{l}^e_{12}]\begin{bmatrix}M_1 & M_2\\ M_2 & M_3\end{bmatrix}\begin{bmatrix}\hat{l}^e_{22}\\ \hat{l}^e_{12}\end{bmatrix}. \qquad (93)$$

The banana-plateau transport coefficients given in Eqs. (85)–(93) satisfy the following symmetry properties:

$$(L^{\rm bp})^{ee}_{21}=(L^{\rm bp})^{ee}_{12},\quad (L^{\rm bp})^{ii}_{21}=(L^{\rm bp})^{ii}_{12},$$
$$(L^{\rm bp})^{ie}_{jk}=\frac{\langle G_e\rangle}{\langle G_i\rangle}(L^{\rm bp})^{ei}_{kj}\quad (j,k=1,2), \qquad (94)$$
$$(L^{\rm bp})^{a}_{Ej}=-\frac{B^\theta\langle G_a\rangle}{\langle B_\zeta\rangle}(L^{\rm bp})^{a}_{jE}\quad (a=e,i;j=1,2).$$

Thus, we confirm that the banana-plateau transport equations have the complete Onsager symmetry in the axisymmetric case where $\langle G_e\rangle=\langle G_i\rangle=\langle B_\zeta\rangle/B^\theta$. The intrinsic ambipolarity of the banana-plateau particle fluxes is expressed by

$$(L^{\rm bp})^{ia}_{1j}=\frac{1}{Z_i}(L^{\rm bp})^{ea}_{1j}(a=e,i;j=1,2),$$

$$(L^{\rm bp})^{i}_{1E}=\frac{1}{Z_i}(L^{\rm bp})^{e}_{1E}, \tag{95}$$

which are rewritten by Eq. (94) as

$$(L^{\rm bp})^{ai}_{j1}=\frac{\langle G_i\rangle}{Z_i\langle G_e\rangle}(L^{\rm bp})^{ae}_{j1}\quad (a=e,i;j=1,2),$$

$$(L^{\rm bp})^{i}_{E1}=\frac{\langle G_i\rangle}{Z_i\langle G_e\rangle}(L^{\rm bp})^{e}_{E1}. \tag{96}$$

Using the above relations, the fluxes $[J^{\rm bp}_{e1}(=Z_iJ^{\rm bp}_{i1}),J^{\rm bp}_{e2},J^{\rm bp}_{i2},J_E]$ are related to the new reduced set of forces $(X^{**}_{e1},X_{e2},X_{i2},X_E)$ by the transport equations

$$\begin{bmatrix} J^{\rm bp}_{e1}\\ J^{\rm bp}_{e2}\\ J^{\rm bp}_{i2}\\ J^{\rm bp}_{E}\end{bmatrix}=\begin{bmatrix} (L^{\rm bp})^{ee}_{11} & (L^{\rm bp})^{ee}_{12} & (L^{\rm bp})^{ei}_{12} & (L^{\rm bp})^{e}_{1E}\\ (L^{\rm bp})^{ee}_{21} & (L^{\rm bp})^{ee}_{22} & (L^{\rm bp})^{ei}_{22} & (L^{\rm bp})^{e}_{2E}\\ (L^{\rm bp})^{ie}_{21} & (L^{\rm bp})^{ie}_{22} & (L^{\rm bp})^{ii}_{22} & (L^{\rm bp})^{i}_{2E}\\ (L^{\rm bp})^{e}_{E1} & (L^{\rm bp})^{e}_{E2} & (L^{\rm bp})^{i}_{E2} & (L^{\rm bp})_{EE}\end{bmatrix}\times\begin{bmatrix} X^{**}_{e1}\\ X_{e2}\\ X_{i2}\\ X_E\end{bmatrix}. \tag{97}$$

Here, the force X^{**}_{e1} is defined by

$$X^{**}_{e1}\equiv X_{e1}+\frac{\langle G_i\rangle}{Z_i\langle G_e\rangle}X_{i1}$$

$$\equiv-\frac{1}{n_e}\left(p'_e+\frac{\langle G_i\rangle}{\langle G_e\rangle}p'_i\right)-e\Phi'\left(\frac{\langle G_i\rangle}{\langle G_e\rangle}-1\right). \tag{98}$$

When $\langle G_e\rangle=\langle G_i\rangle$, X^{**}_{e1} coincides with X^{*}_{e1}, which is proportional to the total pressure gradient $-p'\equiv-(p'_e+p'_i)$, and the radial electric field $-\Phi'$ never affects the banana-plateau transport. However, if the electrons and the ions belong to different collisionality regimes in the nonaxisymmetric case, the banana-plateau radial particle and heat fluxes, and the bootstrap current depend on the radial electric field $-\Phi'$ through X^{**}_{e1} since $\langle G_e\rangle\neq\langle G_i\rangle$.

Next, let us derive the nonaxisymmetric transport equations. For the derivation, it is essential to note that the toroidal viscosities are given in the following form for both of the Pfirsch–Schlüter and plateau regimes:

$$\begin{bmatrix}\langle \mathbf{B}_t\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle\\ \langle \mathbf{B}_t\cdot\nabla\cdot\boldsymbol{\Theta}_a\rangle\end{bmatrix}=\frac{n_am_a}{\tau_{aa}}c_{ta}\begin{bmatrix}\hat{\mu}_{a1} & \hat{\mu}_{a2}\\ \hat{\mu}_{a2} & \hat{\mu}_{a3}\end{bmatrix}\times\left\{\begin{bmatrix}\langle u_{\parallel a}B\rangle\\ \dfrac{2}{5p_a}\langle q_{\parallel a}B\rangle\end{bmatrix}-\langle G_{ta}\rangle\frac{c}{e_a}\begin{bmatrix}X_{a1}\\ X_{a2}\end{bmatrix}\right\}, \tag{99}$$

where c_{ta} and $\langle G_{ta}\rangle$ are given in Appendix C and $\hat{\mu}_{aj}$ $(j=1,2,3)$ are the same as in Eq. (78). Here and hereafter, we consider the toroidal viscosities and the nonaxisymmetric fluxes only for the Pfirsch–Schlüter and plateau regimes since the expressions similar to Eq. (99) have not been obtained yet for the banana regime.

Appendix C shows that the ratio between the toroidal and parallel viscosities c_{ta}/c_a is related to the geometrical factor $\langle G_a\rangle$ in Eq. (78) by the following equation:

$$1-\frac{c_{ta}}{c_a}\frac{\langle B^2\rangle}{B^\zeta\langle B_\zeta\rangle}=\frac{B^\theta\langle G_a\rangle}{\langle B_\zeta\rangle}, \tag{100}$$

which is an essential relation for showing the Onsager symmetry of the total neoclassical transport.

Using Eqs. (78) and (99), the toroidal viscosities are written in terms of the parallel viscosities and the thermodynamic forces as

$$\begin{bmatrix}\langle \mathbf{B}_t\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle\\ \langle \mathbf{B}_t\cdot\nabla\cdot\boldsymbol{\Theta}_a\rangle\end{bmatrix}=\frac{c_{ta}}{c_a}\begin{bmatrix}\langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle\\ \langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\Theta}_a\rangle\end{bmatrix}+\frac{n_am_a}{\tau_{aa}}c_{ta}(\langle G_a\rangle-\langle G_{ta}\rangle)\frac{c}{e_a}\begin{bmatrix}\hat{\mu}_{a1} & \hat{\mu}_{a2}\\ \hat{\mu}_{a2} & \hat{\mu}_{a3}\end{bmatrix}\begin{bmatrix}X_{a1}\\ X_{a2}\end{bmatrix}. \tag{101}$$

Then, from Eqs. (43), (84), and (101), we obtain the nonaxisymmetric transport equations relating $(J^{\rm na}_{e1},J^{\rm na}_{e2},J^{\rm na}_{i1},J^{\rm na}_{i2})$ to $(X_{e1},X_{e2},X_{i1},X_{i2},X_E)$:

$$\begin{bmatrix} J^{\rm na}_{e1}\\ J^{\rm na}_{e2}\\ J^{\rm na}_{i1}\\ J^{\rm na}_{i2}\end{bmatrix}=\begin{bmatrix} (L^{\rm na})^{ee}_{11} & (L^{\rm na})^{ee}_{12} & (L^{\rm na})^{ei}_{11} & (L^{\rm na})^{ei}_{12} & (L^{\rm na})^{e}_{1E}\\ (L^{\rm na})^{ee}_{21} & (L^{\rm na})^{ee}_{22} & (L^{\rm na})^{ei}_{21} & (L^{\rm na})^{ei}_{22} & (L^{\rm na})^{e}_{2E}\\ (L^{\rm na})^{ie}_{11} & (L^{\rm na})^{ie}_{12} & (L^{\rm na})^{ii}_{11} & (L^{\rm na})^{ii}_{12} & (L^{\rm na})^{i}_{1E}\\ (L^{\rm na})^{ie}_{21} & (L^{\rm na})^{ie}_{22} & (L^{\rm na})^{ii}_{21} & (L^{\rm na})^{ii}_{22} & (L^{\rm na})^{i}_{2E}\end{bmatrix}\times\begin{bmatrix} X_{e1}\\ X_{e2}\\ X_{i1}\\ X_{i2}\\ X_E\end{bmatrix}, \tag{102}$$

where the nonaxisymmetric transport coefficients are given by

$$\begin{bmatrix} (L^{\rm na})_{11}^{ee} & (L^{\rm na})_{12}^{ee} & (L^{\rm na})_{11}^{ei} & (L^{\rm na})_{12}^{ei} & (L^{\rm na})_{1E}^{e} \\ (L^{\rm na})_{21}^{ee} & (L^{\rm na})_{22}^{ee} & (L^{\rm na})_{21}^{ei} & (L^{\rm na})_{22}^{ei} & (L^{\rm na})_{2E}^{e} \end{bmatrix} = -\frac{c_{te}}{c_e}\frac{\langle B^2\rangle}{B^\zeta\langle B_\zeta\rangle}\begin{bmatrix} (L^{\rm bp})_{11}^{ee} & (L^{\rm bp})_{12}^{ee} & (L^{\rm bp})_{11}^{ei} & (L^{\rm bp})_{12}^{ei} & (L^{\rm bp})_{1E}^{e} \\ (L^{\rm bp})_{21}^{ee} & (L^{\rm bp})_{22}^{ee} & (L^{\rm bp})_{21}^{ei} & (L^{\rm bp})_{22}^{ei} & (L^{\rm bp})_{2E}^{e} \end{bmatrix}$$
$$+\frac{n_e m_e}{\tau_{ee}}\frac{c^2}{e^2}c_{te}\frac{(\langle G_e\rangle-\langle G_{te}\rangle)}{B^\theta B^\zeta}\begin{bmatrix} \hat{\mu}_{e1} & \hat{\mu}_{e2} & 0 & 0 & 0 \\ \hat{\mu}_{e2} & \hat{\mu}_{e3} & 0 & 0 & 0 \end{bmatrix}, \qquad (103)$$

$$\begin{bmatrix} (L^{\rm na})_{11}^{ie} & (L^{\rm na})_{12}^{ie} & (L^{\rm na})_{11}^{ii} & (L^{\rm na})_{12}^{ii} & (L^{\rm na})_{1E}^{i} \\ (L^{\rm na})_{21}^{ie} & (L^{\rm na})_{22}^{ie} & (L^{\rm na})_{21}^{ii} & (L^{\rm na})_{22}^{ii} & (L^{\rm na})_{2E}^{i} \end{bmatrix} = -\frac{c_{ti}}{c_i}\frac{\langle B^2\rangle}{B^\zeta\langle B_\zeta\rangle}\begin{bmatrix} (L^{\rm bp})_{11}^{ie} & (L^{\rm bp})_{12}^{ie} & (L^{\rm bp})_{11}^{ii} & (L^{\rm bp})_{12}^{ii} & (L^{\rm bp})_{1E}^{i} \\ (L^{\rm bp})_{21}^{ie} & (L^{\rm bp})_{22}^{ie} & (L^{\rm bp})_{21}^{ii} & (L^{\rm bp})_{22}^{ii} & (L^{\rm bp})_{2E}^{i} \end{bmatrix}$$
$$+\frac{n_i m_i}{\tau_{ii}}\frac{c^2}{e_i^2}c_{ti}\frac{(\langle G_i\rangle-\langle G_{ti}\rangle)}{B^\theta B^\zeta}\begin{bmatrix} 0 & 0 & \hat{\mu}_{i1} & \hat{\mu}_{i2} & 0 \\ 0 & 0 & \hat{\mu}_{i2} & \hat{\mu}_{i3} & 0 \end{bmatrix}. \qquad (104)$$

It is found that $(L^{\rm na})_{jk}^{aa}=(L^{\rm na})_{kj}^{aa}$ is always valid although $(L^{\rm na})_{jk}^{ei}=(L^{\rm na})_{kj}^{ie}$ is satisfied only when $c_{te}/c_e=c_{ti}/c_i$. We see from Eq. (100) that the condition $c_{te}/c_e=c_{ti}/c_i$ is equivalent to $\langle G_e\rangle=\langle G_i\rangle$, which holds if the electrons and the ions belong to the same collisionality regime.

Finally, combining the banana-plateau transport equations (84) with the nonaxisymmetric transport equations (102), and using the relation (100), the transport equations for the total of the banana-plateau and nonaxisymmetric fluxes $(J_{e1}^{\rm bn},J_{e2}^{\rm bn},J_{i1}^{\rm bn},J_{i2}^{\rm bn},J_E^{\rm bp})$ are obtained as

$$\begin{bmatrix} J_{e1}^{\rm bn} \\ J_{e2}^{\rm bn} \\ J_{i1}^{\rm bn} \\ J_{i2}^{\rm bn} \\ J_E^{\rm bp} \end{bmatrix} \equiv \begin{bmatrix} J_{e1}^{\rm bp}+J_{e1}^{\rm na} \\ J_{e2}^{\rm bp}+J_{e2}^{\rm na} \\ J_{i1}^{\rm bp}+J_{i1}^{\rm na} \\ J_{i2}^{\rm bp}+J_{i2}^{\rm na} \\ J_E^{\rm bp} \end{bmatrix} = \begin{bmatrix} (L^{\rm bn})_{11}^{ee} & (L^{\rm bn})_{12}^{ee} & (L^{\rm bn})_{11}^{ei} & (L^{\rm bn})_{12}^{ei} & (L^{\rm bn})_{1E}^{e} \\ (L^{\rm bn})_{21}^{ee} & (L^{\rm bn})_{22}^{ee} & (L^{\rm bn})_{21}^{ei} & (L^{\rm bn})_{22}^{ei} & (L^{\rm bn})_{2E}^{e} \\ (L^{\rm bn})_{11}^{ie} & (L^{\rm bn})_{12}^{ie} & (L^{\rm bn})_{11}^{ii} & (L^{\rm bn})_{12}^{ii} & (L^{\rm bn})_{1E}^{i} \\ (L^{\rm bn})_{21}^{ie} & (L^{\rm bn})_{22}^{ie} & (L^{\rm bn})_{21}^{ii} & (L^{\rm bn})_{22}^{ii} & (L^{\rm bn})_{2E}^{i} \\ (L^{\rm bn})_{E1}^{e} & (L^{\rm bn})_{E2}^{e} & (L^{\rm bn})_{E1}^{i} & (L^{\rm bn})_{E2}^{i} & (L^{\rm bn})_{EE} \end{bmatrix}\begin{bmatrix} X_{e1} \\ X_{e2} \\ X_{i1} \\ X_{i2} \\ X_E \end{bmatrix}, \qquad (105)$$

where the total transport coefficients are given by

$$\begin{bmatrix} (L^{\rm bn})_{11}^{ee} & (L^{\rm bn})_{12}^{ee} & (L^{\rm bn})_{11}^{ei} & (L^{\rm bn})_{12}^{ei} & (L^{\rm bn})_{1E}^{e} \\ (L^{\rm bn})_{21}^{ee} & (L^{\rm bn})_{22}^{ee} & (L^{\rm bn})_{21}^{ei} & (L^{\rm bn})_{22}^{ei} & (L^{\rm bn})_{2E}^{e} \end{bmatrix} = \frac{B^\theta\langle G_e\rangle}{\langle B_\zeta\rangle}\begin{bmatrix} (L^{\rm bp})_{11}^{ee} & (L^{\rm bp})_{12}^{ee} & (L^{\rm bp})_{11}^{ei} & (L^{\rm bp})_{12}^{ei} & (L^{\rm bp})_{1E}^{e} \\ (L^{\rm bp})_{21}^{ee} & (L^{\rm bp})_{22}^{ee} & (L^{\rm bp})_{21}^{ei} & (L^{\rm bp})_{22}^{ei} & (L^{\rm bp})_{2E}^{e} \end{bmatrix}$$
$$+\frac{n_e m_e}{\tau_{ee}}\frac{c^2}{e^2}c_{te}\frac{(\langle G_e\rangle-\langle G_{te}\rangle)}{B^\theta B^\zeta}\begin{bmatrix} \hat{\mu}_{e1} & \hat{\mu}_{e2} & 0 & 0 & 0 \\ \hat{\mu}_{e2} & \hat{\mu}_{e3} & 0 & 0 & 0 \end{bmatrix}, \qquad (106)$$

$$\begin{bmatrix} (L^{\rm bn})_{11}^{ie} & (L^{\rm bn})_{12}^{ie} & (L^{\rm bn})_{11}^{ii} & (L^{\rm bn})_{12}^{ii} & (L^{\rm bn})_{1E}^{i} \\ (L^{\rm bn})_{21}^{ie} & (L^{\rm bn})_{22}^{ie} & (L^{\rm bn})_{21}^{ii} & (L^{\rm bn})_{22}^{ii} & (L^{\rm bn})_{2E}^{i} \end{bmatrix} = \frac{B^\theta\langle G_i\rangle}{\langle B_\zeta\rangle}\begin{bmatrix} (L^{\rm bp})_{11}^{ie} & (L^{\rm bp})_{12}^{ie} & (L^{\rm bp})_{11}^{ii} & (L^{\rm bp})_{12}^{ii} & (L^{\rm bp})_{1E}^{i} \\ (L^{\rm bp})_{21}^{ie} & (L^{\rm bp})_{22}^{ie} & (L^{\rm bp})_{21}^{ii} & (L^{\rm bp})_{22}^{ii} & (L^{\rm bp})_{2E}^{i} \end{bmatrix}$$
$$+\frac{n_i m_i}{\tau_{ii}}\frac{c^2}{e_i^2}c_{ti}\frac{(\langle G_i\rangle-\langle G_{ti}\rangle)}{B^\theta B^\zeta}\begin{bmatrix} 0 & 0 & \hat{\mu}_{i1} & \hat{\mu}_{i2} & 0 \\ 0 & 0 & \hat{\mu}_{i2} & \hat{\mu}_{i3} & 0 \end{bmatrix}, \qquad (107)$$

$$[(L^{\rm bn})_{E1}^{e} \quad (L^{\rm bn})_{E2}^{e} \quad (L^{\rm bn})_{E1}^{i} \quad (L^{\rm bn})_{E2}^{i} \quad (L^{\rm bn})_{EE}] = [(L^{\rm bp})_{E1}^{e} \quad (L^{\rm bp})_{E2}^{e} \quad (L^{\rm bp})_{E1}^{i} \quad (L^{\rm bp})_{E2}^{i} \quad (L^{\rm bp})_{EE}]. \qquad (108)$$

Thus, from the above definitions and the symmetry properties given by Eq. (94), we can directly confirm that the total banana-plateau and nonaxisymmetric transport coefficients satisfy the following Onsager symmetry:

$$(L^{\rm bn})_{jk}^{ab}=(L^{\rm bn})_{kj}^{ba}, \quad (L^{\rm bn})_{Ej}^{a}=-(L^{\rm bn})_{jE}^{a} \quad (a,b=e,i;j,k=1,2). \qquad (109)$$

Here, we should note that these transport coefficients contain terms of different orders in $(m_e/m_i)^{1/2}$. As seen from Eqs. (85)–(93) and Eqs. (106)–(108), the ion–ion coefficients $(L^{\rm bn})_{jk}^{ii}$ are $\mathscr{O}[(m_i/m_e)^{1/2}]$ larger than the other coefficients.

When the ambipolarity condition is imposed in the nonaxisymmetric case, we obtain, in the similar way as in Eqs. (71)–(73), the radial electric field:

$$-\Phi'=\frac{p_i'}{n_i e_i}+\frac{X_{i1}}{e_i} = \frac{p_i'}{n_e e}-\left(\sum_{a,b=e,i} e_a e_b (L^{\rm bn})_{11}^{ab}\right)^{-1}\sum_{a=e,i} e_a[(L^{\rm bn})_{11}^{ae}X_{e1}^* + (L^{\rm bn})_{12}^{ae}X_{e2}+(L^{\rm bn})_{12}^{ai}X_{i2}+(L^{\rm bn})_{1E}^{a}X_E] \qquad (110)$$

and the reduced set of the transport equations:

$$
\begin{bmatrix} J_{e1}^{\rm bn} \\ J_{e2}^{\rm bn} \\ J_{i2}^{\rm bn} \\ J_E^{\rm bp} \end{bmatrix} = \begin{bmatrix} (\mathscr{L}^{\rm bn})_{11}^{ee} & (\mathscr{L}^{\rm bn})_{12}^{ee} & (\mathscr{L}^{\rm bn})_{12}^{ei} & (\mathscr{L}^{\rm bn})_{1E}^{e} \\ (\mathscr{L}^{\rm bn})_{21}^{ee} & (\mathscr{L}^{\rm bn})_{22}^{ee} & (\mathscr{L}^{\rm bn})_{22}^{ei} & (\mathscr{L}^{\rm bn})_{2E}^{e} \\ (\mathscr{L}^{\rm bn})_{21}^{ie} & (\mathscr{L}^{\rm bn})_{22}^{ie} & (\mathscr{L}^{\rm bn})_{22}^{ii} & (\mathscr{L}^{\rm bn})_{2E}^{i} \\ (\mathscr{L}^{\rm bn})_{E1}^{e} & (\mathscr{L}^{\rm bn})_{E2}^{e} & (\mathscr{L}^{\rm bn})_{E2}^{i} & (\mathscr{L}^{\rm bn})_{EE} \end{bmatrix} \times \begin{bmatrix} X_{e1}^* \\ X_{e2} \\ X_{i2} \\ X_E \end{bmatrix} \tag{111}
$$

where the transport coefficients are defined by

$$
\begin{aligned}
(\mathscr{L}^{\rm bn})_{jk}^{ab} &= (L^{\rm bn})_{jk}^{ab} - \left(\sum_{A=e,i} e_A (L^{\rm bn})_{j1}^{aA} \right) \left(\sum_{A=e,i} e_A (L^{\rm bn})_{1k}^{Ab} \right) \times \left(\sum_{A,B=e,i} e_A e_B (L^{\rm bn})_{11}^{AB} \right)^{-1}, \\
(\mathscr{L}^{\rm bn})_{jE}^{a} &= (L^{\rm bn})_{jE}^{a} - \left(\sum_{A=e,i} e_A (L^{\rm bn})_{j1}^{aA} \right) \left(\sum_{A=e,i} e_A (L^{\rm bn})_{1E}^{A} \right) \times \left(\sum_{A,B=e,i} e_A e_B (L^{\rm bn})_{11}^{AB} \right)^{-1}, \\
(\mathscr{L}^{\rm bn})_{Ej}^{a} &= (L^{\rm bn})_{Ej}^{a} - \left(\sum_{A=e,i} e_A (L^{\rm bn})_{E1}^{A} \right) \left(\sum_{A=e,i} e_A (L^{\rm bn})_{1j}^{Aa} \right) \times \left(\sum_{A,B=e,i} e_A e_B (L^{\rm bn})_{11}^{AB} \right)^{-1}, \\
(\mathscr{L}^{\rm bn})_{EE} &= (L^{\rm bn})_{EE} - \left(\sum_{A=e,i} e_A (L^{\rm bn})_{E1}^{A} \right) \left(\sum_{A=e,i} e_A (L^{\rm bn})_{1E}^{A} \right) \times \left(\sum_{A,B=e,i} e_A e_B (L^{\rm bn})_{11}^{AB} \right)^{-1}.
\end{aligned} \tag{112}
$$

The Onsager symmetry still holds for the above coefficients:

$$
(\mathscr{L}^{\rm bn})_{jk}^{ab} = (\mathscr{L}^{\rm bn})_{kj}^{ba}, \quad (\mathscr{L}^{\rm bn})_{jE}^{a} = -(\mathscr{L}^{\rm bn})_{Ej}^{a}. \tag{113}
$$

As mentioned earlier, terms of different orders in $(m_e/m_i)^{1/2}$ are included in the transport coefficients $(L^{\rm bn})_{jk}^{ab}$. Therefore, the coefficients $(\mathscr{L}^{\rm bn})_{jk}^{ab}$ for the reduced transport equations also contain different order terms. To the lowest order of $(m_e/m_i)^{1/2}$, Eq. (110) is approximated by

$$
-\Phi' \,.\, \frac{p_i'}{n_e e} - \frac{(L^{\rm na})_{12}^{ii}}{(L^{\rm na})_{11}^{ii}} \frac{X_{i2}}{e_i}, \tag{114}
$$

which is the same one given in Ref. 8 and in Ref. 11. In Ref. 11, Balescu and Fantechi used this approximate expression for the radial electric field instead of Eq. (110) to derive the reduced set of the transport equations. Then, their resultant transport coefficients are different from ours in Eq. (112) and do not satisfy the Onsager symmetry, since part of $\mathscr{O}[(m_e/m_i)^{1/2}]$ terms in Eq. (112) are inconsistently neglected.

V. CONCLUSIONS AND DISCUSSION

In this work, we have investigated the entropy production, the full transport equations, and their Onsager symmetry for the neoclassical transport processes in magnetically confined plasmas with general toroidal configurations. It was clearly shown that, for both the classical and neoclassical transport processes, the kinetic form of the entropy production defined from the linearized collision operator is equivalent to its thermodynamic form written as the inner product of the thermodynamic forces and their conjugate transport fluxes. The entropy production from the gyroangle-dependent distribution function corresponds to the sum of the products of the classical radial particle and heat fluxes and the radial gradient thermodynamic forces, while the magnetic surface average of the entropy production from the gyroangle-averaged distribution function is given by the sum of the products of the thermodynamic forces and their conjugate fluxes which consist of the Pfirsch–Schlüter, banana-plateau, nonaxisymmetric parts of the neoclassical radial fluxes and the parallel current. This equivalence between the kinetic and thermodynamic forms of the entropy production for the full neoclassical transport fluxes were not confirmed by Balescu in Chap. 17 of Ref. 3. The reason is now discussed.

In deriving the thermodynamic form of the entropy production, we used the linearized drift kinetic equation without employing the Hermitian moment expansion of the distribution function. Balescu expressed the kinetic form of the entropy production as the quadratic form of only the vector Hermitian moment part of the distribution function which corresponds to the $l=1$ part of the Legendre polynomial expansion. However, since the neoclassical banana-plateau and nonaxisymmetric fluxes are caused by the parallel and toroidal viscosities, the tensor Hermitian moment part (or the $l=2$ part of the Legendre polynomial expansion) needs to be included for deriving the neoclassical thermodynamic form of the entropy production. Furthermore, as shown in Appendix D, we find that it is necessary to include all the tensor moments including higher order parts with $l=3,4,5,\ldots$, in the kinetic form of the entropy production to derive the neoclassical thermodynamic form in the banana and plateau regimes. This is intuitively understandable by considering the resonant particles responsible for the neoclassical fluxes in the plateau regime. The resonant particle distribution is highly anisotropic in velocity space and is approximated by a delta function in pitch angle so that the all Hermitian moments (or all lth-order Legendre polynomials) are required. As shown in Appendix D, the operator $v_\parallel \mathbf{n}\cdot\nabla$ in the drift kinetic equation (38) introduces the anisotropic distribution in the velocity space and it connects the lth-order moment with $(l\pm1)$th-order moments in contrast with the linearized collision operator which is isotropic in the velocity space and connects the lth-order moment with the same lth-order moment alone. In the Pfirsch–Schlüter regime, the collision operator dominates $v_\parallel \mathbf{n}\cdot\nabla$ and the distribution function has small contributions from higher-order moments representing the anisotropy. Then, in the Pfirsch–Schlüter regime, the $l=1$ vector moment is enough to express the entropy production as in Ref. 3, while the negligibly small viscosityinduced neoclassical fluxes are included in the $l=2$ tensor moment part. On the other hand, as the collision frequency decreases, the operator $v_{\parallel}\mathbf{n}\cdot\nabla$ is comparable to, and then dominates, the collision operator and all l-order moments are required in the kinetic entropy production functional to obtain its neoclassical thermodynamic form in the banana and plateau regimes.

We also proved from the formal solution of the linearized drift kinetic equation with the self-adjoint linearized collision operator that the Onsager symmetry is robustly valid for the neoclassical transport equations for general toroidal plasmas consisting of electrons and multi-species ions with arbitrary collision frequencies. Furthermore, we derived in Sec. IV, in the case of a single ion species, the full banana-plateau transport coefficients for all collisionality regimes and the full nonaxisymmetric transport coefficients for the Pfirsch–Schlüter and plateau regimes. The symmetry properties of these transport matrices were separately examined and the Onsager symmetry for their total transport equations was confirmed. We discussed the effects of the ambipolarity condition on the transport equations in detail for both axisymmetric and nonaxisymmetric configurations. The ambipolarity condition reduces by one the number of the conjugate pairs of the transport fluxes and the thermodynamic forces. In the axisymmetric case, the intrinsic ambipolarity holds and the radial electric field does not affect the transport. On the other hand, for broken toroidal symmetry, a radial current is a function of the thermodynamic forces (X_{aj}, X_E) in which the radial electric field is included. When the ambipolarity condition is imposed in the nonaxisymmetric case, the radial electric field is given by a linear combination of the other thermodynamic forces. We showed that the Onsager symmetry is satisfied whether the conjugate pairs of the fluxes and forces are reduced by the ambipolarity condition or not.

Balescu and Fantechi derived the full neoclassical transport coefficients for the plateau regime in the nonaxisymmetric configuration and claimed that the Onsager symmetry is slightly broken by the nonaxisymmetry. They showed the transport equations only for the reduced pairs of the fluxes and forces, in which the radial electric field is eliminated by the ambipolar condition. There, terms of $\mathcal{O}[(m_e/m_i)^{1/2}]$ were neglected in expressing the radial electric field in terms of the other forces as in Eq. (114). Then, the resultant transport coefficients in Ref. 11 deviate from those in Eq. (112) and the Onsager symmetry is broken in them since part of $\mathcal{O}[(m_e/m_i)^{1/2}]$ terms necessary for the symmetry are dropped.

The banana-plateau and nonaxisymmetric transport equations obtained here are valid whether electrons and ions belong to the same collisionality regime or not. When both species are in the same collisionality regime, we find that the geometrical factors for electrons and ions coincide with each other $\langle G_e\rangle = \langle G_i\rangle$ and that, as far as the radial fluxes and the radial forces are concerned, the Onsager symmetry is separately valid for the banana-plateau transport matrix $(L^{\rm bp})^{ab}_{jk}$ and for the nonaxisymmetric transport matrix $(L^{\rm na})^{ab}_{jk}$. When electron and ion collisionality regimes are different, $\langle G_e\rangle \neq \langle G_i\rangle$ and the mixed electron-ion coefficients in each matrix are not symmetric, although the total matrix $\mathsf{L}^{\rm bn}$ are symmetric. In the latter case, the radial electric field appears in the thermodynamic forces for the banana-plateau radial particle and heat fluxes and the bootstrap current in the nonaxisymmetric systems.

Since we proved the robust validity of the Onsager symmetry for the neoclassical transport equations, even in the nonaxisymmetric cases, this symmetry property can be utilized for the calculation of the nonaxisymmetric transport coefficients in the banana regime which were not given in Sec. IV. For example, from the banana-plateau transport coefficients $(L^{\rm bp})^a_{jE}$ and $(L^{\rm bp})^a_{Ej}$ $(a=e,i;j=1,2)$ for the banana regime given in Sec. IV, we can immediately obtain part of the nonaxisymmetric transport coefficients for the banana regime as $(L^{\rm na})^a_{jE} = -(L^{\rm bp})^a_{jE} - (L^{\rm bp})^a_{Ej}$ $(a=e,i;j=1,2)$.

In the previous work, we investigated the neoclassical and anomalous transport in weakly turbulent plasma and described the entropy production and the Onsager symmetry in electrostatic turbulence.[17] We are also investigating a unified description of the transport equations, the entropy production, and the Onsager symmetry for neoclassical and turbulent processes with both electrostatic and electromagnetic fluctuations, which we will report on in a future work.

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APPENDIX A: HAMADA COORDINATES AND INCOMPRESSIBLE FLOWS

In general toroidal configurations, the magnetic field is written in the contravariant form :

$$\mathbf{B} = B^\theta \frac{\partial \mathbf{x}}{\partial \theta} + B^\zeta \frac{\partial \mathbf{x}}{\partial \zeta}, \tag{A1}$$

where θ and ζ are the poloidal and toroidal angle variables, respectively, and corresponding basis vectors for the contravariant representation are given by $\partial\mathbf{x}/\partial\theta$ and $\partial\mathbf{x}/\partial\zeta$. Throughout this paper, we employ the Hamada coordinates (V,θ,ζ), where the flux label V represents the volume enclosed by the flux surface, and the periods of the angle variables are normalized as $\oint d\theta = \oint d\zeta = 1$. The Jacobian is unity $\nabla V\cdot\nabla\theta\times\nabla\zeta = 1$, so that the flux surface average is simply written as $\langle\cdot\rangle = \oint d\theta \oint d\zeta \cdot$. The poloidal and toroidal fields are given by $\mathbf{B}_p = B^\theta \partial\mathbf{x}/\partial\theta = B^\theta \nabla\zeta\times\nabla V$ and $\mathbf{B}_t = B^\zeta \partial\mathbf{x}/\partial\zeta = B^\zeta \nabla V\times\nabla\theta$. The contravariant poloidal and toroidal components are the surface flux quantities $B^\theta = \mathbf{B}\cdot\nabla\theta = \chi'(V)$ and $B^\zeta = \mathbf{B}\cdot\nabla\zeta = \psi'(V)$, where χ and ψ are the poloidal and toroidal fluxes, respectively, and $' \equiv \partial/\partial V$. The vector product of $\mathbf{B}$ and ∇V is given by the linear combination of $\mathbf{B}$ and $\partial\mathbf{x}/\partial\zeta$ as

$$\mathbf{B}\times\nabla V = \frac{B_\zeta}{B^\theta}\mathbf{B} - \frac{B^2}{B^\theta}\frac{\partial \mathbf{x}}{\partial \zeta}, \tag{A2}$$

where $B_\zeta = \mathbf{B}\cdot\partial\mathbf{x}/\partial\zeta$.

When a solenoidal vector field $\mathbf{U}$ ($\nabla\cdot\mathbf{U}=0$) is tangential to magnetic surfaces $\mathbf{U}\cdot\nabla V=0$ and satisfies $\mathbf{U}\times\mathbf{B}=\nabla K(V)$ with some flux quantity $K(V)$, it is written in the Hamada coordinates as

$$\mathbf{U}=U^\theta\frac{\partial\mathbf{x}}{\partial\theta}+U^\zeta\frac{\partial\mathbf{x}}{\partial\zeta}, \tag{A3}$$

where the both contravariant components $U^\theta=\mathbf{U}\cdot\nabla\theta$ and $U^\zeta=\mathbf{U}\cdot\nabla\zeta$ are flux surface quantities. Since both of the flow velocity $\mathbf{u}_a$ and the heat flow $\mathbf{q}_a$ are incompressible $\nabla\cdot\mathbf{u}_a=\nabla\cdot\mathbf{q}_a=0$ [see Eq. (D12)] and satisfies the above conditions [see Eq. (20)] to the lowest order in δ, all of their contravariant components $u_a^\theta=\mathbf{u}_a\cdot\nabla\theta$, $u_a^\zeta=\mathbf{u}_a\cdot\nabla\zeta$, $q_a^\theta=\mathbf{q}_a\cdot\nabla\theta$, $q_a^\zeta=\mathbf{q}_a\cdot\nabla\zeta$ are flux surface quantities. Also, $\mathbf{u}_a$ and $\mathbf{q}_a$ are separated into the parallel and perpendicular components as

$$\mathbf{u}_a=u_{\parallel a}\mathbf{n}+\mathbf{u}_{\perp a}, \quad \mathbf{q}_a=q_{\parallel a}\mathbf{n}+\mathbf{q}_{\perp a}, \tag{A4}$$

where the perpendicular components $\mathbf{u}_{\perp a}$ and $\mathbf{q}_{\perp a}$ are given by the thermodynamic forces X_{a1} and X_{a2} as in Eq. (20). Then, the contravariant poloidal and toroidal flow components are given by the linear combinations of the parallel flow components and the thermodynamic forces as

$$\begin{aligned}
u_a^\theta&=\frac{B^\theta}{B}u_{\parallel a}-\frac{B_\zeta}{B^2}\frac{cX_{a1}}{e_a},\\
u_a^\zeta&=\frac{B^\zeta}{B}u_{\parallel a}+\frac{B_\theta}{B^2}\frac{cX_{a1}}{e_a},\\
\frac{2}{5}\frac{q_a^\theta}{p_a}&=\frac{B^\theta}{B}\frac{2}{5}\frac{q_{\parallel a}}{p_a}-\frac{B_\zeta}{B^2}\frac{cX_{a2}}{e_a},\\
\frac{2}{5}\frac{q_a^\zeta}{p_a}&=\frac{B^\zeta}{B}\frac{2}{5}\frac{q_{\parallel a}}{p_a}+\frac{B_\theta}{B^2}\frac{cX_{a2}}{e_a}
\end{aligned} \tag{A5}$$

where $B_\theta=\mathbf{B}\cdot\partial\mathbf{x}/\partial\theta$. We find from Eq. (A5) that

$$\begin{aligned}
\langle B^2\rangle u_a^\theta&=B^\theta\langle u_{\parallel a}B\rangle-\langle B_\zeta\rangle\frac{cX_{a1}}{e_a},\\
\langle B^2\rangle u_a^\zeta&=B^\zeta\langle u_{\parallel a}B\rangle+\langle B_\theta\rangle\frac{cX_{a1}}{e_a},\\
\langle B^2\rangle\frac{2}{5}\frac{q_a^\theta}{p_a}&=B^\theta\frac{2}{5}\frac{\langle q_{\parallel a}B\rangle}{p_a}-\langle B_\zeta\rangle\frac{cX_{a2}}{e_a},\\
\langle B^2\rangle\frac{2}{5}\frac{q_a^\zeta}{p_a}&=B^\zeta\frac{2}{5}\frac{\langle q_{\parallel a}B\rangle}{p_a}+\langle B_\theta\rangle\frac{cX_{a2}}{e_a}.
\end{aligned} \tag{A6}$$

The following formulas are obtained from Eqs. (A5) and (A6):

$$\begin{aligned}
\frac{u_{\parallel a}}{B}&=\frac{\langle u_{\parallel a}B\rangle}{\langle B^2\rangle}+\frac{1}{B^\theta}\left(\frac{B_\zeta}{B^2}-\frac{\langle B_\zeta\rangle}{\langle B^2\rangle}\right)\frac{cX_{a1}}{e_a}\\
&=\frac{\langle u_{\parallel a}B\rangle}{\langle B^2\rangle}-\frac{1}{B^\zeta}\left(\frac{B_\theta}{B^2}-\frac{\langle B_\theta\rangle}{\langle B^2\rangle}\right)\frac{cX_{a1}}{e_a},\\
\frac{2}{5p_a}\frac{q_{\parallel a}}{B}&=\frac{2}{5p_a}\frac{\langle q_{\parallel a}B\rangle}{\langle B^2\rangle}+\frac{1}{B^\theta}\left(\frac{B_\zeta}{B^2}-\frac{\langle B_\zeta\rangle}{\langle B^2\rangle}\right)\frac{cX_{a2}}{e_a}\\
&=\frac{2}{5p_a}\frac{\langle q_{\parallel a}B\rangle}{\langle B^2\rangle}-\frac{1}{B^\zeta}\left(\frac{B_\theta}{B^2}-\frac{\langle B_\theta\rangle}{\langle B^2\rangle}\right)\frac{cX_{a2}}{e_a}.
\end{aligned} \tag{A7}$$

APPENDIX B: PFIRSCH–SCHLÜTER TRANSPORT EQUATIONS

The relations between the parallel friction forces and the parallel flows are given by the 13M approximation in the same form as in Eq. (27), and are written as

$$\begin{bmatrix} F_{\parallel a1}\\ -F_{\parallel a2}\end{bmatrix}=\sum_b\begin{bmatrix} l_{11}^{ab} & l_{12}^{ab}\\ l_{21}^{ab} & l_{22}^{ab}\end{bmatrix}\begin{bmatrix} u_{\parallel b}\\ -\frac{2}{5p_b}q_{\parallel b}\end{bmatrix}, \tag{B1}$$

where the coefficients l_{jk}^{ab} are defined in Eq. (28). Then, the Pfirsch–Schlüter particle and heat fluxes $(J_{a1}^{\rm PS},J_{a2}^{\rm PS})$ defined in Eq. (43) are rewritten as

$$\begin{aligned}
\begin{bmatrix} J_{a1}^{\rm PS}\\ J_{a2}^{\rm PS}\end{bmatrix}&=-\frac{c}{e_aB^\theta}\sum_b\begin{bmatrix} l_{11}^{ab} & -l_{12}^{ab}\\ -l_{21}^{ab} & l_{22}^{ab}\end{bmatrix}\\
&\times\begin{bmatrix}\left\langle\frac{u_{\parallel a}}{B}\left(B_\zeta-\langle B_\zeta\rangle\frac{B^2}{\langle B^2\rangle}\right)\right\rangle\\ \left\langle\frac{2}{5p_a}\frac{q_{\parallel a}}{B}\left(B_\zeta-\langle B_\zeta\rangle\frac{B^2}{\langle B^2\rangle}\right)\right\rangle\end{bmatrix}.
\end{aligned} \tag{B2}$$

Substituting Eq. (A7) into Eq. (B2), we obtain the Pfirsch–Schlüter transport equations:

$$\begin{bmatrix} J_{a1}^{\rm PS}\\ J_{a2}^{\rm PS}\end{bmatrix}=\sum_b\begin{bmatrix}(L^{\rm PS})_{11}^{ab} & (L^{\rm PS})_{12}^{ab}\\ (L^{\rm PS})_{21}^{ab} & (L^{\rm PS})_{22}^{ab}\end{bmatrix}\begin{bmatrix} X_{a1}\\ X_{a2}\end{bmatrix}, \tag{B3}$$

where the Pfirsch–Schlüter transport coefficients $(L^{\rm PS})_{jk}^{ab}$ are given by

$$(L^{\rm PS})_{jk}^{ab}=\frac{(-1)^{j+k-1}}{e_ae_b}\left(\frac{c}{B^\theta}\right)^2\left(\left\langle\frac{B_\zeta^2}{B^2}\right\rangle-\frac{\langle B_\zeta\rangle^2}{\langle B^2\rangle}\right)l_{jk}^{ab} \quad (j,k=1,2). \tag{B4}$$

From Eqs. (29) and (B4), we have the Onsager symmetry for the Pfirsch–Schlüter transport as

$$(L^{\rm PS})_{jk}^{ab}=(L^{\rm PS})_{kj}^{ba} \quad (j,k=1,2). \tag{B5}$$

The momentum conservation property described by Eq. (30) reduces to

$$\sum_a e_a(L^{\rm PS})_{1k}^{ab}=0 \quad (k=1,2), \tag{B6}$$

which implies the intrinsic ambipolarity of the Pfirsch–Schlüter particle fluxes. We easily find from Eqs. (32), (B3), (B5), and (B6) that the Pfirsch–Schlüter transport equations for the pairs $(J_{a1}^{\rm PS},X_{a1}^*)_{a\neq I},(J_{a2}^{\rm PS},X_{a2})$ are given by

$$J^{\rm PS}_{a1(a\neq I)}=\sum_{b\neq I}(L^{\rm PS})^{ab}_{11}X^*_{b1}+\sum_b (L^{\rm PS})^{ab}_{12}X_{b2},$$

$$J^{\rm PS}_{a2}=\sum_{b\neq I}(L^{\rm PS})^{ab}_{21}X^*_{b1}+\sum_b (L^{\rm PS})^{ab}_{22}X_{b2}, \tag{B7}$$

which shows that the transport coefficients $(L^{\rm PS})^{ab}_{jk}$ are the same as in Eq. (B3), except for the limitation $(a,j),(b,k) \neq (I,1)$ and that the Onsager symmetry is valid for both of the conjugate pairs.

APPENDIX C: PARALLEL AND TOROIDAL VISCOSITIES

It is shown from the solution of the linearized drift kinetic equation that the parallel viscosities in general toroidal configurations for all collision frequencies in the Pfirsch–Schlüter, plateau, and banana regimes are given in the following form :

$$\begin{bmatrix}\langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle\\ \langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\Theta}_a\rangle\end{bmatrix}=\frac{n_a m_a}{\tau_{aa}}c_a\begin{bmatrix}\hat\mu_{a1} & \hat\mu_{a2}\\ \hat\mu_{a2} & \hat\mu_{a3}\end{bmatrix}\times\left\{\begin{bmatrix}\langle u_{\parallel a}B\rangle\\ \frac{2}{5p_a}\langle q_{\parallel a}B\rangle\end{bmatrix}-\langle G_a\rangle\frac{c}{e_a}\begin{bmatrix}X_{a1}\\ X_{a2}\end{bmatrix}\right\}, \tag{C1}$$

where c_a and $\hat\mu_{aj}$ $(j=1,2,3)$ are the dimensionless parameters for the viscosity coefficients and $\langle G_a\rangle$ represents the geometrical factor. The collision frequency τ_{aa} is defined in Ref. 2. The dimensionless viscosity parameters $\hat\mu_{aj}$ $(j=1,2,3)$ are written as

$$\hat\mu_{aj}=\begin{cases}\{x_a^2(x_a^2-\frac{5}{2})^{j-1}(\nu_T^a\tau_{aa})^{-1}\} & \text{(for the Pfirsch–Schlüter regime)},\\ \delta_{j1}+\frac{1}{2}\delta_{j2}+\frac{13}{4}\delta_{j3} & \text{(for the plateau regime)},\\ \{(x_a^2-\frac{5}{2})^{j-1}(\nu_D^a\tau_{aa})\} & \text{(for the banana regime)},\end{cases} \tag{C2}$$

where the frequencies ν_T^a and ν_D^a are given in Ref. 2, and the velocity-space average $\{\cdot\}\equiv(8/3\sqrt{\pi})\int_0^\infty dx_a x_a^4 e^{-x_a^2}\cdot$ is used. The dimensionless coefficient c_a is given by

$$c_a=\begin{cases}\frac{3}{5}(v_{Ta}\tau_{aa})^2\langle(\mathbf{B}\cdot\nabla\ln B)^2\rangle/\langle B^2\rangle & \text{(for the Pfirsch–Schlüter regime)},\\ \sqrt{\pi}v_{Ta}\tau_{aa}\langle B^2\rangle^{-1/2}\sum_{m,n}2\pi|(\ln B)_{mn}|^2|mB^\theta-nB^\zeta| & \text{(for the plateau regime)},\\ f_t/f_c & \text{(for the banana regime)},\end{cases} \tag{C3}$$

where f_c and $f_t\equiv 1-f_c$ denote the fractions of circulating and trapped particles defined in Ref. 2 (or in Ref. 7), respectively, and $(\ln B)_{mn}$ are the coefficients in the Fourier expansion of $\ln B$:

$$\ln B=\sum_{m,n}(\ln B)_{mn}\exp[2\pi i(m\theta-n\zeta)].$$

The geometrical factor $\langle G_a\rangle$ is written as

$$\langle G_a\rangle=\begin{cases}\langle(\mathbf{B}\cdot\nabla\ln B)^2\rangle^{-1}\left[\langle B_\zeta\rangle\left\langle\frac{\partial\ln B}{\partial\theta}(\mathbf{B}\cdot\nabla\ln B)\right\rangle-\langle B_\theta\rangle\left\langle\frac{\partial\ln B}{\partial\zeta}(\mathbf{B}\cdot\nabla\ln B)\right\rangle\right] & \text{(for the Pfirsch–Schlüter regime)},\\ \left[\sum_{m,n}|(\ln B)_{mn}|^2|mB^\theta-nB^\zeta|\right]^{-1}\sum_{m,n}|(\ln B)_{mn}|^2(m\langle B_\zeta\rangle+n\langle B_\theta\rangle)\frac{mB^\theta-nB^\zeta}{|mB^\theta-nB^\zeta|} & \text{(for the plateau regime)}.\end{cases} \tag{C4}$$

The geometrical factor $\langle G_a\rangle$ for the banana regime is given in Ref. 7 (or in Ref. 9) as $\tilde G_b$. When we put $(\ln B)_{mn}=0$ for all $n\neq 0$ in Eqs. (C3) and (C4), the expressions for axisymmetric systems are reproduced. In the axisymmetric case, $\langle G_a\rangle$ is independent of the collision frequency and is given by $\langle G_a\rangle=\langle B_\zeta\rangle/B^\theta$ for all particle species.

It is shown that the toroidal viscosities are given in the following form for both of the Pfirsch–Schlüter and plateau regimes:

$$\begin{bmatrix}\langle \mathbf{B}_t\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle\\ \langle \mathbf{B}_t\cdot\nabla\cdot\boldsymbol{\Theta}_a\rangle\end{bmatrix}=\frac{n_a m_a}{\tau_{aa}}c_{ta}\begin{bmatrix}\hat\mu_{a1} & \hat\mu_{a2}\\ \hat\mu_{a2} & \hat\mu_{a3}\end{bmatrix}\left\{\begin{bmatrix}\langle u_{\parallel a}B\rangle\\ \frac{2}{5p_a}\langle q_{\parallel a}B\rangle\end{bmatrix}-\langle G_{ta}\rangle\frac{c}{e_a}\begin{bmatrix}X_{a1}\\ X_{a2}\end{bmatrix}\right\}, \tag{C5}$$

where $\hat\mu_{aj}$ $(j=1,2,3)$ are the same as given by Eq. (C2) and

$$c_{ta}=\begin{cases}\frac{3}{5}(v_{Ta}\tau_{aa})^2\langle(\mathbf{B}_t\cdot\nabla\ln B)(\mathbf{B}\cdot\nabla\ln B)\rangle/\langle B^2\rangle & \text{(for the Pfirsch–Schlüter regime)},\\ \sqrt{\pi}v_{Ta}\tau_{aa}\langle B^2\rangle^{-1/2}\sum_{m,n}2\pi|(\ln B)_{mn}|^2(-nB^\zeta)\frac{mB^\theta-nB^\zeta}{|mB^\theta-nB^\zeta|} & \text{(for the plateau regime)},\end{cases} \tag{C6}$$

$$\langle G_{ta}\rangle = \begin{cases} \langle (\mathbf{B}_t\cdot\nabla \ln B)(\mathbf{B}\cdot\nabla\ln B)\rangle^{-1}\left[\langle B_\zeta\rangle\left\langle\frac{\partial \ln B}{\partial\theta}(\mathbf{B}_t\cdot\nabla\ln B)\right\rangle - \langle B_\theta\rangle\left\langle\frac{\partial\ln B}{\partial\zeta}(\mathbf{B}_t\cdot\nabla\ln B)\right\rangle\right] & \text{(for the Pfirsch–Schlüter regime)},\\ \left[\sum_{m,n}|(\ln B)_{mn}|^2\frac{(-nB^\zeta)(mB^\theta-nB^\zeta)}{|mB^\theta-nB^\zeta|}\right]^{-1}\sum_{m,n}|(\ln B)_{mn}|^2(m\langle B_\zeta\rangle+n\langle B_\theta\rangle)\frac{(-nB^\zeta)}{|mB^\theta-nB^\zeta|} & \text{(for the plateau regime)}.\end{cases} \tag{C7}$$

The expressions similar to Eq. (C5) have not been obtained yet for the banana regime. We find from Eqs. (C3), (C4), and (C6) that the ratio between the toroidal and parallel viscosities c_{ta}/c_a is related to the geometrical factor $\langle G_a\rangle$ by the following equation:

$$1-\frac{c_{ta}}{c_a}\frac{\langle B^2\rangle}{B^\zeta\langle B_\zeta\rangle}=\frac{B^\theta\langle G_a\rangle}{\langle B_\zeta\rangle}, \tag{C8}$$

which is an essential relation for showing the Onsager symmetry of the total neoclassical transport. We find from Eqs. (C1) and (C5) that the toroidal viscosities are written in terms of the parallel viscosities and the thermodynamic forces as

$$\begin{bmatrix}\langle \mathbf{B}_t\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle\\ \langle \mathbf{B}_t\cdot\nabla\cdot\boldsymbol{\Theta}_a\rangle\end{bmatrix}=\frac{c_{ta}}{c_a}\begin{bmatrix}\langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle\\ \langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\Theta}_a\rangle\end{bmatrix}+\frac{n_a m_a}{\tau_{aa}}c_{ta}(\langle G_a\rangle - \langle G_{ta}\rangle)\frac{c}{e_a}\begin{bmatrix}\hat\mu_{a1} & \hat\mu_{a2}\\ \hat\mu_{a2} & \hat\mu_{a3}\end{bmatrix}\begin{bmatrix}X_{a1}\\ X_{a2}\end{bmatrix}. \tag{C9}$$

APPENDIX D: DRIFT KINETIC EQUATION AND LEGENDRE POLYNOMIAL EXPANSION

The linearized drift kinetic equation is written as

$$v_\parallel\mathbf{n}\cdot\nabla\bar f_{a1}+\mathbf{v}_{da}\cdot\nabla f_{a0}-\frac{e_a}{T_a}v_\parallel E_\parallel f_{a0}=C_a^L(\bar f_{a1}), \tag{D1}$$

where $\bar f_{a1}$ and f_{a0} are regarded as functions of the phase space variables $(\mathbf{x},E,\mu)$ ($E\equiv\frac12 m_a v^2+e_a\Phi$: the particle's energy; $\mu=m_a v_\perp^2/2B$: the magnetic moment). Using $(\mathbf{x},v,\xi)$ ($\xi\equiv v_\parallel/v$) as the phase space variables instead of $(\mathbf{x},E,\mu)$, we consider the expansion by the Legendre polynomials $P_l(\xi)$ [$P_0(\xi)=1, P_1(\xi)=\xi, P_2(\xi)=\frac32\xi^2-\frac12,\cdots$] for an arbitrary function $F(\mathbf{x},v,\xi)$ as

$$\begin{aligned}F(\mathbf{x},v,\xi)&=\sum_{l=0}^{\infty}F^{(l)}(\mathbf{x},v,\xi),\\ F^{(l)}(\mathbf{x},v,\xi)&=P_l(\xi)\frac{2l+1}{2}\int_{-1}^{1}d\eta P_l(\eta)F(\mathbf{x},v,\eta).\end{aligned} \tag{D2}$$

The lth Lengendre component corresponds to the l-order tensor Hermitian moment part in the Hermitian moment representation employed by Balescu in Ref. 3. The first term in the left-hand side of Eq. (D1) is rewritten in the new phase space variables as

$$v\left(\xi\mathbf{n}\cdot\nabla\bar f_{a1}-(\mathbf{n}\cdot\nabla\ln B)\frac{1-\xi^2}{2}\frac{\partial\bar f_{a1}}{\partial\xi}\right)\equiv\mathscr{A}\bar f_{a1}. \tag{D3}$$

We have the following formulas for the Legendre polynomials:

$$\begin{aligned}\xi P_l(\xi)&=\frac{1}{2l+1}[lP_{l-1}(\xi)+(l+1)P_{l+1}(\xi)],\\ (1-\xi^2)\frac{dP_l(\xi)}{d\xi}&=\frac{l(l+1)}{2l+1}[P_{l-1}(\xi)-P_{l+1}(\xi)].\end{aligned} \tag{D4}$$

We find from Eqs. (D3) and (D4) that the operator $\mathscr{A}$ [$\equiv v_\parallel\mathbf{n}\cdot\nabla$ with $(\mathbf{x},E,\mu)$ as the phase space variables] in the drift kinetic equation (D1) transforms the lth Legendre component to the linear combination of the $(l\pm1)$th components:

$$\mathscr{A}F^{(l)}=(\mathscr{A}F)^{(l-1)}+(\mathscr{A}F)^{(l+1)}. \tag{D5}$$

Contrastively, from the velocity space isotropy of the collision operator described in Eq. (3), the operator C_a^L transforms the lth component to only the lth one. The second and third terms in the right-hand side of Eq. (D1) are rewritten from Eqs. (18) and (20) as

$$\left[\frac23+\frac13P_2(\xi)\right]x_a^2\left[\nabla\cdot\mathbf{u}_{\perp a}+\left(x_a^2-\frac52\right)\frac{2}{5p_a}\nabla\cdot\mathbf{q}_{\perp a}\right]f_{a0} \tag{D6}$$

and

$$-\frac{e_aE_\parallel}{T_a}v\xi f_{a0}, \tag{D7}$$

respectively. The former is proportional to the divergence of the diamagnetic flows and contains only the zeroth and second Legendre polynomial components while the latter is proportional to the parallel electric field and contains only the first Legendre component.

Now, let us write the drift kinetic equation (D1) by each Legendre component, separately. The zeroth Legendre component (or the scalar moment part) of Eq. (D1) is given by

$$\begin{aligned}&(\mathscr{A}\bar f_{a1}^{(l=1)})^{(l=0)}+\frac23x_a^2\left[\nabla\cdot\mathbf{u}_{\perp a}+\left(x_a^2-\frac52\right)\frac{2}{5p_a}\nabla\cdot\mathbf{q}_{\perp a}\right]f_{a0}\\ &=C_a^L(\bar f_{a1}^{(l=0)}).\end{aligned} \tag{D8}$$

Here, the $l=1$ Legendre component $\bar f_{a1}^{(l=1)}$ of the distribution function is expanded by the Laguerre polynomials $L_j^{(3/2)}(x_a^2)$ ($L_0^{(3/2)}(x_a^2)=1, L_1^{(3/2)}(x_a^2)=\frac52-x_a^2,\cdots$) as

$$\bar f_{a1}^{(l=1)}=\frac{2}{v_{Ta}}\xi x_a\left[u_{\parallel a}+\left(x_a^2-\frac52\right)\frac25\frac{q_{\parallel a}}{p_a}\right]f_{a0}+\bar f_{a1}^{(l=1,j\geq2)}, \tag{D9}$$

where $\bar f_{a1}^{(l=1,j\geq2)}$ denotes the sum of the jth Laguerre polynomial components with $j\geq2$, which is neglected in the 13M approximation. The first term of Eq. (D8) is rewritten as

$$
\begin{aligned}
(\mathscr{A}\bar{f}_{a1}^{(l=1)})^{(l=0)} = {} & \frac{2}{3}x_a^2\left\{\nabla\cdot(u_{\parallel a}\mathbf{n}) + \left(x_a^2 - \frac{5}{2}\right)\right. \\
& \left.\times\frac{2}{5p_a}\nabla\cdot(q_{\parallel a}\mathbf{n})\right\}f_{a0} \\
& + (\mathscr{A}\bar{f}_{a1}^{(l=1,j\geq 2)})^{(l=0)}.
\end{aligned} \tag{D10}
$$

Then, Eq. (D8) reduces to

$$
\begin{aligned}
& \frac{2}{3}x_a^2\left\{\nabla\cdot\mathbf{u}_a + \left(x_a^2-\frac{5}{2}\right)\frac{2}{5p_a}\nabla\cdot\mathbf{q}_a\right\}f_{a0} + (\mathscr{A}\bar{f}_{a1}^{(l=1,j\geq 2)})^{(l=0)} \\
& \quad = C_a^L(\bar{f}_{a1}^{(l=0)}).
\end{aligned} \tag{D11}
$$

Integrating Eq. (D11) multiplied by 1 and x_a^2 over the velocity space, we obtain the incompressibility of $\mathbf{u}_a$ and $\mathbf{q}_a$:

$$
\nabla\cdot\mathbf{u}_a = \nabla\cdot\mathbf{q}_a = 0, \tag{D12}
$$

where we used the particle number and energy conservation by the collision operator described in Eq. (2). [Exactly speaking, $\nabla\cdot\mathbf{q}_a = 0$ is valid to the lowest order of the small mass ratio as in Eq. (9).] Thus, we have

$$
(\mathscr{A}\bar{f}_{a1}^{(l=1,j\geq 2)})^{(l=0)} = C_a^L(\bar{f}_{a1}^{(l=0)}), \tag{D13}
$$

from which the contribution from the scalar moment part $\bar{f}_{a1}^{(l=0)}$ of the distribution function to the kinetic form of the entropy production is given by

$$
\begin{aligned}
\bar{\sigma}_a^{(l=0)} &\equiv -\int d^3v\frac{\bar{f}_{a1}^{(l=0)}}{f_{a0}}C_a^L(\bar{f}_{a1}^{(l=0)}) \\
&= -\int d^3v\frac{\bar{f}_{a1}^{(l=0)}}{f_{a0}}\mathscr{A}\bar{f}_{a1}^{(l=1,j\geq 2)}.
\end{aligned} \tag{D14}
$$

In the 13M approximation, $\bar{\sigma}_a^{(l=0)}$ vanishes.

Taking the first Legendre component (or the vector moment part) of Eq. (D1), we have

$$
[\mathscr{A}(\bar{f}_{a1}^{(l=0)} + \bar{f}_{a1}^{(l=2)})]^{(l=1)} - \frac{e_a E_{\parallel a}}{T_a}v\,\xi f_{a0} = C_a^L(\bar{f}_{a1}^{(l=1)}). \tag{D15}
$$

We obtain the following parallel momentum balance equations from velocity integration of Eq. (D15) multiplied by $m_a v\,\xi$ and $m_a v\,\xi(x_a^2-\frac{5}{2})$:

$$
\begin{aligned}
& \mathbf{n}\cdot(\nabla p_{a1} + \nabla\cdot\boldsymbol{\pi}_a) - n_a e_a E_\parallel = F_{\parallel a1}, \\
& \mathbf{n}\cdot(\nabla\theta_{a1} + \nabla\cdot\boldsymbol{\Theta}_a) = F_{\parallel a2}.
\end{aligned} \tag{D16}
$$

Using Eqs. (D9) and (D15), the contribution from the vector moment part $\bar{f}_{a1}^{(l=1)}$ to the kinetic form of the entropy production is given by

$$
\begin{aligned}
\bar{\sigma}_a^{(l=1)} &\equiv -\int d^3v\frac{\bar{f}_{a1}^{(l=1)}}{f_{a0}}C_a^L(\bar{f}_{a1}^{(l=1)}) \\
&= \frac{1}{T_a}n_a e_a u_{\parallel a}E_\parallel - \int d^3v\frac{\bar{f}_{a1}^{(l=1)}}{f_{a0}}\mathscr{A}(\bar{f}_{a1}^{(l=0)} + \bar{f}_{a1}^{(l=2)}) \\
&= \frac{1}{T_a}n_a e_a u_{\parallel a}E_\parallel - \frac{1}{T_a}\left\{u_{\parallel a}\mathbf{n}\cdot(\nabla p_{a1} + \nabla\cdot\boldsymbol{\pi}_a)\right. \\
&\quad \left. + \frac{2}{5}\frac{q_{\parallel a}}{p_a}\mathbf{n}\cdot(\nabla\theta_{a1} + \nabla\cdot\boldsymbol{\Theta}_a)\right\} \\
&\quad - \int d^3v\frac{\bar{f}_{a1}^{(l=1,j\geq 2)}}{f_{a0}}\mathscr{A}(\bar{f}_{a1}^{(l=0)} + \bar{f}_{a1}^{(l=2)}),
\end{aligned} \tag{D17}
$$

where the last integral vanishes in the 13M approximation.

The second Legendre component (or the tensor moment part) of Eq. (D1) is given by

$$
\begin{aligned}
& [\mathscr{A}(\bar{f}_{a1}^{(l=1)} + \bar{f}_{a1}^{(l=3)})]^{(l=2)} + \tfrac{1}{3}P_2(\xi)x_a^2 \\
& \quad \times\left\{\nabla\cdot\mathbf{u}_{\perp a} + \left(x_a^2-\frac{5}{2}\right)\frac{2}{5p_a}\nabla\cdot\mathbf{q}_{\perp a}\right\}f_{a0} \\
& \quad = C_a^L(\bar{f}_{a1}^{(l=2)}),
\end{aligned} \tag{D18}
$$

which is rewritten by Eqs. (D9) and (D12) as

$$
\begin{aligned}
& 2P_2(\xi)x_a^2\left\{\mathbf{u}_a + \left(x_a^2-\frac{5}{2}\right)\frac{2}{5p_a}\mathbf{q}_a\right\}\cdot(\nabla\ln B)f_{a0} + [\mathscr{A}(\bar{f}_{a1}^{(l=1,j\geq 2)} \\
& \quad + \bar{f}_{a1}^{(l=3)})]^{(l=2)} = C_a^L(\bar{f}_{a1}^{(l=2)}).
\end{aligned} \tag{D19}
$$

The contribution from the tensor moment part $\bar{f}_{a1}^{(l=2)}$ to the kinetic form of the entropy production is given by

$$
\begin{aligned}
\bar{\sigma}_a^{(l=2)} &\equiv -\int d^3v\frac{\bar{f}_{a1}^{(l=2)}}{f_{a0}}C_a^L(\bar{f}_{a1}^{(l=2)}) \\
&= -\frac{1}{T_a}\left\{(p_{\parallel a}-p_{\perp a})\mathbf{u}_a + (\Theta_{\parallel a}-\Theta_{\perp a})\frac{2}{5p_a}\mathbf{q}_a\right\} \\
&\quad \cdot\nabla\ln B - \int d^3v\frac{\bar{f}_{a1}^{(l=2)}}{f_{a0}}\mathscr{A}(\bar{f}_{a1}^{(l=1,j\geq 2)} + \bar{f}_{a1}^{(l=3)}).
\end{aligned} \tag{D20}
$$

Similarly, for higher-order tensor moment parts with $l\geq 3$, we have

$$
[\mathscr{A}(\bar{f}_{a1}^{(l-1)} + \bar{f}_{a1}^{(l+1)})]^{(l)} = C_a^L(\bar{f}_{a1}^{(l)}) \quad (\text{for } l\geq 3), \tag{D21}
$$

from which we obtain

$$
\begin{aligned}
\bar{\sigma}_a^{(l)} &\equiv -\int d^3v\frac{\bar{f}_{a1}^{(l)}}{f_{a0}}C_a^L(\bar{f}_{a1}^{(l)}) \\
&= -\int d^3v\frac{\bar{f}_{a1}^{(l)}}{f_{a0}}\mathscr{A}(\bar{f}_{a1}^{(l-1)} + \bar{f}_{a1}^{(l+1)}) \quad (\text{for } l\geq 3).
\end{aligned} \tag{D22}
$$

Taking the magnetic surface average of each order moment part of the entropy production in Eqs. (D14), (D17), (D20), and (D22) gives

$$
\begin{aligned}
\langle\bar{\sigma}_a^{(l=0)}\rangle &= -\left\langle\int d^3v\frac{\bar{f}_{a1}^{(l=0)}}{f_{a0}}\mathscr{A}\bar{f}_{a1}^{(l=1,j\geq 2)}\right\rangle \\
\langle\bar{\sigma}_a^{(l=1)}\rangle &= \frac{1}{T_a}n_a e_a\langle u_{\parallel a}E_\parallel\rangle - \frac{1}{T_a}\left\langle u_{\parallel a}\mathbf{n}\cdot(\nabla p_{a1} + \nabla\cdot\boldsymbol{\pi}_a)\right. \\
&\quad \left. + \frac{2}{5}\frac{q_{\parallel a}}{p_a}\mathbf{n}\cdot(\nabla\theta_{a1} + \nabla\cdot\boldsymbol{\Theta}_a)\right\rangle \\
&\quad - \left\langle\int d^3v\frac{\bar{f}_{a1}^{(l=1,j\geq 2)}}{f_{a0}}\mathscr{A}(\bar{f}_{a1}^{(l=0)} + \bar{f}_{a1}^{(l=2)})\right\rangle,
\end{aligned} \tag{D23}
$$

$$\langle\bar{\sigma}_a^{(l=2)}\rangle = -\frac{1}{T_a}\left\langle\left[(p_{\parallel a}-p_{\perp a})\mathbf{u}_a+(\Theta_{\parallel a}-\Theta_{\perp a})\frac{2}{5p_a}\mathbf{q}_a\right]\cdot\nabla\ln B\right\rangle - \left\langle\int d^3v\frac{\bar{f}_{a1}^{(l=2)}}{f_{a0}}\mathscr{A}(\bar{f}_{a1}^{(l=1,j\geq 2)}+\bar{f}_{a1}^{(l=3)})\right\rangle,$$

$$\langle\bar{\sigma}_a^{(l)}\rangle = -\left\langle\int d^3v\frac{\bar{f}_{a1}^{(l)}}{f_{a0}}\mathscr{A}(\bar{f}_{a1}^{(l-1)}+\bar{f}_{a1}^{(l+1)})\right\rangle \quad (\text{for } l\geq 3),$$

which are summed up to

$$\langle\bar{\sigma}_a\rangle = \sum_{l=0}^{\infty}\langle\bar{\sigma}_a^{(l)}\rangle = \frac{1}{T_a}n_a e_a\langle u_{\parallel a}E_\parallel\rangle - \frac{1}{T_a}\left\langle u_{\parallel a}\mathbf{n}\cdot(\nabla p_{1a}+\nabla\cdot\boldsymbol{\pi}_a)+\frac{2}{5}\frac{q_{\parallel a}}{p_a}\mathbf{n}\cdot(\nabla\theta_{1a}+\nabla\cdot\boldsymbol{\Theta}_a)\right\rangle - \frac{1}{T_a}\left\langle\left[(p_{\parallel a}-p_{\perp a})\mathbf{u}_a+(\Theta_{\parallel a}-\Theta_{\perp a})\frac{2}{5p_a}\mathbf{q}_a\right]\cdot\nabla\ln B\right\rangle. \tag{D24}$$

Here, we have used the following cancellation formula for arbitrary functions χ and ψ of $(\mathbf{x},v,\xi)$:

$$\left\langle\int d^3v\frac{\chi}{f_{a0}}\mathscr{A}\psi\right\rangle + \left\langle\int d^3v\frac{\psi}{f_{a0}}\mathscr{A}\chi\right\rangle = 0, \tag{D25}$$

which is derived from the following properties of $\mathscr{A}$:

$$\mathscr{A}(\chi\psi) = \chi\mathscr{A}\psi+\psi\mathscr{A}\chi, \quad \left\langle\int d^3v\,\mathscr{A}F(\mathbf{x},v,\xi)\right\rangle = 0,$$
$$\mathscr{A}G(V,v) = 0, \tag{D26}$$

where F and G are arbitrary functions of $(\mathbf{x},v,\xi)$ and (V,v), respectively. After some calculations, Eq. (D24) is rewritten in the same thermodynamic form as in Eq. (46):

$$T_a\langle\bar{\sigma}_a\rangle = J_{a1}^{\text{ncl}}X_{a1}+J_{a2}^{\text{ncl}}X_{a2}+J_{a3}X_{a3}, \tag{D27}$$

of which the species sum is given by

$$\sum_a T_a\langle\bar{\sigma}_a\rangle = \sum_a (J_{a1}^{\text{ncl}}X_{a1}+J_{a2}^{\text{ncl}}X_{a2})+J_E X_E. \tag{D28}$$

It should be noted that the above neoclassical thermodynamic forms of the entropy production are derived from the kinetic form only through the magnetic surface average and sum of all the l-th tensor moment contributions of the distribution function. In Chap. 17 of Ref. 3, Balescu used only the vector Hermitian moments of the distribution function to calculate the kinetic form of the entropy production, which is written in our notation as

$$\sum_a T_a(\langle\tilde{\sigma}_a\rangle+\langle\bar{\sigma}_a^{(l=1)}\rangle). \tag{D29}$$

Here, as is understandable from the form of $\tilde{f}_{a1}$ shown in Eq. (17), the contribution of the vector Hermitian moments contains $\tilde{\sigma}_a$ which is defined by Eq. (16) and written in the classical thermodynamic form of Eq. (21). In the Pfirsch–Schlüter regime, the higher-order moments representing higher anisotropy become small so that the $l=2$ tensor moments corresponding to the viscosities (and therefore to the banana-plateau and nonaxisymmetric transport fluxes) are smaller than the $l=1$ vector moment corresponding to the classical and Pfirsch–Schlüter fluxes by a small factor $v_{Ta}\tau_{aa}/L$ (L: the scale length of the magnetic configuration). In this case, the $l\geq 3$ moments are further smaller and the total entropy production is approximated in the lowest order of $v_{Ta}\tau_{aa}/L$ with the 13M approximation by Eq. (D29) as

$$\sum_a T_a(\langle\tilde{\sigma}_a\rangle+\langle\bar{\sigma}_a\rangle) \simeq \sum_a T_a(\langle\tilde{\sigma}_a\rangle+\langle\bar{\sigma}_a^{(l=1)}\rangle) \simeq \sum_a ((\langle J_{a1}^{\text{cl}}\rangle+J_{a1}^{\text{PS}})X_{a1}+(\langle J_{a2}^{\text{cl}}\rangle+J_{a2}^{\text{PS}})X_{a2})+J_E^{\text{cl}}X_E \quad (\text{for the Pfirsch–Schlüter regime}), \tag{D30}$$

which takes the thermodynamic form consisting of the products of the thermodynamic forces and their conjugate classical and Pfirsch–Schlüter fluxes. [Here, it is noted that the classical fluxes J_{aj}^{cl} $(j=1,2)$ is defined not by the magnetic surface average but by the local quantity in Eq. (22).] However, for small collision frequencies as in the plateau and banana regimes, contributions from higher-order tensor moments with $l\geq 2$ are comparable to the $l=1$ vector moment and are indispensable in order to derive the complete neoclassical thermodynamic form including the banana-plateau and nonaxisymmetric transport fluxes.

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