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Viscous Dusty Fluid Flow with Constant Velocity Magnitude

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Abstract: We consider the viscous dusty fluid, where the velocity of the dust particle is everywhere parallel to that of the fluid with velocity magnitude of the fluid is constant along each individual streamline. Also it is assumed that number density of the dust particle is constant and the dust particles are uniform in size and shape and bulk concentration of the dust is small. Hodograph and Legendre transform of stream function is employed to get the solutions and the geometry of streamlines for these flows by using the resulting partial differential equations when the Jacobian is zero and nonzero cases. In each case the variation of pressure is analyzed graphically.

Keywords: Dusty Fluids; Hodograph and Legendre Transform; Velocity Components; Pressure Function; Streamlines
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1. Introduction

Cosmic dust is widely present in space, where gas and dust clouds are primary precursors for planetary systems. The zodiacal light, seen in the sky on a dark night, is produced by sunlight reflected from particles of dust in orbit around the sun. The tails of comets

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are produced by emissions of dust and ionized gas from the body of the comet. Dust also covers solid planetary bodies, and vast dust storms can occur on Mars that can cover almost the entire planet. Interstellar dust is found between the stars, and high concentrations can produce diffuse nebulae and reflection nebulae. Dust samples returned from outer space could provide information about conditions in the early solar system. Several spacecraft have been launched in an attempt to gather samples of dust and other materials. Among these was stardust, which flew past comet wild 2 in 2004 and returned a capsule of the remains of the comet to the U.S. in January 2006. The Japanese Hayabusa spacecraft is currently on a mission to collect samples of dust from the surface of an asteroid.

In the present paper some assumptions are made on Saffman model and the basic equations of fluid phase and dust phase are written into a convenient form by using suitable hodograph transformation. Further it is assumed that velocity vector of fluid is everywhere parallel to that of dust velocity and fluid is flowing with constant velocity magnitude. By introducing stream function and Legendre transform of this stream function, flow equations are recasted in the transformed function. By assuming the Jacobian is zero and nonzero, exact solutions to flow variables are obtained. It is shown that the flow is irrotational and streamlines are concentric circles when Jacobian is nonzero and flow is rotational and stream lines are straight lines when it is zero.

P.G.Saffman [1] discussed stability of the laminar flow of a dust gas in which the dust particles are uniformly distributed. Marble [2] discussed the dynamics of dusty gas. R.M.Barron [3, 4] studied two dimensional steady flow of a dusty gas. He obtained solutions to flow variables in orthogonal curvilinear co-ordinate system. Also by considering the dust particle distribution to be variable and the velocity of dust particle is everywhere parallel to velocity of fluid and proved that the possible flows are radial and streamlines are straight lines. He established that the dust particle distribution can not be uniform in radial flow and possible stream lines are only parallel straight lines. He found solutions to flow variables in natural coordinate system where the coordinate axes are the stream lines η is a constant and their orthogonal trajectories are curves $\xi = constant$.

O.P.Chandana et al [5] studied rotational plane flow of viscous fluid in the hodograph plane using Legendre transformation of the stream function. Satter [6] by assuming velocity magnitude is constant along each stream line, given solutions to flow variables of steady plane MHD flow of viscous incompressible fluid of infinite electrical conductivity when magnetic field vector is constantly inclined to velocity vector. M.H.Hadaman et al [7] analyzed the squeezing flow dust fluid and concluded the introduction of dust to fluid squeeze film increase the load carrying capacity of the squeeze. C.S.Bagewadi et al [8] studied the flow of dust gas using Frenet frame field system and found solutions to flow variables using Laplace transformation method. In [9] they studied the geometry of streamlines on spherical surface, inverse surface and parallel surface and that the streamlines are concentric circles on this surface by the method of metric coefficients. Siddabasappa et al [12] obtained the exact solutions to dusty gas flow variables including viscosity and compressibility in different surfaces like spherical, centro, Beltrami surfaces using fundamental magnitudes and differential geometry. In [13, 14] the importance and application of flows, helps us to analyze waste water treatment, corrosive particles in engine, oil flows, air pollution, smoke emission from vehicles, emission of fine particles from cement industries, nuclear reactors, filtration, etc. Also, it gives the information about water pollution like rain fall in space, flow of blood, pumping of water in pipes. The possible presence of solid particles such as ash or soot in combustion MHD generators and plasma MHD accelerators and their effect on the performance of such devices led to studies of particulate suspensions in conducting fluids in the presence of magnetic fields. For example in an MHD generator, coal mixed with seed is fed into a combustor. The coal and seed mixture is burned in oxygen and combustion gas expands through a nozzle before it enters the generator section. The gas mixture flowing through the MHD channel consists of a condensable vapour (slag) and a non condensable gas mixed with seed coal combustion products. Both the slag and the non condensable gas are electrically conducting. The presence of slag and seeded particles significantly influences the flow in the MHD channel. This field has important applications in areas as cooling systems, centrifugal separation of matter from fluid, petroleum industry, purification of crude oil, electrostatic precipitation, polymer technology and fluid droplets springs.

2. Governing Equation

Based on Saffman model of the steady motion of a incompressible fluid the governing equations of the flow are

For fluid phase

$$\nabla.\vec{u} = 0 \qquad \text{continuity equation} \tag{1}$$

$$\rho(\vec{u}.\nabla)\vec{u} + \nabla p = \mu\nabla^2\vec{u} + KN(\vec{v} - \vec{u}) \qquad \text{momentum equation} \tag{2}$$

For dust phase

$$\nabla.(N\vec{v}) = 0 \qquad \text{continuity equation} \tag{3}$$

$$(\vec{v}.\nabla)\vec{v} = \frac{K}{m}(\vec{u} - \vec{v}) \qquad \text{momentum equation} \tag{4}$$

where
$\vec{u}$- velocity of the fluid $=(u_1, u_2)$
$\vec{v}$-velocity of the dust particle$=(v_1, v_2)$
ρ- density of fluid
p-fluid pressure
μ- viscosity of fluid
N-number density of dust particle per unit volume (constant)
K-stock's coefficient of resistance ($6\pi a\mu$) for spherical dust particles and 'a' is the average radius of dust particles
m-average mass of dust particles.
In the present situation the dust particles are assumed to be spherical and uniform in size

and shape and are uniformly distributed throughout the fluid. The bulk concentration of the dust is small $\frac{m}{K} = \tau$ may be called the relaxation time, the dust particle $\frac{mN}{\rho} = f$-mass concentration of dust particle. The last term in equation (2) represents the force due to the relative motion between fluid and dust particles.
Let the velocity of the fluid be every where particle to dust particle velocity so that

$$\vec{v} = \frac{\alpha}{N}\vec{u} \tag{5}$$

where α is to be determined. Using $\vec{v}(v_1, v_2)$ and $\vec{u}(u_1, u_2)$ equation (5) reduces to

$$v_1 = \frac{\alpha}{N}u_1 \qquad and \qquad v_2 = \frac{\alpha}{N}u_2. \tag{6}$$

Equations (1)-(4) using (6) reduces to the following system of six equations.

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0 \tag{7}$$

$$\rho(u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_2}{\partial x}) + \frac{\partial p}{\partial x} = \mu(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}) + \rho u_2(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}) + KN(v_1 - u_1) \tag{8}$$

$$\rho(u_1\frac{\partial u_1}{\partial y} + u_2\frac{\partial u_2}{\partial y}) + \frac{\partial p}{\partial y} = \mu(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2}) + \rho u_1(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}) + KN(v_2 - u_2) \tag{9}$$

$$\frac{\partial(Nv_1)}{\partial x} + \frac{\partial(Nv_2)}{\partial y} = 0 \tag{10}$$

$$(v_1\frac{\partial v_1}{\partial x} + v_2\frac{\partial v_2}{\partial x}) - v_2(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}) = \frac{K}{m}(u_1 - v_1) \tag{11}$$

$$(v_1\frac{\partial v_1}{\partial y} + v_2\frac{\partial v_2}{\partial y}) + v_1(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}) = \frac{K}{m}(u_2 - v_2). \tag{12}$$

By assuming number density of the dust particle is constant and introducing vorticity function $\xi(x, y)$ and energy function $h(x, y)$ as

$$\xi(x, y) = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \tag{13}$$

$$h(x, y) = p + \frac{1}{2}\rho u^2 \qquad where \qquad u^2 = u_1^2 + u_2^2 \tag{14}$$

the system of equations (7)-(12) reduces to the system,

$$\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}=0 \tag{15}$$

$$\mu\frac{\partial \xi}{\partial y}-\rho u_2\xi+KN(u_1-v_1)=-\frac{\partial h}{\partial x} \tag{16}$$

$$\mu\frac{\partial \xi}{\partial x}-\rho u_1\xi+KN(v_2-u_2)=\frac{\partial h}{\partial y} \tag{17}$$

$$u_1\frac{\partial \alpha}{\partial x}+u_2\frac{\partial \alpha}{\partial y}=0 \tag{18}$$

$$m\alpha^2\left(u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_1}{\partial y}\right)=KN(N-\alpha)u_1 \tag{19}$$

$$m\alpha^2\left(u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_2}{\partial y}\right)=KN(N-\alpha)u_2 \tag{20}$$

Equations (6), (13), (14) and (15) - (20) is a system of first order nine partial differential equation in eight unknown functions $u_1(x,y)$, $u_2(x,y)$, $v_1(x,y)$, $v_2(x,y)$, $\xi(x,y)$, $h(x,y)$, $p(x,y)$ and $\alpha(x,y)$. where $\alpha(x,y)$ is calculated from (18) and v_1 and v_2 from (6). Reducing the order of differential equation from two to one is successfully done in (11). Equation (13) gives vorticity and $h(x,y)$ is form (16) and (17). Lastly the pressure function $p(x,y)$ is obtained, using equation (14). $\tau=\frac{m}{K}$ may be discussed using equations (19) and (20). Using integrability condition on $h(x,y)$ from (16) and (17) we have

$$\mu\left[\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}\right]-\rho\left[u_1\frac{\partial \xi}{\partial x}+u_2\frac{\partial \xi}{\partial y}\right]-K(N-\alpha)\left[\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y}\right]$$
$$+K\left\{u_1\left[N-\frac{\partial \alpha}{\partial y}\right]-u_2\left[N-\frac{\partial \alpha}{\partial x}\right]\right\}=0 \tag{21}$$

Now equations (6), (13), (14), (15), (18), (19), (20) and (21) is system of eight partial differential equations in eight unknowns u_1, u_2, v_1, v_2, ξ, h, p and α.

Flows with constant velocity magnitude;

Now the fluid is flowing with constant velocity magnitude along each individual streamline. We must have

$$\vec{u}.grad\ u^2=0$$

so that

$$u_1^2\frac{\partial u_1}{\partial x}+u_1u_2\left(\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial x}\right)+u_2^2\frac{\partial u_2}{\partial y}=0 \tag{22}$$

Hence in the flow with constant velocity magnitude u_1 and u_2 must satisfy the equation (15) and (22). Once a solution of u_1 and u_2 are determined from (15) and (22), the pressure function is found from the definition of energy function. Now consider the following case

1. $J\neq 0$ in the entire region of flow
2. $J=0$ in the entire region of flow

3. $J \neq 0$ in a part of the region and $J = 0$ in the remaining part of the region where J is the Jacobian.

I when $J \neq 0$ in the entire region of flow

Hodograph transformations

Letting the function $u_1 = u_1(x, y)$ and $u_2 = u_2(x, y)$ to be such that in the region of flow the Jacobian

$$J = \frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x} \neq 0, \qquad 0 < |J| < \infty \tag{23}$$

We may consider x and y as functions of u_1 and u_2 by means of $x = x(u_1, u_2)$ and $y = y(u_1, u_2)$, we have the relations

$$\begin{aligned} \frac{\partial u_1}{\partial x} &= J\frac{\partial y}{\partial u_2}, \qquad \frac{\partial u_1}{\partial y} = -J\frac{\partial x}{\partial u_2}, \\ \frac{\partial u_2}{\partial x} &= -J\frac{\partial y}{\partial u_1}, \qquad \frac{\partial u_2}{\partial y} = J\frac{\partial x}{\partial u_1}. \end{aligned} \tag{24}$$

With the application of transformation (23) and (24) for the first order partial derivatives appearing in the above system we have the partial differential equations in the (u, v) plane.

$$\frac{\partial x}{\partial u_1} + \frac{\partial y}{\partial u_2} = 0 \tag{25}$$

$$\begin{aligned} &\mu\left[J\frac{\partial(w_1, y)}{\partial(u_1, u_2)} + J\frac{\partial(x, w_2)}{\partial(u_1, u_2)}\right] - \rho\left[u_1 J\frac{\partial(\xi, y)}{\partial(u_1, u_2)} + u_2 J\frac{\partial(x, \xi)}{\partial(u_1, u_2)}\right] \\ &-\frac{K}{\rho}(N - \alpha)\left(\frac{\partial^2 L}{\partial u_1^2} + \frac{\partial^2 L}{\partial u_2^2}\right) + u_1\left[N - J\frac{\partial(x, \alpha)}{\partial(u_1, u_2)}\right] \\ &-u_2\left[N - J\frac{\partial(\alpha, y)}{\partial(u_1, u_2)}\right] = 0, \end{aligned} \tag{26}$$

$$J\left[u_1\frac{\partial(\alpha, y)}{\partial(u_1, u_2)} + u_2\frac{\partial(x, \alpha)}{\partial(u_1, u_2)}\right] = 0 \tag{27}$$

$$Jm\alpha^2\left[u_1\frac{\partial y}{\partial u_2} - u_2\frac{\partial x}{\partial u_2}\right] = KN(N - \alpha)u_1 \tag{28}$$

$$Jm\alpha^2\left[u_2\frac{\partial x}{\partial u_1} - u_1\frac{\partial y}{\partial u_1}\right] = KN(N - \alpha)u_2 \tag{29}$$

$$\xi = J\left[\frac{\partial x}{\partial u_2} - \frac{\partial y}{\partial u_1}\right] \tag{30}$$

$$J = \left(\frac{\partial y}{\partial u_2}\frac{\partial x}{\partial u_1} - \frac{\partial x}{\partial u_2}\frac{\partial y}{\partial u_1}\right)^{-1} = j \tag{31}$$

$$u_1^2\frac{\partial y}{\partial u_2} - u_1 u_2\left(\frac{\partial x}{\partial u_2} - \frac{\partial y}{\partial u_1}\right) + u_2^2\frac{\partial x}{\partial u_1} = 0 \tag{32}$$

Equations in Legendre transform function

The equation of continuity (15) implies the existence of a stream function $\Psi(x, y)$ such that

$$d\Psi = -u_2 dx + u_1 dy \qquad or \qquad \frac{\partial \Psi}{\partial x} = -u_2, \;\; \frac{\partial \Psi}{\partial y} = u_1. \tag{33}$$

Likewise equation (25) implies the existence of a function $L(x, y)$ called the Legendre transform function of the stream function $\Psi(x, y)$ such that

$$dL = -y du_1 + x du_2 \qquad or \qquad \frac{\partial L}{\partial u_1} = -y, \quad \frac{\partial L}{\partial u_2} = x. \tag{34}$$

Introducing $L(u_1, u_2)$ into the system (25)-(32), it follows that (25) is identically satisfied and the system may be replaced by

$$\xi = j\left[\frac{\partial^2 L}{\partial u_1^2} + \frac{\partial^2 L}{\partial u_2^2}\right] \tag{35}$$

$$J^{-1} = \left[\frac{\partial^2 L}{\partial u_1^2}\frac{\partial^2 L}{\partial u_2^2} - \left(\frac{\partial^2 L}{\partial u_1 \partial u_2}\right)^2\right] = j \tag{36}$$

$$\begin{aligned} &\mu\left(\frac{\partial(w_1, -\frac{\partial L}{\partial u_1})}{\partial(u_1.u_2)} + \frac{\partial(\frac{\partial L}{\partial u_2}, w_2)}{\partial(u_1, u_2)}\right) - \left(u_1\frac{\partial(\xi, -\frac{\partial L}{\partial u_1})}{\partial(u_1, u_2)} + u_2\frac{\partial(\frac{\partial L}{\partial u_2}, \xi)}{\partial(u_1, u_2)}\right) \\ &-\frac{K}{\rho}(N-\alpha)\left(\frac{\partial^2 L}{\partial u_1^2} + \frac{\partial^2 L}{\partial u_2^2}\right) + K\left\{u_1\left(\frac{N}{J} - \frac{\partial(\frac{\partial L}{\partial u_2}, \alpha)}{\partial(u_1, u_2)}\right)\right. \\ &\left.-u_2\left(\frac{N}{J} - \frac{\partial(\alpha, -\frac{\partial L}{\partial u_1})}{\partial(u_1, u_2)}\right)\right\} = 0 \end{aligned} \tag{37}$$

$$J\left(u_1\frac{\partial(\alpha, -\frac{\partial L}{\partial u_1})}{\partial(u_1, u_2)} + u_2\frac{\partial(\frac{\partial L}{\partial u_2}, \alpha)}{\partial(u_1, u_2)}\right) = 0 \tag{38}$$

$$-Jm\alpha^2\left(u_1\frac{\partial^2 L}{\partial u_2 \partial u_1} + u_2\frac{\partial^2 L}{\partial u_2^2}\right) = KN(N-\alpha)u_1 \tag{39}$$

$$Jm\alpha^2\left(u_1\frac{\partial^2 L}{\partial u_1^2} + u_2\frac{\partial^2 L}{\partial u_1 \partial u_2}\right) = KN(N-\alpha)u_2 \tag{40}$$

$$u_1 u_2\frac{\partial^2 L}{\partial u_1^2} + (u_2^2 - u_1^2)\frac{\partial^2 L}{\partial u_1 \partial u_2} - u_1 u_2\frac{\partial^2 L}{\partial u_2^2} = 0 \tag{41}$$

Polar coordinates:
By employing Polar coordinates (q, θ) in the hodograph plane, defined by the relations

$$u_1 = q\cos\theta, \quad u_2 = q\sin\theta, \quad q = \sqrt{u_1^2 + u_2^2}, \quad \theta = \tan^{-1}(\frac{u_2}{u_1}) \tag{42}$$

and defining $L^*(q, \theta)$, $\xi^*(q, \theta)$, $J^*(q, \theta)$ to be the Legendre transform, vorticity and jacobian functions in (q, θ) coordinates. Using

$$\frac{\partial(F, G)}{\partial(u_1, u_2)} = \frac{\partial(F^*, G^*)}{\partial(q, \theta)} \qquad and \qquad \frac{\partial(q, \theta)}{\partial(u_1, u_2)} = \frac{1}{q}\frac{\partial(F^*, G^*)}{\partial(q, \theta)} \tag{43}$$

where $F(u_1,u_2)=F^*(q,\theta),\qquad G(u_1,u_2)=G^*(q,\theta)$ are continuously differentiable functions, the equations (37) becomes

$$\begin{aligned}&\mu\left(\frac{\partial(w_1,-\cos\theta\frac{\partial L}{\partial q}+\frac{\sin\theta}{q}\frac{\partial L}{\partial\theta})}{\partial(u_1,u_2)}+\frac{\partial(\sin\theta\frac{\partial L}{\partial q}+\frac{\cos\theta}{q}\frac{\partial L}{\partial\theta},w_2)}{\partial(u_1,u_2)}\right)\\&-\left(u_1\frac{\partial(\xi,-\cos\theta\frac{\partial L}{\partial q}+\frac{\sin\theta}{q}\frac{\partial L}{\partial\theta})}{\partial(u_1,u_2)}+u_2\frac{\partial(\sin\theta\frac{\partial L}{\partial q}+\frac{\cos\theta}{q}\frac{\partial L}{\partial\theta},\xi)}{\partial(u_1,u_2)}\right)\\&-\frac{qK}{\rho}(N-\alpha)\left(\frac{\partial^2L}{\partial q^2}+\frac{1}{q}\frac{\partial L}{\partial q}+\frac{1}{q^2}\frac{\partial^2L}{\partial\theta^2}\right)+K\left[u_1\left(\frac{N}{J}-\frac{\partial(\sin\theta\frac{\partial L}{\partial q}+\frac{\cos\theta}{q}\frac{\partial L}{\partial\theta},\alpha)}{\partial(u_1,u_2)}\right)\right.\\&\left.-u_2\left(\frac{N}{J}-\frac{\partial(\alpha,-\cos\theta\frac{\partial L}{\partial q}+\frac{\sin\theta}{q}\frac{\partial L}{\partial\theta})}{\partial(u_1,u_2)}\right)\right]=0\end{aligned}\tag{44}$$

Since the equation (44) holds identically for all values of q, equating the coefficients of different powers of q, we have

$$\begin{aligned}&\mu\left(\frac{\partial(w_1,-\cos\theta\frac{\partial L}{\partial q}+\frac{\sin\theta}{q}\frac{\partial L}{\partial\theta})}{\partial(u_1,u_2)}+\frac{\partial(\sin\theta\frac{\partial L}{\partial q}+\frac{\cos\theta}{q}\frac{\partial L}{\partial\theta},w_2)}{\partial(u_1,u_2)}\right)\\&-\left(u_1\frac{\partial(\xi,-\cos\theta\frac{\partial L}{\partial q}+\frac{\sin\theta}{q}\frac{\partial L}{\partial\theta})}{\partial(u_1,u_2)}+u_2\frac{\partial(\sin\theta\frac{\partial L}{\partial q}+\frac{\cos\theta}{q}\frac{\partial L}{\partial\theta},\xi)}{\partial(u_1,u_2)}\right)\\&+k\left[u_1\left(\frac{N}{J}-\frac{\partial(\sin\theta\frac{\partial L}{\partial q}+\frac{\cos\theta}{q}\frac{\partial L}{\partial\theta},\alpha)}{\partial(u_1,u_2)}\right)\right.\\&\left.-u_2\left(\frac{N}{J}-\frac{\partial(\alpha,-\cos\theta\frac{\partial L}{\partial q}+\frac{\sin\theta}{q}\frac{\partial L}{\partial\theta})}{\partial(u_1,u_2)}\right)\right]=0\end{aligned}\tag{45}$$

$$k(N-\alpha)\left(\frac{\partial^2L}{\partial q^2}+\frac{1}{q}\frac{\partial L}{\partial q}+\frac{1}{q^2}\frac{\partial^2L}{\partial\theta^2}\right)=0\tag{46}$$

and the equation (41) in (q,θ) form is

$$q\frac{\partial^2L^*}{\partial q\partial\theta}-\frac{\partial L^*}{\partial\theta}=0\tag{47}$$

Now we use (44) and (47) to get the expression for L. Once $L(q,\theta)$ is defined we can express it in (u,v) from with the help of (42), J is evaluated from (36) satisfies $0<|J|<\infty$. The solutions for the velocity components u_1 and u_2 are obtained by solving equations $x=\frac{\partial L}{\partial u_2},\qquad y=-\frac{\partial L}{\partial u_1}$. After obtaining velocity components $\alpha(x,y)$ can be evaluated by solving (18), then the velocity component of dust are from equation (6). Having obtained the velocity components in the physical plane vorticity and energy functions are determined from the vorticity and linear momentum equations in the system of equations (13), (16) and (17). Finally the pressure function is evaluated from (14).

Solutions to flow variables

Assuming the most general solution of (47) in the form

$$L^*(q,\theta)=q\phi(\theta)+\chi(q)\tag{48}$$

where ϕ and χ are arbitrary functions of their arguments. Now equation (48) in (46) gives us

$$q\chi'' + \chi' + (\phi'' + \phi) = 0. \tag{49}$$

Set

$$q\chi'' + \chi' = \lambda \qquad \textit{and} \qquad \phi'' + \phi = -\lambda \tag{50}$$

where λ is constant and primes denote differentiation with respect to the arguments. Now from (50) we have

$$\chi = \lambda q - \lambda_1 \ln q + \lambda_2 \qquad \textit{and} \qquad \phi = A\cos\theta + B\sin\theta - \lambda \tag{51}$$

where $A, B, \lambda_1, \lambda_2$ are arbitrary constants. Equations (51) in (48) gives us

$$L^*(q,\theta) = Aq\cos\theta + Bq\sin\theta - \lambda_1 \ln q + \lambda_2.$$

The above equation in u_1 and u_2 plane using (42) is

$$L(u_1, u_2) = Au_1 + Bu_2 - \frac{\lambda_1}{2}\ln(u_1^2 + u_2^2) + \lambda_2. \tag{52}$$

From (34) and (52) we have

$$x = B - \frac{u_2\lambda_1}{u_1^2 + u_2^2} \qquad \textit{and} \qquad y = -A + \frac{u_1\lambda_1}{u_1^2 + u_2^2}$$

and hence

$$u_1 = \frac{\lambda_3(y + A)}{\lambda_1} \qquad \textit{and} \qquad u_2 = \frac{\lambda_3(B - x)}{\lambda_1} \tag{53}$$

provided

$$u_1^2 + u_2^2 = \lambda_1^{-2}\left[(y + A)^2 + (B - x)^2\right] = \lambda_3(\textit{constant})$$

Using (53) in (13) and (23) the vorticity and jacobian are

$$\xi = \frac{-2\lambda_3}{\lambda_1} \qquad \textit{and} \qquad J = \frac{2\lambda_3}{\lambda_1} \tag{54}$$

This shows that the flow is irrotational. From (18) and (53) we have

$$(y + A)\frac{\partial\alpha}{\partial x} + (B - x)\frac{\partial\alpha}{\partial y} = 0 \tag{55}$$

the general solution of this equation is

$$\alpha(x, y) = a_1 x + \frac{(y + A)}{B - x} a_1 xy + a_2 \tag{56}$$

where a_1 and a_2 are constants. From (53), (56) and (6), velocity components of dust particles are given by

$$v_1 = \frac{\lambda_3(y+A)}{N\lambda_1}\left[(a_1x+a_2)+a_1xy(y+a)(B-x)^{-1}\right]$$
$$v_2 = \frac{\lambda_3}{N\lambda_1}(a_1x+a_2)(B-x)+\frac{\lambda_3 a_1 xy(y+A)}{N\lambda_1} \tag{57}$$

The integrability condition on h from (16), (17) and using (6), the pressure function $p(x,y)$ is given by

$$p(x,y) = \rho\frac{\lambda_3^2}{\lambda_1^2}\left(y^2+2Ay-Bx+\frac{x^2}{2}\right)-K\frac{\lambda_3}{\lambda_1}[(y+A)(N-C)x$$
$$-(y+A)\alpha\frac{x^3}{3}-\alpha x^2-(C-N)-(N-C)x+\alpha\frac{x^3}{3}+(3x$$
$$+2B\log(x-B))\left(a\frac{y^3}{3}+aA\frac{y^2}{2}\right)+(x+B\log(x-B))\Bigg]$$
$$+K\frac{(y+A)^2}{\lambda_1}\lambda_3\alpha y\left[x+B\log(x-B)\right]$$
$$-\frac{1}{2}\rho\frac{\lambda_3^2}{\lambda_1^2}\left[(y+A)^2+(B-x)^2\right] \tag{58}$$

The streamlines are given by $(x-B)^2+(y+A)^2=$ constant. Hence streamlines are concentric circles. The variation of pressure for different densities is shown in figure (1) and (2).

II When the Jacobian is zero

Consider $J=\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y}-\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}=0$ in the entire region of flow. In this case u_1 is a function of u_2 or u_2 is a function of u_1. Consider the case when u_2 is a function of u_1
Let

$$u_2 = f(u_1) \tag{59}$$

where 'f' is an arbitrary of u_1. Using (59) in (15) and (22) we get

$$\frac{\partial u_1}{\partial x}+f'(u_1)\frac{\partial u_1}{\partial y}=0 \tag{60}$$
$$(u_1^2+u_1ff')\frac{\partial u_1}{\partial x}+(u_1f+f^2f')\frac{\partial u_1}{\partial y}=0 \tag{61}$$

Eliminating $\frac{\partial u_1}{\partial x}$ from (60) and (61), we have

$$(u_1f'-f)(u_1+ff')\frac{\partial u_1}{\partial y}=0 \tag{62}$$

implies that either

$$(i)\ \frac{\partial u_1}{\partial y}=0 \quad or \qquad (ii)\ (u_1f'-f) \quad or \qquad (iii)\ (u_1+ff')=0$$

(i) If $\frac{\partial u_1}{\partial y} = 0$ gives us $\frac{\partial u_1}{\partial x} = 0$. Therefore $u_1 =$ constant, $u_2 =$ constant

$$Let \quad u_1 = c_1 \qquad and \qquad u_2 = c_2 \tag{63}$$

From (18)

$$\alpha = b_1 \left[x - \frac{c_1}{c_2} y \right] + b_2 \tag{64}$$

and therefore

$$\xi = 0 \qquad and \qquad j = 0$$

$$\begin{aligned} v_1 &= \frac{b_1 c_1 (c_2 x - c_1 y) + c_2 b_2}{N c_2}, \qquad v_2 = \frac{b_1 (c_2 x - c_1 y) + b_2}{N} \\ h(x,y) &= akc_2 xy + K(c_3 - N)(c_1 x + c_2 y) + \frac{Kc_1}{2} a(x^2 - y^2) - c_5 \end{aligned} \tag{65}$$

and hence pressure function is

$$\begin{aligned} p(x,y) = &\; aKc_2 xy + K(c_3 - N)(c_1 x + c_2 y) + \frac{Kc_1}{2} a(x^2 - y^2) \\ &- c_5 - \frac{\rho}{2}(c_1^2 + c_2^2). \end{aligned} \tag{66}$$

The variation of p is graphed in figure (3) and (4). In this case streamlines are straight lines given by

$$c_2 x - c_1 y = constant$$

. **(ii)** Next $u_1 f' - f = 0$ this gives $f = d_1 u_1$. Put $f' = d_1$ in (63), we have

$$\frac{\partial u_1}{\partial x} + d_1 \frac{\partial u_1}{\partial y} = 0 \tag{67}$$

The general solution of this equation is

$$u_1 = g(d_1 x - y) \tag{68}$$

where g is an arbitrary. By taking one particular value $u_1 = d_1x - y$ we have the following exact solutions to flow variables,

$$
\begin{aligned}
&u_1 = d_1x - y, \qquad u_2 = d_1(d_1x - y),\\
&\xi = d_1^2 + 1, \quad j = 0, \qquad \alpha(x,y) = d_2\left[x - \frac{y}{d_1}\right] + d_3 \qquad (69)\\
&v_1 = \frac{d_2(d_1x-y)^2}{d_1N} + \frac{d_3}{N}(d_1x - y),\\
&v_2 = \frac{d_2(d_1x-y)^2}{d_1N} + \frac{d_3d_1}{N}d_1(d_1x - y)\\
&h(x,y) = -\rho u_2(d_1^2+1)x - K\left(d_1\frac{x^2}{2} - yx\right)N + K\left(d_1\frac{x^2}{2} - yx\right)\\
&\left\{d_2\left(x - \frac{y}{d_1}\right) + d_3\right\} + K(d_1x-y)\left(\frac{x^2}{2} - \frac{xy}{d_1}\right) - \Big(d_3x - \rho d_1^2 +\\
&\rho - \frac{Kx}{d_1} - ax - d_3 + N\Big)\left(d_1xy - \frac{y^2}{2}\right) - \left(d_1\frac{xy^2}{2} - \frac{y^3}{3}\right)d_2\\
&-Kx\left(N - \frac{3d_2x}{2} - 2d_3 - \frac{x}{2}\right)y - \frac{Kx}{d_1}(d_2+1)\frac{y^2}{2}
\end{aligned}
$$

The pressure expression is

$$
\begin{aligned}
&p(x,y) = -\rho u_2(d_1^2+1)x - K\left(d_1\frac{x^2}{2} - yx\right)N + K\left(d_1\frac{x^2}{2} - yx\right)\\
&\left\{d_2\left(x - \frac{y}{d_1}\right) + d_3\right\} + K(d_1x-y)\left(\frac{x^2}{2} - \frac{xy}{d_1}\right) - \Big(d_3x - \rho d_1^2 +\\
&\rho - \frac{Kx}{d_1} - ax - d_3 + N\Big)\left(d_1xy - \frac{y^2}{2}\right) - \left(d_1\frac{xy^2}{2} - \frac{y^3}{3}\right)d_2\\
&-Kx\left(N - \frac{3d_2x}{2} - 2d_3 - \frac{x}{2}\right)y - \frac{kx}{d_1}(d_2+1)\frac{y^2}{2}\\
&-\frac{1}{2}\left\{(d_1x-y)^2 + d_1^2(d_1x-y)^2\right\} \qquad (70)
\end{aligned}
$$

In this case also, the streamlines are straight lines are given by $y - d_1x =$ constant. The variation of pressure is graphically shown in figure (5) and (6). **(iii)** In the last case $u_1 + ff' = 0$ we have $u_1 = d_1x - y$, $u_2 = d_1(d_1x - y)$, $\xi = 0$, $j = 0$ are the solutions and the geometry of streamlines is similar to **(ii)**.

III $J \neq 0$ a part of the region and $J = 0$ in the remaining part of the region

From I and II, we see that the streamlines are concentric circles, when $J \neq 0$ and they are parallel straight lines when $J = 0$. Therefore there exists no common streamline pattern. Also if $J \neq 0$ the flow is irrotational and if $J = 0$ the flow is rotational as shown in case (i) and in (ii) it is rotational if $d_1^2 = -1$, otherwise it is irrotational. The velocity vector of fluid when $J \neq 0$ is $u_1 = \frac{\lambda_3(y+A)}{\lambda_1}$ and $u_2 = \frac{\lambda_3(B-x)}{\lambda_1}$ and when $J = 0$ is $u_1 = C_1$, $u_2 = C_2$ which are constants. We see that there is a discontinuity in the velocity field as we cross from one region to another. Therefore such flows cannot exists.

3. Conclusion

B.J.Gireesha et al., [10] discussed the flow of an unsteady dusty fluid under varying pressure gradient using differential geometry technique. But in our work it is used Hodograph method and it is taken that the magnitude of velocity of fluid is constant along each individual streamline. In [10] analytical solutions to velocity components of both fluid and dust phase are obtained using laplace transform technique. But in this article it is used the technique of Legendre transformation. In their paper solutions are obtained in terms of binormal vector but ours is the physical plane. In this article we also found solutions to pressure and vorticity function. We also discussed variation of velocity components, further it is analyzed the pressure variation graphically. When Jacobian is non zero, the pressure variation is shown in figure (1) and (2).It is observed that the density increases with increase of pressure. Such situations may be seen in oceans etc., When the Jacobian is zero, we have three possibilities. The variation of pressure in the first possibilities is parabolic when y=constant and it is inverted when x=constant, as in figure (3) and (4). Here as density increases pressure decreases as we observe in air and some gases. The second possibility is graphed in figure (5) and (6). Here also pressure increases with variation of density. In the third possibility the solutions and pressure variation is similar to second possibility. When the Jacobian is non zero the streamlines are concentric circles and when it is zero the streamlines are straight lines. From equations (57), (65) and (69) it is observed that as the number density of the dust particle increases the velocity of the dust phase decreases. When Jacobian is non zero the flow is irrotational. When it is zero the flow is rotational. We have such situations arised in free air, oceans etc.,

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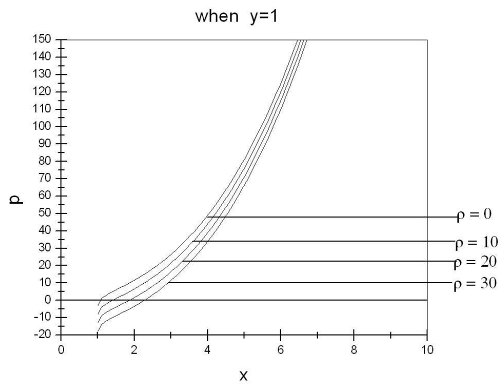


Fig. 1

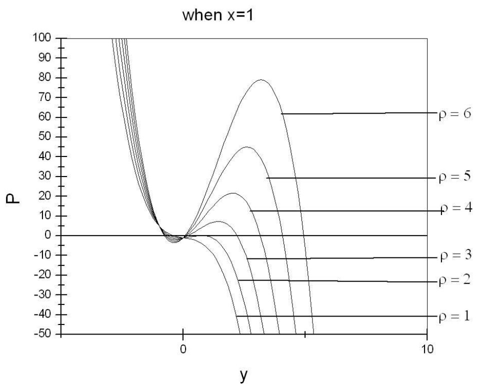


Fig. 2

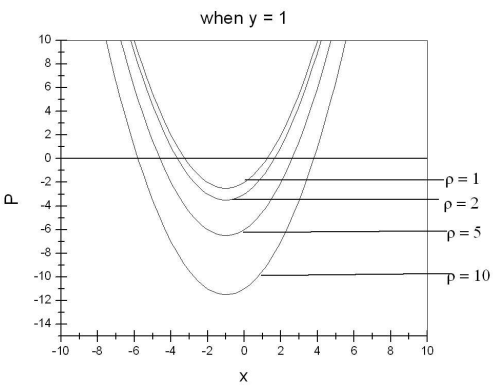
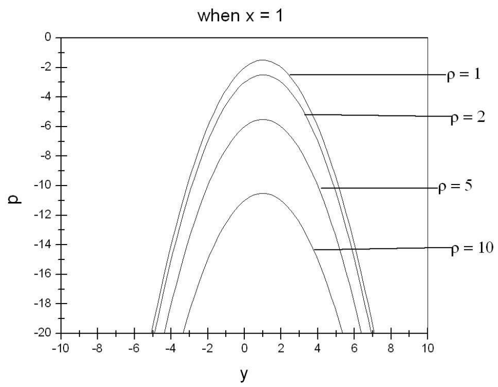


Fig. 3



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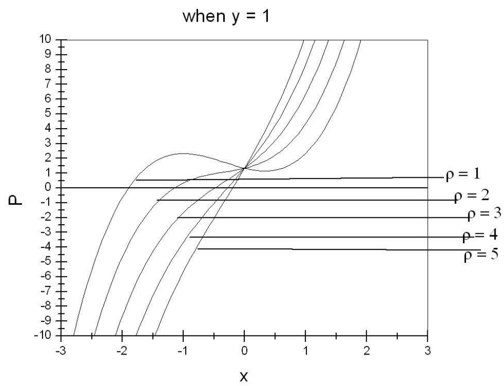

Fig. 5

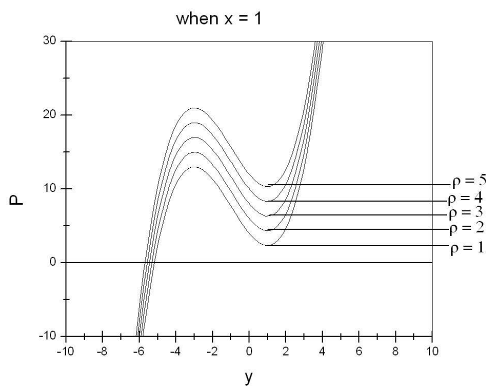

Fig. 6