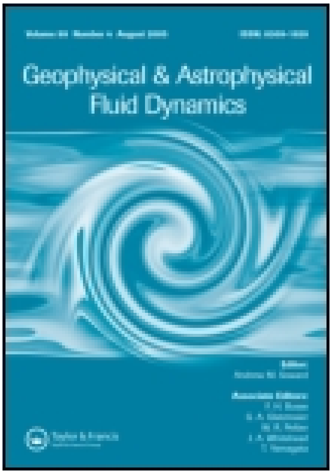

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THE STABILITY OF STRATIFIED CONDUCTING SHEAR FLOW IN AN ALIGNED MAGNETIC FIELD

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The stability of a horizontally stratified, electrically conducting fluid permeated by a uniform magnetic field aligned with the motion is investigated. The resulting linear stability problem for the special case of constant density gradient and linear shear in an unbounded fluid is reduced to the study of a third order differential equation in time. In the absence of dissipation, the linear shear eventually causes hybrid Alfvén-gravity waves to decay algebraically. The effect of the shear is to shorten the vertical length scale. So with the addition of even small diffusivity, dissipation is strongly stabilising and all modes eventually collapse exponentially, generally at a fast rate. The evolution from wave motion to exponential decay is examined for particular limiting cases. When the fluid is bounded by horizontal planes a nonlinear stability criterion is derived using the energy method.

KEY WORDS: Linear shear flow, Alfvén waves, gravity waves.

1. INTRODUCTION

The stability of a horizontally stratified, electrically conducting, shear flow in the presence of an externally applied magnetic field is of interest both in geophysics and astrophysics. In general, other competing effects are present, which may be of greater or lesser importance: most notable in the case of the Earth's fluid core is the Coriolis force. Nevertheless, a detailed understanding of the physical processes involved naturally emerges from a systematic study of a hierarchy of models of varying complexity. Here we adopt such an approach and build on the earlier investigations of Knobloch (1984), Lerner and Knobloch (1985) (subsequently referred to as K84, LK85 respectively). Shear flow is considered in both those papers; the former includes horizontal stratification alone, whereas the latter restricts attention to fluid permeated by a uniform horizontal magnetic field. In this paper, however, we address the more complicated problem, which occurs when both the stratification and magnetic field are present simultaneously. As suggested above the natural application of this study is to regions in the Earth's core, where there is significant differential rotation and a strong azimuthal magnetic field aligned with the flow. For sufficiently strong magnetic fields it is legitimate to neglect the Coriolis force in a first approximation. The more complicated problem, which arises when the Coriolis force plays an active role, will not be discussed here.

Typically, when instabilities are localised, they are of two types. One is a wall mode localised close to the boundary of the system, upon which the mode's existence is linked. The other is an internal mode localised far from any boundary. Its existence is independent of the boundaries and its nature is not influenced by boundary conditions in any essential way. Here, following K84, LK85, we restrict attention to the latter case and consider disturbances localised on a length scale L in regions, where the shear, magnetic field and density are smooth and vary on length scales large compared to L. Within this framework we undertake a Boussinesq analysis with the shear σ (see (2.3a)), the magnetic field B_0 (see (2.3b)), and the temperature gradient $-\beta$ (see (2.3c)) all assumed to be constant. Our particular system consists of plane Couette flow aligned with a uniform horizontal magnetic field and horizontal stratification, which are characterised by constant values the Alfvén velocity A (see (2.7a)) and Brunt-Väisälä frequency N (see (2.7b)). In this paper we study the stability of this system in the presence of viscous (ν), Ohmic (η), and thermal (κ) dissipations (see (2.1) and below).

The stability of electrically conducting and nonconducting parallel shear flows with and without density stratification has been studied by a number of researchers (Miles, 1961; Eliassen *et al.*, 1953; Stuart, 1954; Hains, 1965; Hunt, 1966; Koppel, 1964). Since the linearised equations have coefficients, which are independent of time t, conventional stability theory considers modes with a time dependence proportional to $\exp(\lambda t)$, where the complex growth rate λ is a constant. The problem for each mode is reduced to solving partial differential equations involving the space coordinates alone, in which λ is an eigenvalue. The system is unstable if there are growing modes characterised by $\mathscr{R}e\,\lambda>0$.

For our problem with linear shear velocity σz (see (2.3a)) we follow K84, LK85 and adopt an alternative approach with its origins in the early work of Lord Kelvin (1887). It relies on the fact that a single Fourier harmonic $\exp[i\mathbf{K}\cdot\mathbf{x}]$ (position $\mathbf{x}$) can remain as such and move with the fluid when the wave vector $\mathbf{K}$ has the time dependent form

$$\mathbf{K}(t)=(k,l,m-\sigma kt), \tag{1.1}$$

where k,l,m are constants, because then

$$\left(\frac{\partial}{\partial t}+\sigma z\,\frac{\partial}{\partial x}\right)(\mathbf{K}\cdot\mathbf{x})=0. \tag{1.2}$$

We also note that, as usual with Fourier methods, we have

$$\nabla(e^{i\mathbf{K}\cdot\mathbf{x}})=i\mathbf{K}\,e^{i\mathbf{K}\cdot\mathbf{x}}. \tag{1.3}$$

Consequently the evolution of an arbitrary initial disturbance can be investigated by first taking its Fourier transform (see (2.8b)) and then considering the evolution of each Fourier amplitude $\hat{\theta}(k,l,m,t)$, which composes the complete solution (2.8a) below. By this device we obtain a set of three first order ordinary differential equations (see (2.6)), which govern the amplitude evolution. This is a standard technique, which has been applied to a wide variety of viscous flow and MHD problems (see, for example, Craik and Criminale, 1986; Craik, 1988; Philips, 1966; Zeldovich *et al.*, 1984). The only complication is the fact that some of the coefficients in the resulting differential equations are not constants because $\mathbf{K}$ is a linear function of time t (see particularly $p(t)=|\mathbf{K}(t)|^2$ defined by (2.7c) below). For problems of this type with finite diffusivities the secular increase in the magnitude $|\mathbf{K}(t)|$ of the wave vector as $t\to\infty$ is strongly stabilising, because of the enhanced dissipation. It sometimes has dramatic consequences. For example, in the case of the thermal instability in a rapidly rotating self-gravitating fluid sphere, Soward (1977 (4.16)) showed that the localised modes isolated earlier by Roberts (1968) and Busse (1970) evolve according to the ansatz (1.1) and necessarily decay irrespective of the magnitude of the Rayleigh number, which measures the magnitude of the adverse density gradient $N^2<0$. Though the origin of that instability does not stem from any shear, the essential mechanism, that of shortening a length scale, is the same in both cases. The result, therefore, is quite consistent with the conclusion of K84 that, in the case of stratified shear flow, all modes are stabilised as $t\to\infty$ even when $N^2<0$.

This paper is organised to closely parallel the earlier developments of the special cases of no magnetic field in K84 and no stratification in LK85. In Section 2 the mathematical problem is formulated. We show in Sections 2.1 and 2.2 how the initial value problem can be resolved by considering the temporal evolution of modes whose spatial structure is proportional to $\exp[i\mathbf{K}(t)\cdot\mathbf{x}]$. K84 and LK85 found that most of the important solutions of their second order system of differential equations could be obtained by use of the WKBJ-approximation. Accordingly, since our more complicated third order system has very few simple solutions, we outline the WKBJ-approximation as it applies here in Section 2.3 and employ the technique throughout Section 3. The perfect fluid case with no dissipation is considered in Section 3.1. The temporal development of hybrid Alfvén-gravity waves is considered and all modes even in the presence of unstable stratification are shown to decay algebraically as $t\to\infty$. With the inclusion of weak diffusion in Section 3.2 their evolution from lightly damped waves to their final period of rapid exponential decay is carefully followed. Special cases, which occur when some or all of the diffusivities are equal, are discussed in the Appendix.

From the point of view of stability, one of the most significant results to emerge is that the final period of decay is dominated by the mode linked to the smallest diffusivity. It is either viscous (when $\nu<\eta$ and κ), or magnetic (when $\eta<\nu$ and κ), or thermal (when $\kappa<\nu$ and η). If viscous, the mode coincides with the dominant viscous modes isolated in both K84 and LK85. If magnetic (thermal), the mode coincides with the dominant magnetic (thermal) mode isolated in LK85 (K84). This means that for an arbitrary initial disturbance the final period of decay is given correct to leading order by the results of one or both of K84 and LK85. Consequently their bounds on the decay rates are directly applicable to our problem and show that the response to any initial disturbance ultimately decays, even when the fluid is unstably stratified ($N^2<0$). In Section 4, we use the energy method of Joseph (1976) to establish a sufficient condition for nonlinear stability of a particular bounded system. In Section 5, we summarise the main results concerning the final period of algebraic decay for the case of vanishing diffusivities and of exponential decay for the case of finite diffusivities.

2. MATHEMATICAL FORMULATION

2.1 The Governing Equations

We consider an electrically conducting fluid, density ρ, moving with velocity $\mathbf{u}$ in the presence of a magnetic field $\mathbf{B}$, under the influence of gravity $\mathbf{g}$. Small density changes are caused by variations in the temperature T. We assume that the fluid is Boussinesq, for which motion is governed by the equations

$$D\mathbf{u}/Dt=-\rho^{-1}\nabla P+\mathbf{g}+(\rho\mu)^{-1}(\mathbf{B}\cdot\nabla)\mathbf{B}+\nu\nabla^2\mathbf{u}, \qquad \nabla\cdot\mathbf{u}=0, \tag{2.1a, b}$$

$$D\mathbf{B}/Dt=(\mathbf{B}\cdot\nabla)\mathbf{u}+\eta\nabla^2\mathbf{B}, \qquad \nabla\cdot\mathbf{B}=0, \tag{2.1c, d}$$

$$DT/Dt=\kappa\nabla^2 T, \qquad \rho=\rho_e[1-\alpha(T-T_e)], \tag{2.1e, f}$$

where

$$D/Dt=\partial/\partial t+\mathbf{u}\cdot\nabla, \tag{2.2}$$

is the material derivative, P is the total pressure, α is the coefficient of expansion, μ is the magnetic permeability, ν is the kinematic viscosity, η is the magnetic diffusivity and κ is the thermal diffusivity. The constant density ρ_e corresponds to some reference temperature T_e.

We consider an unbounded fluid which, relative to rectangular Cartesian coordinates x, y, z, has gravity directed in the negative z-direction, $\mathbf{g}=(0,0,-g)$. In the basic unperturbed equilibrium state the flow velocity $\mathbf{u}_0$, magnetic field $\mathbf{B}_0$ and temperature T_0 are given by

$$\mathbf{u}_0=(\sigma z,0,0), \qquad \mathbf{B}_0=(B_0,0,0), \qquad T_0=-\beta z, \tag{2.3a, b, c}$$

in which the shear σ, magnetic field B_0 and temperature gradient $-\beta$ are all constants. We study the stability of our system by adding to the basic state (2.3) the perturbations

$$\mathbf{u}'=(u'_x,u'_y,u'), \qquad \mathbf{B}'=B_0(b'_x,b'_y,b'), \qquad T'=-\beta\theta', \tag{2.4a, b, c}$$

and linearising the governing equations (2.1).

Following LK85 and Phillips (1966) we consider perturbations of the form

$$\theta'=\hat{\theta}(k,l,m,t)\exp\{i[kx+ly+(m-\sigma kt)z]\}, \tag{2.5}$$

with similar expressions for the components of $\mathbf{u}'$ and $\mathbf{b}'$. From the governing equations and use of the properties (1.2) and (1.3), it is readily established that θ' and the vertical components u', b' of (2.4a, b) satisfy the system of equations

$$(D+\nu p)(p\hat{u}) = ikA^2 p\hat{b} + N^2 a^2 \hat{\theta}, \tag{2.6a}$$

$$(D+\eta p)\hat{b} = ik\hat{u}, \tag{2.6b}$$

$$(D+\kappa p)\hat{\theta} = -\hat{u}, \tag{2.6c}$$

in which

$$A^2 = B_0^2/\mu\rho_0, \qquad N^2 = -(g/\rho_0)(d\rho_0/dz) = -\alpha\beta g, \tag{2.7a, b}$$

$$p(t) = a^2 + (m-\sigma kt)^2, \qquad a^2 = k^2 + l^2, \qquad D \equiv d/dt, \tag{2.7c, d, e}$$

and ρ_0 is the equilibrium density. Here A is the Alfvén velocity, and N is the Brunt-Väisälä frequency, which is real ($N^2 > 0$) when the fluid is stably stratified. It is a simple matter to eliminate $\hat{\theta}$ from (2.6a, c), and then use (2.6b) to eliminate $\hat{u}$. In this way a third order differential equation similar to (2.28) in LK85 may be derived. The equation is cumbersome and we find it more convenient to study the primitive set (2.6) of three first order equations.

2.2 *The General Solution*

In principle the evolution of any initial disturbance can be determined by considering the superposition of modes of the form (2.5). Indeed at time t, the solution is given by

$$\theta'(x, y, z, t) = \int\int\int \hat{\theta}(k, l, m, t) \exp\{i[kx + ly + (m-\sigma kt)z]\}\, dk\, dl\, dm, \tag{2.8a}$$

with similar expressions for u' and b'. Here $\hat{u}$, $\hat{b}$ and $\hat{\theta}$ evolve according to (2.6) and their initial values are given by

$$\hat{\theta}(k, l, m, 0) = (2\pi)^{-3} \int\int\int \theta'(x, y, z, 0) \exp\{-i[kx + ly + mz]\}\, dx\, dy\, dz, \tag{2.8b}$$

with similar expressions for $\hat{\mathbf{u}}$ and $\hat{\mathbf{b}}$. Unfortunately except for very special values of the wave vector (k, l, m) it is not possible to provide analytic solutions of (2.6). Nevertheless, as $t \to \infty$, WKBJ-type asymptotic solutions given by (3.26) below are adequate to settle the question of stability. Strictly we should undertake the analyses of K84 [Section 2.3] and LK85 [Section 3] to place upper bounds on the decay rate. Nevertheless, the exponential factors defined by (3.25) and the algebraic coefficients in the complete asymptotic representation (3.26) are similar to either those given in (2.42) in LK85 for the non-stratified problem or those given in (2.20) in K84 for the non-magnetic problem. Indeed all our algebraic coefficients listed in (5.3), (5.4), (5.5) for the final period of decay coincide with theirs.

Consequently stability to an arbitrary initial disturbance is also guaranteed for our more general system which includes both the magnetic field and stratification.

2.3 *The WKBJ-Approximation and Turning Points*

The effect of the shear after a sufficiently long time is to tilt the waves and shorten the z-length scale indefinitely. This means that the magnitude of p defined by (2.7c) tends to infinity with time t. In general this leads to a rapid oscillation or exponential decay which can be described within the framework of the WKBJ-approximation. This technique can be usefully employed for our problem and so we set

$$\hat{\theta}=\tilde{\theta}(t)\,e^{\int \chi\,dt}, \qquad (\chi=\chi(t)), \tag{2.9}$$

with similar expressions for $\hat{u}$ and $\hat{b}$. Substitution into (2.6) yields the alternative set of equations

$$\begin{bmatrix} (\chi+\nu p)p & -ikA^2p & -N^2a^2 \\ -ik & \chi+\eta p & 0 \\ 1 & 0 & \chi+\kappa p \end{bmatrix}\begin{bmatrix} \tilde{u} \\ \tilde{b} \\ \tilde{\theta} \end{bmatrix}=-D\begin{bmatrix} p\tilde{u} \\ \tilde{b} \\ \tilde{\theta} \end{bmatrix}. \tag{2.10}$$

When χ is large, in the sense that

$$\chi^{-1}D\ll 1, \tag{2.11}$$

we may neglect the right-hand side of (2.10). The condition that the remaining homogeneous equation has a solution is that χ satisfies the cubic

$$F(\chi,t)\equiv p(\chi+\nu p)+k^2A^2p(\chi+\eta p)^{-1}+N^2a^2(\chi+\kappa p)^{-1}=0. \tag{2.12}$$

When there are three distinct roots $\chi=\chi^{(i)}(i=1,2,3)$ the complete solution of the governing equations can be expressed in the form

$$\hat{\theta}=\sum_{i=1}^{3}\tilde{\theta}^{(i)}\,e^{\int \chi^{(i)}dt} \tag{2.13}$$

with similar expressions for $\hat{u}$ and $\hat{b}$. For each value of $\chi^{(i)}$ the corresponding values of $\tilde{u}^{(i)}$ and $\tilde{b}^{(i)}$ are according to (2.10) given by

$$\tilde{b}=ik(\chi+\eta p)^{-1}\tilde{u}, \qquad \tilde{\theta}=-(\chi+\kappa p)^{-1}\tilde{u}. \tag{2.14a, b}$$

To obtain a higher order approximation of the solution it is necessary to solve the inhomogeneous equation (2.10) with the right-hand side given by (2.14). Since for each $\chi^{(i)}$ the matrix on the left of (2.10) is singular, a solution is only possible when the consistency condition

$$D(p\tilde{u})+ikA^2p(\chi+\eta p)^{-1}D\tilde{b}+N^2a^2(\chi+\kappa p)^{-1}D\tilde{\theta}=0 \tag{2.15}$$

is met. Upon substitution of (2.14) into (2.15), we obtain a first order ordinary differential equation governing the evolution of $\tilde{u}$, and hence also $\tilde{b}$ and $\tilde{\theta}$ by (2.14). This is the standard WKBJ-method as applied to matrix problems.

The WKBJ-method fails at those critical instants $t=t_c$ (say) at which (2.12) has repeated roots

$$\chi^{(1)}(t_c)=\chi^{(2)}(t_c)=\chi_c \quad \text{(say)}. \tag{2.16}$$

Provided that $\chi^{(3)}(t_c)\neq\chi_c$, a solution may be sought in the neighbourhood of $t=t_c$ in the form

$$\hat{\theta}=\bar{\theta}(\tau)e^{\chi_c\tau}+\tilde{\theta}^{(3)}e^{\int\chi^{(3)}dt}, \qquad (\tau=t-t_c). \tag{2.17}$$

The amplitude equation

$$d^2\bar{\theta}/d\tau^2-\lambda^3\tau\bar{\theta}=0, \tag{2.18}$$

governing the modulation of $\bar{\theta}$, where the constant λ is defined by (2.21) below, may be derived by modifying the expansion procedure which led to (2.15). Its essential features are, however, evident from the properties of the dispersion relation (2.12). In particular, we note that, in addition to the requirement $F_c=0$, we must also have (because of the repeated roots (2.16))

$$(\partial F/\partial\chi)_c=0, \tag{2.19}$$

where the subscript c is used to denote the value at $(\chi,t)=(\chi_c,t_c)$. This means that the leading order terms in the Taylor series expansion of F are

$$F=\tfrac{1}{2}(\chi-\chi_c)^2(\partial^2F/\partial\chi^2)_c+\tau(\partial F/\partial t)_c. \tag{2.20}$$

The WKBJ-dispersion relation for (2.18) coincides with (2.20) when

$$\lambda=\{-2(\partial F/\partial t)_c/(\partial^2F/\partial\chi^2)_c\}^{1/3}. \tag{2.21}$$

The Eq. (2.18) provides a classical *turning point* problem (see, for example, Heading, 1962) whose general solution is

$$\bar{\theta}=\mathscr{A}Ai(\lambda\tau)+\mathscr{B}Bi(\lambda\tau), \tag{2.22}$$

where Ai, Bi are the two independent solutions of Airy's equation. The two constants $\mathscr{A},\mathscr{B}$ are fixed by matching $\bar{\theta}\exp(\chi_c\tau)$ with the incoming WKBJ-solution

$$\hat{\theta}=\sum_{i=1}^{2}\tilde{\theta}^{(i)}e^{\int\chi^{(i)}dt}, \tag{2.23}$$

as $\tau \to -\infty$. The two amplitudes of the outgoing WKBJ-solution are then fixed by a further matching with (2.22) as $\tau \to +\infty$. This provides the solution of the transmission problem across the *turning point*. The important feature here is that a single mode of the type (2.9) arrives ($t < t_c$) at the *turning point* and excites both the new modes, which leave ($t > t_c$). This means that a single mode by itself cannot provide a valid approximation to the governing equations when turning points are crossed. Instead combinations of modes must be considered and their transmission properties determined. For special values of the parameters of the problem, triple rather than repeated roots of (2.12) can occur. For those exceptional cases a separate analysis of the *turning point* problem is required.

In the following section we apply the WKBJ-method to our problem in particular limiting cases and take note of the occurrence of *turning points*.

3. THE ASYMPTOTIC SOLUTION

3.1 The Non-Dissipative Case

When all the diffusivities vanish,

$$\nu = \eta = \kappa = 0, \tag{3.1}$$

the dispersion relation reduces to a quadratic with two solutions

$$\chi^{(\pm)} = \pm i\omega_{AN}, \tag{3.2}$$

where

$$\omega_{AN} = (\omega_A^2 + \omega_N^2)^{1/2}, \tag{3.3a}$$

is the hybrid Alfvén-gravity wave frequency and

$$\omega_A^2 = k^2 A^2, \qquad \omega_N^2 = N^2 a^2/p. \tag{3.3b, c}$$

The local stability of the system characterised by (3.2) depends on the sign of ω_{AN}^2. Evidently instability is only possible for unstably stratified fluids ($N^2 < 0$) and then only during the time interval $t_0 - t_1 < t < t_0 + t_1$. Here $t_0 = m/\sigma k$, while $t_0 \pm t_1$ are the critical times at which

$$p(t_0 \pm t_1) = a^2 + (\sigma k t_1)^2 = p_c, \tag{3.4a}$$

where

$$p_c = -(N^2/A^2)(a^2/k^2). \tag{3.4b}$$

Exponential growth occurs for $t_0 - t_1 < t < t_0 + t_1$. The coincidence of the roots (3.2) at $t(=t_c) = t_0 \pm t_1$ signals the *turning points*. At times outside the interval $t_0 - t_1 < t < t_0 + t_1$ the two modes emerge with wave-like structure. Of course, when the fluid is stably stratified ($N^2 > 0$), the value of p_c defined by (3.4b) is negative and (3.4a) has no real solution. There are then no *turning points*.

The amplitude modulation is determined by (2.15) which, with the help of (2.14) and (3.2), reduces to

$$D\{(p/\omega_{AN})\tilde{u}^2\}=0. \tag{3.5}$$

It follows that the complete solution is

$$\tilde{u}^{(\pm)}=(\Omega^{(\pm)}\omega_{AN}/p)^{1/2},\qquad \tilde{b}^{(\pm)}=\pm k(\Omega^{(\pm)}/\omega_{AN}\,p)^{1/2},\qquad \tilde{\theta}^{(\pm)}=\pm i(\Omega^{(\pm)}/\omega_{AN}\,p)^{1/2}, \tag{3.6a, b, c}$$

where $\Omega^{(+)}$ and $\Omega^{(-)}$ are arbitrary constant frequencies linked to each of the modes $\chi^{(+)}$ and $\chi^{(-)}$ (see (3.2)). As $t\to\infty$, the gravity frequency ω_N tends to zero and we are left with the non-stratified result given previously by (2.35) in LK85 for which $|\tilde{u}^{(\pm)}|\sim t^{-1}$.

When the magnetic field is weak, in the sense that

$$|N^2|\gg k^2A^2,\qquad \text{or equivalently}\quad |p_c|\gg a^2, \tag{3.7a, b}$$

and the stratification is strong, in the sense that

$$|Ri|\equiv|N^2|/\sigma^2\gg k^2/a^2, \tag{3.8}$$

where Ri is the Richardson number, there is a limited period of time when the WKBJ-approximation is valid and magnetic effects are negligible: $p\ll|p_c|$. In the case of stable stratification, $N^2>0$, this period of algebraic decay is characterised by the solution given by (2.15b) in K84, for which $|\tilde{u}^{(\pm)}|\sim t^{-3/2}$. The transition from the $t^{-3/2}$ to the t^{-1} power law occurs when $p(t)=O(|p_c|)$.

When the fluid is unstably stratified, $N^2<0$, it is of some interest to determine the growth and decay characterised by (3.2). If we consider time sufficiently large that

$$p\gg a^2, \tag{3.9}$$

we may approximate p by $(m-\sigma kt)^2$ (see (2.7c)). The resulting integral of (3.2) is

$$\int\chi^{(\pm)}\,dt\sim\pm(-Ri)^{1/2}\left(\frac{a}{k}\right)\left\{\left[\frac{p_c-p}{p_c}\right]^{1/2}+\frac{1}{2}\ln\left[\frac{p_c^{1/2}-(p_c-p)^{1/2}}{p_c^{1/2}+(p_c-p)^{1/2}}\right]\right\}. \tag{3.10}$$

When t exceeds t_c ($p>p_c$), the value of (3.10) is purely imaginary and the algebraic decay predicted by (3.6) is applicable. On the other hand, when t is less than t_c ($p<p_c$), the two values of (3.10) are real and exponential growth ensues. When the magnetic field is weak and the stratification is strong as defined by (3.7) and (3.8) above, there is a period of time during which buoyancy forces dominate. That occurs when $|t-t_0|$ is much less than t_1 for which the approximation $p\ll p_c$ in (3.10) gives

$$\int \chi^{(\pm)}\,dt \sim \pm\tfrac{1}{2}(-Ri)^{1/2}(a/k)\ln(p/4p_c), \tag{3.11a}$$

and determines the algebraic growth

$$e^{\int \chi^{(\pm)}dt} \sim t^{\pm(-Ri)^{1/2}(a/k)}. \tag{3.11b}$$

This result is compatible with (2.12) in K84 within the framework of the approximation (3.8).

3.2 *The Case of Small Diffusivity*

When ν, η, κ are finite but small in the sense that the diffusion times $(\nu a^2)^{-1}$, $(\eta a^2)^{-1}$, $(\kappa a^2)^{-1}$ are long compared to the dynamic times $|N|^{-1}$, $|kA|^{-1}$, the wave motions described in the previous subsection are weakly damped. Accordingly it is convenient to express $\chi^{(i)}$ $(i=1,2,3)$ in the form

$$\chi^{(\pm)} = -\tfrac{1}{2}(\nu+\lambda)p \pm i\omega, \qquad \chi^{(3)} = -(\eta+\kappa-\lambda)p, \tag{3.12a, b}$$

where we have used the properties of the sum of the roots of the cubic (2.12). The further two properties concerning the product of roots yield

$$\frac{\omega_A^2}{(\kappa-\lambda)p} + \frac{\omega_N^2}{(\eta-\lambda)p} = (\nu-\eta-\kappa+\lambda)p, \tag{3.13a}$$

$$\omega^2 - \omega_{AN}^2 = -[\tfrac{1}{2}(\nu-\lambda)p]^2 + (\kappa-\lambda)(\eta-\lambda)p^2. \tag{3.13b}$$

Once λ is determined as the real root of the cubic (3.13a), the corresponding value of the frequency ω can be determined from (3.13b). For completeness we note that with the help of (2.12) and (2.14) the amplitude evolution equation (2.15) may be cast in the form

$$\frac{2D\tilde{u}}{\tilde{u}} = -\left\{\frac{D\psi - (\nu-\eta)\omega_A^2\psi_\eta D\psi_\eta - (\nu-\kappa)\omega_N^2\psi_\kappa D\psi_\kappa}{\psi - \tfrac{1}{2}(\nu-\eta)\omega_A^2\psi_\eta^2 - \tfrac{1}{2}(\nu-\kappa)\omega_N^2\psi_\kappa^2}\right\}, \tag{3.14}$$

where

$$\psi_\eta = \psi/(\chi+\eta p), \qquad \psi_\kappa = \psi/(\chi+\kappa p), \qquad \psi = p/(\chi+\nu p). \tag{3.15a, b}$$

In the absence of dissipation, (3.13b) and (3.14) reduce to (3.2) and (3.5). It is not feasible, in general, to solve the cubic or to perform the integration in (3.14). Instead we consider below certain limiting cases.

We begin by remarking on the nature of the early time solution valid when

$$(\lambda p)^2 \ll \omega_{AN}^2. \tag{3.16}$$

At that stage the right hand sides of (3.13) can be neglected giving

$$\omega^2 = \omega_{AN}^2, \qquad \lambda = (\eta\omega_A^2 + \kappa\omega_N^2)/\omega_{AN}^2. \tag{3.17a, b}$$

correct to leading order. Evidently, the three modes (3.12) are damped when the fluid is stably stratified, $N^2>0$. In the case of unstable stratification, resistive instabilities are possible in certain regimes and correspond to the classical problem of Bénard convection. They may be either direct modes, $\chi^{(3)}>0$, or overstable modes $\mathscr{Re}[\chi^{(\pm)}]>0$. We are reminded, however, that in these circumstances the cross rolls, $k=0$, are the most unstable, as they are influenced neither by the magnetic field nor the shear (see also Kuo, 1963, in the non-magnetic context).

At intermediate times when

$$(\lambda p)^2 = O(\omega_{AN}^2), \tag{3.18}$$

there is a critical time at which the two damped oscillations (3.12a) merge. Their coalescence and re-emergence as a pair of exponential modes is characterised by the turning point problem explained in Subsection 2.3. In the strongly stratified case

$$|\omega_N|^2 \gg |\omega_A|^2, \tag{3.19a}$$

which is applicable for times satisfying

$$p \ll |p_c|, \tag{3.19b}$$

the solution of (3.13) gives the approximate results

$$\lambda=\kappa, \qquad \omega^2=\omega_N^2-[\tfrac{1}{2}(\nu-\kappa)p]^2. \tag{3.20a, b}$$

When the fluid is stably stratified ($N^2>0$) and $|\tfrac{1}{2}(\nu-\kappa)p|<\omega_N$, the two modes (3.12a) are gravity waves damped by viscous and thermal diffusion. After coalescence $|\tfrac{1}{2}(\nu-\kappa)p|>\omega_N$, they cease to be oscillatory and they finally decay as the viscous and thermal modes described by (3.25a, c) below. The other mode (3.12b) is characterised by $\chi^{(3)}=-\eta p$ and corresponds throughout to the magnetic mode (3.25b).

In the strong magnetic field case

$$|\omega_A|^2 \gg |\omega_N|^2, \tag{3.21a}$$

which is always applicable at sufficiently late times when

$$p \gg |p_c|, \tag{3.21b}$$

the solution of (3.13) gives the approximate results

$$\lambda=\eta, \qquad \omega^2=\omega_A^2-[\tfrac{1}{2}(\nu-\eta)p]^2. \tag{3.22a, b}$$

Indeed with the terms ω_N^2 in (3.14) ignored, the equation is easily integrated. Further reductions involving (2.12), (3.15) and (3.22) yield the compact result

$$\tilde{u}^{(\pm)}=\{\Omega^{(\pm)}\omega_A^2/p\omega\}^{1/2}, \tag{3.23a}$$

which matches with the early time solution (3.6) so fixing the constants $\Omega^{(\pm)}$. Now the two modes (3.12a) are Alfvén waves damped by viscous and Ohmic diffusion. After coalescence $|\frac{1}{2}(\nu-\eta)p|>\omega_A$, they cease to be oscillatory and they finally decay as the viscous and magnetic modes described by (3.25a, b) below. Matching with the late time amplitude modulations given by (3.26) below is achieved when the integrals $\int \chi\, dt$ have the same lower limit and

$$\tilde{u}^{(\nu)}=A^{(\nu)}\left\{\frac{|\nu-\eta|}{2|\omega|p}\right\}^{1/2}, \qquad \tilde{u}^{(\eta)}=\frac{\omega_A^2}{\eta-\nu}A^{(\eta)}\left\{\frac{|\nu-\eta|}{2|\omega|p}\right\}^{1/2}. \tag{3.23b}$$

The other mode is characterised by $\chi^{(3)}=-\kappa p$ and corresponds throughout to the thermal mode (3.25c). Note also that (3.23) exhibits the asymptotic behaviour $\tilde{u}=O[|t-t_c|^{-1/2}]$ in the neighbourhood of the *turning point* $t=t_c$ in accord with the asymptotic behaviour of the Airy functions (2.22) as $|t-t_c|\to\infty$.

At late times when

$$(\lambda p)^2 \gg \omega_{AN}^2, \tag{3.24}$$

the final period of decay sets in and three modes—viscous, magnetic and thermal—can be identified. The corresponding values of χ have the expansions

$$\chi^{(\nu)}=-\nu p+\left\{\frac{\omega_A^2}{(\nu-\eta)p}+\frac{\omega_N^2}{(\nu-\kappa)p}\right\}+\cdots, \tag{3.25a}$$

$$\chi^{(\eta)}=-\eta p+\omega_A^2[(\eta-\nu)p]^{-1}+\cdots, \tag{3.25b}$$

$$\chi^{(\kappa)}=-\kappa p+\omega_N^2[(\kappa-\nu)p]^{-1}+\cdots, \tag{3.25c}$$

respectively as $p\to\infty$, valid whenever ν,η,κ are all unequal. To the same order of accuracy the corresponding forms for the complete solutions given by (2.13) to (2.15) are

$$\begin{bmatrix} \hat{u} \\ \hat{b}/ik \\ -\hat{\theta} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{p} & \dfrac{\omega_A^2}{(\eta-\nu)p} & \dfrac{\omega_N^2}{(\kappa-\nu)p} \\ -\dfrac{1}{(\nu-\eta)p^2} & 1 & -\dfrac{\omega_N^2}{(\kappa-\eta)(\kappa-\nu)p^2} \\ -\dfrac{1}{(\nu-\kappa)p^2} & -\dfrac{\omega_A^2}{(\eta-\kappa)(\eta-\nu)p^2} & 1 \end{bmatrix} \begin{bmatrix} A^{(\nu)}e^{\int\chi^{(\nu)}dt} \\ A^{(\eta)}e^{\int\chi^{(\eta)}dt} \\ A^{(\kappa)}e^{\int\chi^{(\kappa)}dt} \end{bmatrix}, \tag{3.26}$$

for distinct ν,η,κ. It is now a relatively straightforward matter to show from (2.15) that to leading order

$$DA^{(\nu)}=DA^{(\eta)}=DA^{(\kappa)}=0, \tag{3.27}$$

implying that $A^{(\nu)}$, $A^{(\eta)}$ and $A^{(\kappa)}$ are constants. This leads to the asymptotic behaviour summarised in (5.3) to (5.5) below. Exceptional cases, which arise when some or all of the diffusivities are equal, are discussed in the Appendix.

According to (3.26) the mode, which dominates as $t \to \infty$, is identified by the smallest of ν, κ, η. We see that the magnetic mode is given by LK85 (Eq. (2.42) with $\eta < \nu$ and lower sign), the thermal mode is given by K84 (Eq. (2.20) with $\kappa < \nu$ and $\beta = 2$), and the viscous mode is given either by LK85 (Eq. (2.42) with $\nu < \eta$ and lower sign) or K84 (Eq. (2.20) with $\nu < \kappa$ and $\beta = 1$). In each case we see from (3.26) that the diffusive decay, which characterises the mode, is linked with the mechanism driving it. So, for example, Ohmic diffusion corresponding to $\hat{b}^{(\eta)}$ leads to Lorentz forces, which ultimately drive motion $\hat{u}^{(\eta)}$ and lead to thermal perturbations $\hat{\theta}^{(\eta)}$.

Furthermore the decay is fast, since

$$\int p\,dt \sim \begin{cases} \frac{1}{3}\sigma^2 k^2 t^3, & \text{for } |\sigma k| t \gg |m|, \qquad (3.28a) \\ (a^2 + m^2)t, & \text{for } |\sigma k| t \ll |m|, \qquad (3.28b) \end{cases}$$

as $t \to \infty$. Equations (3.19) and (3.20) in LK85 distinguish these two modes of asymptotic decay for moderate and small k. The slower decay rates for modes aligned with the shear means that it is those modes in the Fourier representation (2.8), which persist for the longest time as they are the least influenced by the shear. In any event L84 (Section 2.3) and LK85 (Section 3) argue that an arbitrary initial disturbance will decay. Since we have shown that our dominant mode must coincide with one of theirs it follows that their results apply to our case also implying stability.

4. FINITE AMPLITUDE DISTURBANCES IN THE PRESENCE OF BOUNDARIES

Our analysis so far relies on the fact that any boundaries, which exist, are sufficiently far away from the disturbance that they can be ignored. Here we discuss briefly the case of fluid contained between rigid, isothermal boundaries at $z = 0$, H, which move with the fluid. We also suppose, like LK85 before, that the magnetic field perturbations vanish, thus yielding the boundary conditions

$$\mathbf{u}' = 0, \qquad \mathbf{B}' = 0, \qquad T' = 0 \quad \text{on } z = 0, H. \tag{4.1}$$

All disturbances are assumed to vanish at infinity ($|x^2 + y^2| \to \infty$).

A sufficient condition for nonlinear stability may be established by considering the total energy

$$E(t) = \langle \tfrac{1}{2}(|\mathbf{u}'|^2 + A^2 |\mathbf{b}'|^2 + N^2 \theta'^2) \rangle, \tag{4.2}$$

in the fluctuating disturbance, where here the angle brackets are used to denote integration throughout the volume ($\langle\ \rangle = \int dx\,dy\,dz$). The equation for the evolution of (4.2) is obtained from the perturbation equations (2.1) by multiplying (2.1a) by $\mathbf{u}'$, (2.1c) by $(\rho\mu)^{-1}\mathbf{b}'$, (2.1e) by $(N^2/\beta^2)T'$ and adding the results together. Integration throughout the volume, application of the divergence condition and use of (4.1) yields the extension,

$$dE/dt = E\{-\nu I^{(\nu)} - \eta I^{(\eta)} - \kappa I^{(\kappa)} + \sigma I^{(\sigma)}\}, \tag{4.3}$$

of (4.7) in LK85, where

$$EI^{(\nu)} = \langle |\nabla \mathbf{u}'|^2 \rangle, \qquad EI^{(\eta)} = A^2 \langle |\nabla \mathbf{b}'|^2 \rangle, \tag{4.4a, b}$$

$$EI^{(\kappa)} = N^2 \langle |\nabla \theta'|^2 \rangle, \qquad EI^{(\sigma)} = \langle A^2 b_x' b' - u_x' u' \rangle. \tag{4.4c, d}$$

In the case of stable stratification the bound on the rate of change of E, which follows, is a simple modification of (4.16) in LK85, namely

$$dE/dt \leqq -2E\{H^{-2}(\nu + \eta + \kappa) - |\sigma|\}. \tag{4.5}$$

From (4.5) we conclude that the system is asymptotically stable when

$$(\nu + \eta + \kappa)/H^2 > |\sigma|. \tag{4.6}$$

We note that without shear ($\sigma = 0$) this condition is always met in a dissipative system. On the other hand, this is not the case for $\sigma \neq 0$ and so suggests that the linear shear may decrease the stability of the system.

5. SUMMARY

Here we summarise the nature of the final decay of disturbances both without and with dissipation.

In the absence of diffusion the results (2.12) of K84 and (2.35) of LK85 show that the amplitude of the vertical velocity has the asymptotic behaviour

$$|\hat{u}^{(\pm)}| \sim t^{\lambda}, \tag{5.1}$$

where the extreme values of λ are

$$\lambda = \lambda_{N\pm} = \mathscr{R}e[-\tfrac{3}{2} \pm (\tfrac{1}{4} - (Ri)a^2/k^2)^{1/2}], \tag{5.2a}$$

when there is no magnetic field, and

$$\lambda = \lambda_A = -1, \tag{5.2b}$$

when there is no stratification. These limiting cases have been identified by our analysis. In the case of stable stratification ($Ri = N^2/\sigma < 0$), we see from (5.2a) that without magnetic field the amplitude of the vertical velocity only decays ($\lambda_{N+} < 0$)

when $(-Ri)<\frac{1}{4}(k/a)^2$ and decays most rapidly $[\lambda_{N+}(=\lambda_A)=-1]$ when $Ri=0$. Accordingly, the growth and decay rates of gravity modes are controlled by the magnitude of the shear, in contrast to the Alfvén waves (see (5.2b)), for which $\lambda_A=-1$ is independent of σ. Nevertheless we stress that, when both stratification and magnetic fields are present, magnetic effects ultimately dominate and the final algebraic decay is characterised by (5.2b).

With the inclusion of distinct diffusivities ν, η, κ the three modes of final decay are the viscous mode characterised by

$$\tilde{u}^{(\nu)} \sim t^{-2} e^{-\nu s}, \qquad \tilde{b}^{(\nu)} \sim t^{-4} e^{-\nu s}, \qquad \tilde{\theta}^{(\nu)} \sim t^{-4} e^{-\nu s}, \tag{5.3a, b, c}$$

the magnetic mode characterised by

$$\tilde{u}^{(\eta)} \sim t^{-2} e^{-\eta s}, \qquad \tilde{b}^{(\eta)} \sim e^{-\eta s}, \qquad \tilde{\theta}^{(\eta)} \sim t^{-4} e^{-\eta s}, \tag{5.4a, b, c}$$

and the thermal mode characterised by

$$\tilde{u}^{(\kappa)} \sim t^{-4} e^{-\kappa s}, \qquad \tilde{b}^{(\kappa)} \sim t^{-6} e^{-\kappa s}, \qquad \tilde{\theta}^{(\kappa)} \sim e^{-\kappa s}, \tag{5.5a, b, c}$$

where in each case

$$s=\tfrac{1}{3}\sigma^2 k^2 t^3, \tag{5.6}$$

provided that $\sigma k \neq 0$. As explained in Section 3 above, each of these modes are identified by K84 and LK85. This fact is used to argue the stability with respect to an arbitrary initial disturbance. It means that, given any initial disturbance at $t=0$, the solution remains bounded for $t>0$ and tends to zero as $t \to \infty$.

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APPENDIX

Here the small diffusivity results of Section 3.2 for late times are extended to encompass the exceptional results, which occur when some or all of the diffusivities ν, η, κ are equal.

The Case $\eta = \kappa$

In this limit the dispersion relation (3.13a) has the trivial solution

$$\lambda = \eta = \kappa, \tag{A.1}$$

and so, with (3.13b), the three values of $\chi^{(i)}$ given by (3.12) are

$$\chi^{(\pm)} = -\tfrac{1}{2}(\nu+\lambda)p \pm \{[\tfrac{1}{2}(\nu-\lambda)p]^2 - \omega_{AN}^2\}^{1/2}, \qquad \chi^{(3)} = -\lambda p. \tag{A.2a, b}$$

At late times when

$$|(\nu-\lambda)p|^2 \gg \omega_{AN}^2, \tag{A.3}$$

we may identify the viscous decay mode characterised by $\chi^{(-)} \sim \chi^{(\nu)}$ (see (3.25a)) together with its corresponding solution given by (3.26). On the other hand, the thermal and magnetic modes remain linked because $\chi^{(+)} \sim \chi^{(3)}$.

To determine the nature of the hybrid modes it is convenient to restrict attention to

$$\chi = \chi^{(3)} = -\lambda p, \tag{A.4}$$

in (2.10). Accordingly it may be reduced to the second order equation

$$p^{-1}D(pD\tilde{T}) + (\nu - \lambda)pD\tilde{T} + \omega_{AN}^2\tilde{T} = (\omega_A^2 - \omega_N^2)\tilde{S}_0, \tag{A.5a}$$

for $\tilde{T}$, in which $\tilde{S}_0$ is a constant and

$$\tilde{u} = D\tilde{T}, \qquad \tilde{b} = ik(\tilde{T} - \tilde{S}_0), \qquad \tilde{\theta} = -\tilde{T} - \tilde{S}_0. \tag{A.5b}$$

The rapidly varying WKBJ-solution obtained by balancing the first two terms on the left of (A.5a) corresponds to the viscous mode $\chi^{(-)} \sim \chi^{(\nu)}$ isolated above. The remaining hybrid modes obtained by neglecting the highest derivative $p^{-1}D(pD\tilde{T})$ at lowest order have the asymptotic representation

$$\tilde{T} \sim \begin{cases} \tilde{S}_0\{(\omega_A^2 - \omega_N^2)/\omega_{AN}^2\}, & (\tilde{S}_0 \neq 0), \quad \text{(A.6a)} \\[2ex] \tilde{T}_0 \exp\left\{\dfrac{-1}{(\nu-\lambda)} \displaystyle\int \dfrac{\omega_{AN}^2}{p}\, dt\right\}, & (\tilde{S}_0 = 0), \quad \text{(A.6b)} \end{cases}$$

where $\tilde{T}_0$ is a constant.

The special case,

$$(\lambda =)\nu = \kappa = \eta, \tag{A.7}$$

must be treated separately. Though (A.6a) continues to hold, the other solution is obviously inapplicable, since $\nu = \lambda$. So with the term $(\nu - \lambda)pD\tilde{T}$ absent in (A.5a), we must consider the two distinct solutions of that homogeneous equation obtained when $\tilde{S}_0 = 0$. Those solutions are traced most easily from (A.2a), which gives

$$\chi^{(\pm)} \sim -\lambda p \pm i\omega_{AN}. \tag{A.8}$$

These modes, though damped, continue to oscillate at the Alfvén-gravity wave frequency indefinitely as $t \to \infty$.

The Case $\nu = \eta(\neq \kappa)$

In this limit the dispersion relation (3.13a) has the asymptotic solution

$$\lambda \sim \nu = \eta, \tag{A.9}$$

valid at sufficiently late time (see (A.12a) below). Accordingly we may identify the thermal mode $\chi^{(3)}=\chi^{(\kappa)}\sim-\kappa p$ together with its corresponding solution given by (3.26).

To determine the nature of the other hybrid viscous, magnetic modes, we set

$$\chi=-\nu p(=-\eta p) \tag{A.10}$$

in (2.10). It reduces to the pair of equations

$$D[p^{-1}D(p\tilde{u})]+\omega_A^2\tilde{u}=D(\omega_N^2\tilde{\theta}), \tag{A.11a}$$

$$D\tilde{\theta}-(\nu-\kappa)p\tilde{\theta}=-\tilde{u}, \tag{A.11b}$$

where

$$D\tilde{b}=ik\tilde{u}. \tag{A.11c}$$

When

$$|(\nu-\kappa)p|^2+\omega_A^2\gg\omega_N^4/\omega_A^2, \tag{A.12a}$$

we may neglect the right hand side of (A.11a) at leading order. The corresponding asymptotic solution is

$$\tilde{u}\sim\tilde{u}_0\exp\{\pm i\textstyle\int\omega_A\,dt\}, \tag{A.12b}$$

where $\tilde{u}_0$ is a constant. With the choice (A.9) for λ, these hybrid modes correspond to

$$\chi^{(\pm)}\sim-\nu p\pm i\omega_A, \tag{A.12c}$$

in (3.12a). Like (A.8) above, it defines a damped (Alfvén) wave which persists indefinitely as $t\to\infty$. Moreover it agrees with (A.8) when $\nu=\eta=\kappa$, provided that $\omega_A^2\gg\omega_N^2$ as required by (A.12a).

The Case $\nu=\kappa(\neq\eta)$

This case is very similar to the previous case and so we will not describe it in detail. With $\lambda\sim\nu=\kappa$ we may identify the magnetic mode $\chi^{(3)}=\chi^{(\eta)}\sim-\eta p$. In addition, when

$$|(\nu-\eta)p|^2+\omega_N^2\gg\omega_A^4/\omega_N^2, \tag{A.13a}$$

we may identify the hybrid viscous, gravity mode characterised by

$$\chi^{(\pm)}\sim-\nu p\pm i\omega_N, \tag{A.13b}$$

which define damped gravity waves.

Note that all cases have been discussed within the framework of stable stratification for which $N^2>0$. When $N^2<0$, the gravity frequency ω_N is pure imaginary but since $|\omega_N|\to0$ as $t\to\infty$ all the modes discussed are eventually damped.