Surface tension driven convection in viscoelastic liquids with thermorheological effect [☆]

P.G. Siddheshwar [a,*], G.N. Sekhar [b], G. Jayalatha [c]

[a] Department of Mathematics, Central College Campus, Bangalore University, Bangalore-560001, India
[b] Department of Mathematics, B M S College of Engineering, Bangalore-560019, India
[c] Department of Mathematics, R V College of Engineering, Bangalore-560059, India

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ABSTRACT

Oscillatory onset of convection is studied numerically for Rivlin–Ericksen, Maxwell and Jeffreys liquids by considering free–free and rigid–free isothermal/adiabatic boundaries. The effect of variable viscosity parameter is shown to destabilize the system. The problem reveals the stabilizing nature of strain retardation parameter and the destabilizing nature of stress relaxation parameter, on the onset of convection. The Maxwell liquids are found to be more unstable than the one subscribing to the Jeffreys description whereas the Rivlin–Ericksen liquids are comparatively more stable. Rigid-free adiabatic boundary combination is found to give rise to a most stable system, whereas the free isothermal free adiabatic combination gives rise to a most unstable system.

1. Introduction

Rayleigh–Bénard–Marangoni convection problems in constant viscosity Newtonian liquids have received widespread attention due to their implications in heat transfer and in other engineering applications (see Chandrasekhar [1]; Platten and Legros [2]). In practical problems, many a time, non-Newtonian liquids are used as working media, especially the viscoelastic ones. In view of this several works have appeared on Rayleigh–Bénard convection in these liquids.

Vest and Arpaci [3] have studied the conditions under which thermally induced overstability occurs in a viscoelastic liquid. It is found that overstability would occur at the lowest possible adverse temperature gradient at which the rate of change of kinetic energy can balance, in a synchronous manner, the periodically varying rates of energy dissipation by the shear stresses and the energy release by the buoyancy force, assuming that stationary convection has not been initialized.

Sokolov and Tanner [4] have studied the Rayleigh–Bénard convection in a general viscoelastic liquid using an integral form of constitutive equations. It is shown that under certain conditions the liquid system is overstable. The theoretical results have been applied to a Maxwell liquid and to some real viscoelastic solutions. Siginer and Valenzuela-Rendón [5] have studied the natural convection of viscoelastic liquids.

Dávalos-Orozco and Vázquez-Luis [6] have investigated natural convection in a viscoelastic liquid with deformable free surface considering the Oldroyd's viscoelastic constitutive equation with relaxation and retardation times and with the Maxwell's model as a

particular case. It is found that, for different values of the Galileo number, for relaxation times that are large enough, the curves of the critical Rayleigh numbers are lower than those of the stationary convection and those of the overstability of the Newtonian liquid with deformable free surface.

Mardones et al. [7] have studied thermal convection thresholds in viscoelastic solutions. The threshold for oscillatory motions in the Rayleigh–Bénard experiments with viscoelastic binary liquids is explicitly determined as a function of separation ratio and rheological parameters. It is shown that the critical oscillation frequency may differ by several orders of magnitude on varying separation ratio and Deborah number.

Siddheshwar [8] has studied oscillatory convection in viscoelastic ferromagnetic and dielectric liquids of Rivlin–Ericksen, Maxwell and Oldroyd types. It is found that the Maxwell liquids are more unstable than the one subscribing to the Oldroyd description whereas the Rivlin–Ericksen liquid is comparatively more stable.

Demir [9] has considered a two-dimensional unsteady Rayleigh–Bénard convective motion of a viscoelastic liquid in a square cavity. For the first time, it is shown that shear-thinning and elastic numbers have an influence in shaping the flow field and determining the heat transfer characteristics with respect to the Rayleigh numbers and their combined effect acts to increase and decrease the heat transfer as represented by the local Nusselt number.

Abu-Ramadan et al. [10] have studied chaotic thermal convection of viscoelastic liquids. The viscoelastic flow in the context of the Rayleigh–Bénard thermal convection set-up is examined using a four-dimensional non-linear dynamical system resulting from a truncated Fourier representation of the conservation and constitutive equations, for an Oldroyd-B liquid. It is found that stress relaxation and strain

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Nomenclature

a	dimensionless wave number
d	thickness of the liquid
l, m	horizontal wave numbers
M	Marangoni number
Pr	Prandtl number
p	pressure
Q	Elastic ratio $\left(\frac{\lambda_2}{\lambda_1}\right)$
q_i	velocity components (u, v, w)
T	temperature
T_0	constant temperature of the upper boundary
t	time
V	variable viscosity parameter

Greek symbols

δ	small positive constant
κ	thermal diffusivity
Λ_1	Deborah number
Λ_2	scaled strain–retardation parameter
λ_1	stress relaxation coefficient
λ_2	strain retardation coefficient
μ	temperature dependent viscosity
ω	frequency
ρ_0	reference density
σ	surface tension

Subscripts

a	average quantity
b	basic state
c	critical quantity
o	oscillatory quantity
oc	oscillatory critical quantity
s	stationary quantity
sc	stationary critical quantity

Superscripts

'	dimensional quantity
*	dimensionless quantity

retardation parameters alter the flow behavior with the stress relaxation precipitating the onset of chaos.

Park and Park [11] have considered the Rayleigh–Bénard convection of viscoelastic liquids in arbitrary finite domains. It is shown that the domain shape can change the viscoelastic parameter values where a Hopf bifurcation occurs in the Rayleigh–Bénard convection.

Siddheshwar and Srikrishna [12] have considered thermal instability in a layer of dilute polymeric liquids when boundaries are subjected to imposed time-periodic boundary temperatures. Periodicity and amplitude criterion for thermal instability is determined. The qualitative effects of various governing parameters on the convective system are discussed.

The works surveyed so far concern instability due to buoyancy. In what follows we review the literature pertaining to Bénard–Marangoni convection, i.e., instability due to surface tension. Getachew and Rosenblat [13] studied the Marangoni instability and calculated the critical Marangoni number as a function of Prandtl number, heat transfer coefficient and elasticity parameters. It is shown that at larger elasticities oscillatory convection is the first mode of instability to appear. Lebon and Cloot [14] have also studied the Marangoni instability problem in viscoelastic liquids.

Few papers have appeared on coupled Bénard–Marangoni and Rayleigh–Bénard convections. Dauby et al. [15] studied linear coupled buoyancy and thermocapillary instabilities in a Maxwell viscoelastic liquid. It is shown that beyond a critical value of the relaxation time, the instability appearing in a liquid layer with a free upper surface, subjected to a temperature-dependent surface tension, takes the form of oscillations.

Lebon et al. [16] considered Bénard–Marangoni instability in a viscoelastic Jeffreys liquid layer and calculated the critical temperature difference between the boundaries at the onset of convection. They have also discussed the role of various viscometric coefficients.

Parmentier et al. [17] analyzed the Bénard–Marangoni instability in viscoelastic liquids considering a general rheological model and Jeffreys model as a particular case. Both linear and weakly nonlinear analyses are presented. The critical values for the temperature gradient, wave number and oscillation frequencies corresponding to the onset of convection are determined from a linear approach. After motion has set in, particular patterns are predicted taking the form of either rolls, or hexagon, or squares. By means of a nonlinear technique, restricted to steady situations, it is determined under which specific conditions of one pattern is preferred.

The above works address the problem of convection in viscoelastic liquids with constant viscosity. It is well known that viscosity varies with temperature. In literature various μ-T models have been reported by Palm [18] and Jensen [19], Torrance and Turcotte [20], Stengel et al. [21], Nield [22], Straughan [23] and Siddheshwar [24]. The Nield thermorheological model [22] is an apt description for variable viscosity in view of the fact that the μ-T curve is a decreasing function and is concave upwards. The problem on the Rayleigh–Bénard convection in viscoelastic liquids with variable viscosity is considered by Sekhar and Jayalatha [25]. In view of their utility in practical situations, in this paper we consider the Bénard–Marangoni convection in three viscoelastic liquids, viz., Rivlin–Ericksen, Maxwell and Jeffreys liquids with temperature-dependent viscosity.

2. Mathematical formulation for Bénard–Marangoni convection

We consider an infinite horizontal layer of a Jeffreys liquid of thickness d (see Fig. 1). The upper plane at $z = d$ and the lower plane at $z = 0$ are maintained at constant temperatures T_0 and $T_0 + \Delta T$ respectively. The viscosity μ of the Jeffreys liquid is assumed to depend on the temperature T and in the present paper the Nield model [22] is used in studying the Bénard–Marangoni convection. The governing equations for the Jeffreys liquid with variable viscosity are:
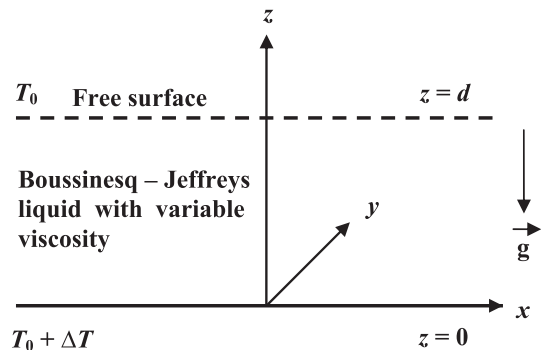
$$q_{i,i} = 0, \tag{1}$$

Fig. 1. Schematic of the problem.

$$\rho_0 \frac{\partial q_i}{\partial t} = -p_{,i} + \tau'_{ij,j}, \tag{2}$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \tau'_{ij} = \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \left[\mu(T)\left(q_{i,j} + q_{j,i}\right)\right], \tag{3}$$

$$\frac{\partial T}{\partial t} + q_i T_{,i} = \kappa T_{,ii}, \tag{4}$$

where $q_i = (u, v, w)$ are the components of the velocity of the liquid, ρ_0 is the density at the reference temperature T_0, p is the pressure, $\mu(T)$ is the temperature-dependent viscosity of the liquid, λ_1 is the stress relaxation coefficient, λ_2 is the strain retardation coefficient, T is the temperature and κ is the thermal diffusivity. All the assumptions that lead to the Oberbeck-Boussinesq system are assumed (see Rajagopal et al. [26]).

The Nield model [22] of shear viscosity as a function of temperature yields the thermorheological equation of state in the form:

$$\mu(T) = \frac{\mu_0}{1 + \delta(T - T_0)}, \tag{6}$$

where $0 < \delta < 1$ and μ_0 is the viscosity at $T = T_0$.

The quiescent basic state has a solution in the form:

$$\left.\begin{array}{l} q_{ib} = (0,0,0), T = T_b(z) = T_0 + \Delta T\left(1 - \dfrac{z}{d}\right), \\[2mm] \mu = \mu_b(z) = \dfrac{\mu_0}{[1 + \delta(T_b - T_0)]} = \dfrac{\mu_0}{\left[1 + \delta\Delta T\left(1 - \dfrac{z}{d}\right)\right]}, \\[2mm] \text{and} \\[1mm] p = p_b(z) = \text{constant}. \end{array}\right\} \tag{7}$$

We now perturb the basic state. Following the classical procedure of linear stability analysis and taking d as the characteristic length, d^2/κ as the characteristic time and ΔT as the characteristic temperature, the linearized dimensionless equations governing small perturbations turn out to be

$$\begin{aligned} \mathrm{Pr}^{-1}\left(1 + \Lambda_1 \frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\left(\nabla^2 w'\right) &= \left(1 + Q\Lambda_1 \frac{\partial}{\partial t}\right) \\ &\times \left[g(z)\nabla^4 w' + 2Vg^2(z)\nabla^2\left(\frac{\partial w'}{\partial z}\right) + 2V^2 g^3(z)\left(\frac{\partial^2 w'}{\partial z^2} - \nabla_1^2 w'\right)\right], \end{aligned} \tag{8}$$

$$\frac{\partial T'}{\partial t} - \nabla^2 T' = w', \tag{9}$$

where

$$g(z) = [1 + V(1-z)]^{-1}, \tag{10}$$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla_1^2 + \frac{\partial^2}{\partial z^2}$ and primes refer to the perturbed quantities.

The dimensionless parameters appearing in Eqs. (8) and (9) are the following:

$\mathrm{Pr} = \dfrac{\mu_0}{\rho_0 \kappa}$ (Prandtl number),

$V = \delta\Delta T$ (Variable viscosity parameter or thermorheological parameter),

$\Lambda_1 = \lambda_1 \dfrac{\kappa}{d^2}$ (Scaled stress−relaxation parameter or Deborah number),

$\Lambda_2 = \lambda_2 \dfrac{\kappa}{d^2}$ (Scaled strain−retardation parameter)

and

$Q = \dfrac{\lambda_2}{\lambda_1}$ (Elasticity ratio).

We assume the perturbations to be periodic waves and hence seek the solution of Eqs. (8) and (9) in the separable form:

$$\left.\begin{array}{l} w'(x,y,z,t) = e^{i\omega t} w(z) e^{i\,(lx + my)} \\ T'(x,y,z,t) = e^{i\omega t} T(z) e^{i\,(lx + my)} \end{array}\right\}, \tag{11}$$

where l and m are the horizontal components of wave number and $a^2 = l^2 + m^2$, $w(z)$, and $T(z)$ are the amplitudes of the perturbations of velocity and temperature respectively and ω is the frequency.

Substituting Eq. (11) into Eqs. (8) and (9) and using D to denote the non-dimensional derivative operator (d/dz), we obtain the governing equations for the Jeffreys model in the form:

$$\begin{aligned} F(\omega)\Big[g(z)\left(D^2 - a^2\right)^2 w &+ 2Vg^2(z)\left(D^2 - a^2\right)Dw \\ &+ 2V^2 g^3(z)\left(D^2 + a^2\right)w\Big] - \frac{i\omega}{\mathrm{Pr}}\left(D^2 - a^2\right)w = 0 \end{aligned} \tag{12}$$

$$\left(D^2 - a^2 - i\omega\right)T + w = 0, \tag{13}$$

where $F(\omega) = \dfrac{1 + iQ\Lambda_1\omega}{1 + i\Lambda_1\omega}$.

We consider four different boundary combinations, RIFA, FIFA, RAFA and FAFA, in studying the Bénard–Marangoni convection and these are mentioned in Table 1. The derivation of the surface-tension boundary condition for variable-viscosity viscoelastic liquids is given in the Appendix. We employ the single term Galerkin technique to compute the critical values of the Marangoni number and wave number as the method is good enough for making qualitative predictions. To that end we choose $w(z)$ and $T(z)$ in the form:

$$\left.\begin{array}{l} w(z) = A\, w_1(z) \\ T(z) = B\, T_1(z) \end{array}\right\}, \tag{14}$$

where A and B are the constants and w_1 and T_1 are chosen to satisfy the given boundary conditions. The details on these functions are given in Table 1. Multiplying Eq. (12) by w, Eq. (13) by T and integrating with respect to z between $z = 0$ to $z = 1$, using Eq. (14) in the resulting equation and then using the condition for a non-trivial solution of the resulting homogeneous equations in A and B, we get the following expression for the eigen value:

$$M = \frac{-x_1 x_5\,\omega^2 N_2 - x_1\, x_4\, N_1 + \frac{x_2 x_5 \omega^2}{\mathrm{Pr}}}{Dw(1)T(1)a^2 x_3} - i\omega \frac{N_3}{Dw(1)T(1)a^2 x_3}, \tag{15}$$

where

$$\begin{aligned} x_1 ={}& <g(z)\left(D^2 w_1\right)^2> + 2V<g^2(z)Dw_1 D^2 w_1> + a^4<g(z)w_1^2> \\ &- 2a^2<g(z)w_1 D^2 w_1> + 2V<g^2(z)w_1 D^3 w_1> \\ &- 2\,Va^2<g^2(z)w_1 Dw_1> + 4V^2<g^3(z)w_1 D^2 w_1> \\ &+ 2\,V^2 a^2<g^3(z)w_1^2>, \end{aligned}$$

$$x_2 = -<w_1\, D^2\, w_1> + a^2<w_1^2>,$$

$$x_3 = <w_1 T_1>,\quad x_4 = -<T_1 D^2 T_1> + a^2<T_1^2>,\quad x_5 = <T_1^2>,$$

$$N_1 = \frac{1 + \Lambda_1^2 Q\omega^2}{1 + \Lambda_1^2\omega^2},\quad N_2 = \frac{\Lambda_1(1-Q)}{1 + \Lambda_1^2\omega^2},\quad N_3 = x_1 x_5 N_1 - x_1 x_4 N_2 + \frac{x_2 x_4}{\mathrm{Pr}}.$$

The eigen value M, given by Eq. (15), is real and therefore the imaginary part of Eq. (15) must be zero, ie., $\omega N_3 = 0$. Note that the other

quantities in the denominator of the imaginary part cannot be zero. We have here two possibilities: either $\omega = 0$ ($N_3 \neq 0$) or $N_3 = 0$ ($\omega \neq 0$). The oscillatory instability is the most interesting case in the case of viscoelastic liquids and hence we consider only that in what follows.

Oscillatory Instability (when $\omega \neq 0$ and $N_3 = 0$):

Eq. (15) in this case is given by

$$M_o = \frac{-x_1 x_5 \omega^2 N_2 - x_1 x_4 N_1 + \frac{x_2 x_5 \omega^2}{Pr}}{Dw(1)T(1)a^2 x_3},$$
(16)

where ω^2 can be obtained by taking $N_3 = 0$ and this gives us:

$$\omega^2 = -\frac{Pr x_1 [\Lambda_1 (Q-1) x_4 - x_5] + x_2 x_4}{\Lambda_1^2 [x_2 x_4 + Pr Q x_1 x_5]}.$$
(17)

In the present example when we take $N_3 = 0$, we get an expression for ω^2 and this indicates the possibility of oscillatory convection. In problems wherein the principle of exchange of stabilities is valid, it is found that N_3 is independent of ω. Based on this observation we may make an inference on the validity or otherwise of the principle of exchange of stabilities for the three types of viscoelastic liquids considered in the paper. The following table documents the validity or otherwise of the principle of exchange of stabilities in the three different liquids studied in the paper.

Q	Type of liquid	Principle of exchange of stabilities
∞	Rivlin–Ericksen	Valid
0	Maxwell	Not valid
$0 < Q < 1$	Jeffreys	Not valid

The validity of the principle of exchange of stabilities in the case of the Rivlin–Ericksen liquid is proved in the section on "Results and discussions". We now move over to the discussion of the computed results obtained by the Galerkin technique for all four boundary combinations.

Table 1
Different boundary combinations and corresponding trial functions for Bénard–Marangoni convection.

Case	Boundary	Boundary condition (BC)		Acronym for boundary condition	Trial functions
1	$z=0$	$w = D\,w = 0$ $T=0$ $(1+Q\Lambda_1 i\,\omega)D^2 w +$	Rigid isothermal	RIFA	$w_1 = z^3 - z^2,$
	$z=1$	$Ma^2[1+V(1-z)]$ $(1+\Lambda_1 i\,\omega)T=0$ $DT=0$	Free adiabatic		$T_1 = z^2 - 2z.$
2	$z=0$	$w = D^2 w = 0$ $T=0$	Free isothermal	FIFA	$w_1 = z^3 - z,$
	$z=1$	$(1+Q\Lambda_1 i\,\omega)$ $D^2 w + Ma^2$ $[1+V(1-z)]$ $(1+\Lambda_1 i\,\omega)$ $T=0$ $DT=0$	Free adiabatic		$T_1 = z^2 - 2z.$
3	$z=0$	$w = D\,w = 0$ $DT=0$	Rigid adiabatic	RAFA	$w_1 = z^3 - z^2,$
	$z=1$	$(1+Q\Lambda_1 i\,\omega)$ $D^2 w + Ma^2[1+V(1-z)]$ $(1+\Lambda_1 i\,\omega)T=0$ $DT=0$	Free adiabatic		$T_1 = 1.$
4	$z=0$	$w = D^2 w = 0$ $DT=0$	Free adiabatic	FAFA	$w_1 = z^3 - z,$
	$z=1$	$(1+Q\Lambda_1 i\,\omega)$ $D^2 w + Ma^2[1+V(1-z)]$ $(1+\Lambda_1 i\,\omega)T=0$ $DT=0$	Free adiabatic		$T_1 = 1.$

3. Results and discussion

The paper deals with a linear stability analysis of the Bénard–Marangoni convection in Rivlin–Ericksen, Maxwell and Jeffreys liquids. In our calculations we have assumed a value of the Prandtl number larger than that assumed in the Newtonian liquids due to the fact that the magnitude of the dynamic viscosity in viscoelastic liquids is much greater than that of the Newtonian liquids. We now discuss the results on the Bénard–Marangoni convection that manifests via the oscillatory mode.

From Eq. (17) it is clear that the stationary convection is possible when $Q > 1$ or when $Q < 1$ for some range of parameters. Oscillatory convection may be possible only when $Q < 1$. This is true for all four boundary combinations considered and is also true in the case of constant viscosity.

The critical value of the wave number a_c and the frequency ω_c, which in turn gives the critical Marangoni number M_c for the onset of convection, was computed for various parameter ranges. The computations revealed that the effect of increasing Λ_1 and Pr is the decreasing of M_{oc} for Maxwell and Jeffreys liquids. The effect of increasing Q is the increasing of M_{oc}. We also find that for Maxwell and Jeffreys liquids M_{oc} decreases with increasing V, a_c and ω_c increase with increasing V for all the considered boundary combinations.

In the case of the Rivlin–Ericksen liquid that subscribes to the limitations discussed by Dunn and Rajagopal [27], oscillatory convection is not possible. This can be shown as follows:

For the Rivlin–Ericksen liquid, we have $\Lambda_1 = 0$ and hence M given by Eq. (15) takes the form:

$$M = \frac{-x_1 x_4 + \Lambda_2 x_1 x_5 \omega^2 + \frac{x_2 x_5 \omega^2}{Pr}}{Dw(1)T(1)a^2 x_3} - i\omega \left[\frac{\Lambda_2 x_1 x_4 + x_1 x_5 + \frac{x_2 x_4}{Pr}}{Dw(1)T(1)a^2 x_3}\right].$$

Clearly for M to be real the only possibility is $\omega = 0$, as can be seen quite easily from the imaginary part of M. Hence, we can conclude that the principle of exchange of stabilities is valid in the case of the Rivlin–Ericksen liquid. Thus the Rivlin–Ericksen liquid behaves like a Newtonian liquid while in a Bénard–Marangoni situation.

We now move over to the discussion on some general results.

General results:

(a) Comparing the results of the Rivlin–Ericksen (and Newtonian), Maxwell, and Jeffreys liquids the following observation is true (for all V):

$$M_{sc}^{Newtonian} = M_{sc}^{Rivlin-Ericksen} > M_{oc}^{Jeffreys} > M_{oc}^{Maxwell},$$
$$a_c^{Newtonian} = a_c^{Rivlin-Ericksen} < a_c^{Jeffreys} < a_c^{Maxwell},$$
$$\omega_c^{Jeffreys} < \omega_c^{Maxwell}.$$

(b) Comparing the results of all the considered boundary combinations, the following observation is true (for all V) in the case of Maxwell and Jeffreys liquids and is illustrated in Figs. 2–4:

$$M_{oc}^{RAFA} > M_{oc}^{RIFA} > M_{oc}^{FAFA} > M_{oc}^{FIFA},$$
$$a_c^{RIFA} > a_c^{RAFA} > a_c^{FIFA} > a_c^{FAFA},$$
$$\omega_c^{RIFA} > \omega_c^{FIFA} > \omega_c^{RAFA} > \omega_c^{FAFA}.$$

4. Conclusions

The results indicate that the effect of variable viscosity parameter is the destabilizing of the system. The results pertaining to the Jeffreys
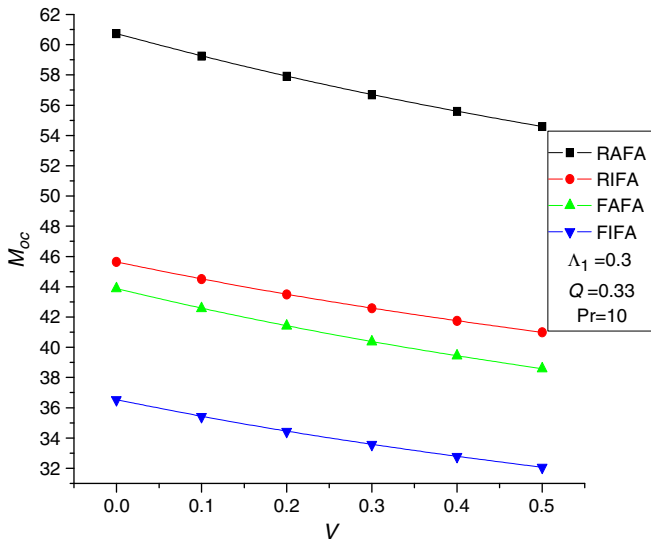
Fig. 2. Plot of M_{oc} vs. V for different boundary combinations, for fixed values of Λ_1, Q and Pr.

Fig. 4. Plot of ω_c vs. V for different boundary combinations, for fixed values of Λ_1, Q and Pr.

liquid lead to those of the Rivlin–Ericksen (and Newtonian liquids) and Maxwell liquids by suitable limiting processes. It is observed that the Maxwell liquids are found to be more unstable than the one subscribing to the Jeffreys description whereas the Rivlin–Ericksen liquids are comparatively more stable. It is clear from the obtained results that the Rivlin–Ericksen liquids behave like a Newtonian liquid when in a Bénard–Marangoni situation. This is true both in the absence and presence of temperature-dependence of viscosity. When $Q = 1$ our results coincide with those related results reported in Nield [28] and Siddheshwar et al. [29] for the Bénard–Marangoni convection.

At this point we note that our conclusions are based on the choice of T_0 (temperature of the upper boundary) as the reference temperature. As pointed out by Nield [22] the study can be carried out with $T_a = \frac{(T_0 + \Delta T) + T_0}{2}$ (average temperature of the two boundaries) as the r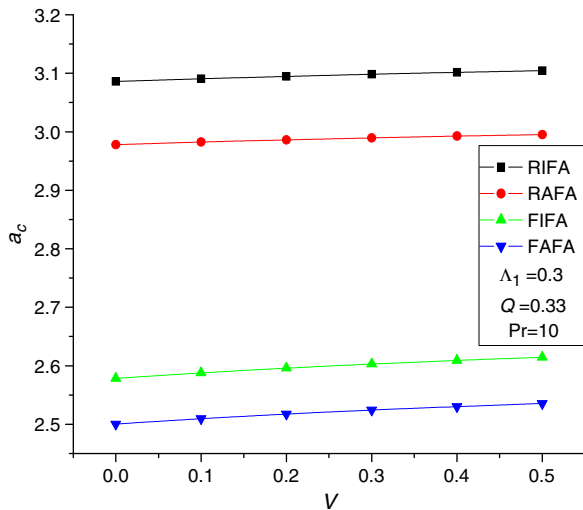eference temperature. With T_0 replaced by T_a in the above analysis we conclude the following in respect of all the considered boundary combinations:

(1) M_{oc} increases as V increases but the variation is weak as pointed out by Nield [22].
(2) a_c and ω_c increase as V increases but only in the second decimal digit.

The effect of each of Λ_1, Q and Pr on M_{oc}, a_c and ω_c is the same as what is seen when T_0 is used.

These results concerning viscosity variation with T_a as the reference temperature are in tune with qualitative observations made by Vanishree and Siddheshwar [30]. Further, we note here that a nonlinear realm of convection in variable-viscosity liquids has not been investigated so far and this work is under progress. Some such studies in constant viscosity viscoelastic liquids are reported in Siddheshwar et al. [31].

Appendix A. Boundary condition for Bénard–Marangoni convection

In the presence of temperature-dependent surface-tension, $\sigma(T)$, the boundary conditions for a variable viscosity, viscoelastic liquid can be obtained by equating the shear stresses at the surface to the variations of surface-tension, i.e.,

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right)\tau_{xz} = \left(1 + \lambda_2 \frac{\partial}{\partial t}\right)\left[\mu(T)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] = \left(1 + \lambda_1 \frac{\partial}{\partial t}\right)\frac{\partial \sigma}{\partial x}$$

(A1)

Fig. 3. Plot of a_c vs. V for different scombination scombinations, for fixed values of Λ_1, Q and Pr.

and

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\tau_{yz} = \left(1+\lambda_2\frac{\partial}{\partial t}\right)\left[\mu(T)\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right] = \left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial\sigma}{\partial y}.$$

(A2)

Here the variation of viscosity with temperature is given by Eq. (6).

Let $l_1 = \left(1+\lambda_1\frac{\partial}{\partial t}\right)$ and $l_2 = \left(1+\lambda_2\frac{\partial}{\partial t}\right)$.

Since $w = 0$ at $z = 0, d$ the following is also true:

$$\frac{\partial w}{\partial x} = 0, \frac{\partial w}{\partial y} = 0 \text{ at } z = 0, d.$$

(A3)

Differentiating Eq. (A1) w. r. t. x, Eq. (A2) w. r. t. y and using Eq. (A3) before adding the resulting expressions, we get

$$l_2\left[\mu\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\partial\mu}{\partial x}\frac{\partial u}{\partial z}+\frac{\partial\mu}{\partial y}\frac{\partial v}{\partial z}\right] = l_1\nabla_1^2\sigma.$$

(A4)

Using the continuity Eq. (1), Eq. (A4) now reduces to

$$l_2\left[-\mu\frac{\partial^2 w}{\partial z^2}+\frac{\partial\mu}{\partial x}\frac{\partial u}{\partial z}+\frac{\partial\mu}{\partial y}\frac{\partial v}{\partial z}\right] = l_1\nabla_1^2\sigma.$$

(A5)

The temperature-dependent surface tension, $\sigma(T)$, is assumed to have the form:

$$\sigma(T) = \sigma(T_0)-\sigma_T(T-T_0),$$

(A6)

where $\sigma_T = -\frac{d\sigma}{dT}|_{T=T_0}$.

Substituting Eq. (A6) in Eq. (A5), we get

$$l_2\left[\mu(T)\frac{\partial^2 w}{\partial z^2}-\frac{\partial\mu}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial\mu}{\partial y}\frac{\partial v}{\partial z}\right] = \sigma_T\nabla_1^2 T.$$

(A7)

In the perturbed state, Eq. (A7) takes the form:

$$l_2\mu_b\frac{\partial^2 w'}{\partial z^2} = l_1\sigma_T\nabla_1^2 T',$$

(A8)

where prime denotes the infinitesimal perturbed quantities and in arriving at the Eq. (A8) use has been made of the basic state solution. Non-dimensionalizing Eq. (A8) using the procedure explained in Section 2 and using the non-dimensional form of the expression for μ_b, we get

$$L_2\frac{\partial^2 w}{\partial z^2} = M[1 + V(1-z)]L_1\nabla_1^2 T,$$

(A9)

where the dimensionless parameter $M = \frac{\sigma_T\Delta T d}{\mu_0\kappa}$ is the Marangoni number, $L_1 = \left(1+\Lambda_1\frac{\partial}{\partial t}\right)$ and

$$L_2 = \left(1+Q\Lambda_1\frac{\partial}{\partial t}\right).$$

Using the normal mode solution (Eq. (10)) in Eq. (A9), we get

$$(1+Q\Lambda_1 i\omega)D^2 w + Ma^2[1 + V(1-z)](1 + \Lambda_1 i\omega)T = 0.$$

(A10)

Eq. (A10) is the required boundary condition for the Bénard–Marangoni convection at the upper boundary, $z = 1$. It is obvious from Eq. (A10) that the variable viscosity parameter V does not affect the boundary condition on the surface that has a surface tension variation. In other words, one can continue using the condition derived by Lebon et al. [16] in the case of variable viscosity liquids also.

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