Journal of Informatics and Mathematical Sciences
Vol. 7, No. 2, pp. 109–120, 2015
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications

http://www.rgnpublications.com

η-Einstein (k,μ)-Space Forms

Research Article

Savithri Shashidhar* and H. G. Nagaraja

Department of Mathematics, Bangalore University, Central College Campus, Bengaluru 560 001, India
***Corresponding author:** savithrishashi@yahoo.co.in

Abstract. In this paper we obtain the scalar curvatures of a (k,μ)-space form under h-projective, ϕ-projective semi symmetric and h-Weyl and ϕ-Weyl semisymmetry conditions.

Keywords. (k,μ)-space form; h-projective; ϕ-projective semi symmetric; h-Weyl; ϕ-Weyl semi symmetry

MSC. 53C40; 53C55; 53C25

Received: May 5, 2015 **Accepted:** August 28, 2015

1. Introduction

A class of (k,μ) contact metric manifolds [3] is of interest as it contains both the classes of Sasakian and non-Sasakian cases. The contact metric manifolds for which the characteristic vector field ξ belongs to (k,μ)-nullity distribution for some real numbers k and μ are called (k,μ) contact metric manifolds. A full classification of (k,μ)-contact metric manifolds was given by Boeckx [4] and many authors [2] studied (k,μ)-contact metric manifolds. T. Koufogiorgos proved in [1] that if a (k,μ)-space M has constant ϕ-sectional curvature c and dimension greater than 3, the curvature tensor of this (k,μ)-space form is given by

$$\begin{aligned}4R(X,Y)Z = {}&[(c+3)\{g(Y,Z)X-g(X,Z)Y\}+(c+3-4k)\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X\\&+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\}+(c-1)\{2g(X,\phi Y)\phi Z+g(X,\phi Z)\phi Y\\&-g(Y,\phi Z)\phi X\}-2\{g(hX,Z)hY-g(hY,Z)hX+2g(X,Z)hY-2g(Y,Z)hX\\&-2\eta(X)\eta(Z)hY+2\eta(Y)\eta(Z)hX+2g(hX,Z)Y-2g(hY,Z)X\\&+2g(hY,Z)\eta(X)\xi-2g(hX,Z)\eta(Y)\xi-g(\phi hX,Z)\phi hY+g(\phi hY,Z)\phi hX\}\\&+4\mu\{\eta(Y)\eta(Z)hX-\eta(X)\eta(Z)hY+g(hY,Z)\eta(X)\xi-g(hX,Z)\eta(Y)\xi\}],\end{aligned} \tag{1.1}$$

for any vector fields X, Y, Z, where $2h=L_\xi\phi$ and L is the usual Lie derivative.

The projective curvature tensor is an important tensor from the differential geometric point of view. An $(2n+1)$-dimensional Riemannian manifold M is locally projectively flat if their exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in Euclidean space. It is well-known that for $n \geq 1$, M is locally projectively flat if and only if the projective curvature tensor P vanishes. Here P is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{1.2}$$

where S is the Ricci tensor of M.

In an $(2n+1)$-dimensional Riemannian manifold, the conformal curvature tensor C is given by

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ - g(X,Z)QY] + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1.3}$$

where r is a scalar curvature and Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$.

The paper is organised as follows. In Section 2 we give some preliminary results of (k,μ)-space forms. Section 3 deals with h-projective and ϕ-projective semi-symmetric non-Sasakian (k,μ)-space forms. Section 4 is devoted to the study of h-Weyl and ϕ-Weyl semi-symmetric non-Sasakian (k,μ)-space forms. In all the cases the manifold becomes an η-Einstein manifold and we obtain scalar curvatures of (k,μ)-space forms.

2. Preliminaries

A $(2n+1)$-dimensional differential manifold M is said to admit an almost contact metric structure (ϕ,ξ,η,g) if it satisfies the following relations

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \tag{2.1}$$

$$\eta(\xi) = 1, g(X,\xi) = \eta(X), \quad \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(\phi X,Y) = -g(X,\phi Y), \quad g(\phi X,X) = 0, \tag{2.4}$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y), \tag{2.5}$$

$$g(X,\phi Y) = d\eta(X,Y), \tag{2.6}$$

for all vector fields X, Y on M. In a contact metric manifold the $(1,1)$ tensor field h defined by $h = \frac{1}{2}L_\xi \phi$, where L denotes the Lie differentiation, is a symmetric operator anti-commutative with ϕ and satisfies $h\xi = 0$, $h\phi = -\phi h$, $Tr\,h = Tr\,\phi h = 0$. Moreover in any contact metric manifold, we have $\nabla_X \xi = -\phi X - \phi h X$. In [3] Blair et al. introduced a class of contact metric manifold M which satisfy

$$R(X,Y)\xi = k\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \tag{2.7}$$

where k and μ are real constants. This class of contact metric manifolds are called (k,μ) manifolds.

Also in a (k,μ)-contact metric manifold, the following relations hold ([3], [4]):

$$h^2 = (k-1)\phi^2, k \le 1, \tag{2.8}$$

$$S(X,Y) = \frac{1}{4}\Big[(c(2n+1)+6n+4k-5)g(X,Y) - (c(2n+1)+6n+4k-5-8nk)\eta(X)\eta(Y) + (8-8n+4\mu)g(Y,hX)\Big], \tag{2.9}$$

$$r = \frac{n}{2}[c(2n+1)+6n+4k-5]+2nk, \tag{2.10}$$

$$S(X,\xi) = 8nk\eta(X),\ S(\xi,\xi) = 8nk,$$

where S is the Ricci tensor of the type $(0,2)$ and r is the scalar curvature of the manifold.

If $\mu = 0$, the (k,μ)-nullity distribution $N(k,\mu)$ is reduced to the k-nullity distribution [5], where k-nullity distribution $N(k)$ of a Riemannian manifold M is defined by

$$N(k): p \to N_p(k) = \{W \in T_p(M)/R(X,Y)W = k(g(Y,W)X - g(X,W)Y)\}.$$

If $\xi \in N(k)$, then we call M a $N(k)$-contact metric manifold.

The class of (k,μ)-contact metric manifolds contain both the class of Sasakian ($k=1$ and $h=0$) and non-Sasakian ($k \ne 1$ and $h \ne 0$) manifolds. Throughout the paper we denote by M^{2n+1}, a $(2n+1)$-dimensional non-Sasakian (k,μ)-space form. A contact metric manifold is said to be η-Einstein if $Q = aId + b\eta \otimes \xi$, where a, b are smooth functions on M^{2n+1}.

3. h-Projectively and ϕ-Projectively Semi Symmetric Non-Sasakian (k,μ)-Space Form

Definition 3.1. *A (k,μ)-space form M is said to be h-projectively semi-symmetric if $P(X,Y)\cdot h = 0$ holds in M.*

We now prove the following theorem.

Theorem 3.1. *Let M be a non-Sasakian (k,μ)-space form. If M is h-projectively semi-symmetric, then M is an η-Einstein manifold.*

Proof. Let M be an $(2n+1)$-dimensional h-projectively semi symmetric non-Sasakian (k,μ)-space form. The condition $P(X,Y)\cdot h = 0$ turns into

$$(P(X,Y)\cdot h)Z = P(X,Y)hZ - hP(X,Y)Z = 0. \tag{3.1}$$

From (1.1), we have

$$\begin{aligned}
&R(X,Y)hZ - hR(X,Y)Z \\
&= \frac{1}{4}[(c+3)\{g(Y,hZ)X - g(X,hZ)Y - g(Y,Z)hX + g(X,Z)hY\} \\
&\quad + (c+3-4k)\{g(X,hZ)\eta(Y)\xi - g(Y,hZ)\eta(X)\xi - \eta(X)\eta(Z)hY \\
&\quad + \eta(Y)\eta(Z)hX\} + (c-1)\{g(X,\phi hZ)\phi Y - g(Y,\phi hZ)\phi X \\
&\quad - g(X,\phi Z)h\phi Y + g(Y,\phi Z)h\phi X\} - 2\{g(hX,hZ)hY - g(hY,hZ)hX
\end{aligned}$$

$$
\begin{aligned}
&+2g(X,hZ)hY-2g(Y,hZ)hX+2g(hX,hZ)Y-2g(hY,hZ)X\\
&+2g(hY,hZ)\eta(X)\xi-2g(hX,hZ)\eta(Y)\xi-g(\phi hX,hZ)\phi hY\\
&+g(\phi hY,hZ)\phi hX+g(hX,Z)h^2Y-g(hY,Z)h^2X+2g(X,Z)h^2Y\\
&-2g(Y,Z)h^2X-2\eta(X)\eta(Z)h^2Y+2\eta(Y)\eta(Z)h^2X-g(\phi hX,Z)h\phi hY\\
&+g(\phi hY,Z)h\phi hX+2g(hX,Z)hY-2g(hY,Z)hX\}+4\mu\{g(hY,hZ)\eta(X)\xi\\
&-g(hX,hZ)\eta(Y)\xi+\eta(X)\eta(Z)h^2Y-\eta(Y)\eta(Z)h^2X\}]. \qquad (3.2)
\end{aligned}
$$

for any vector fields X, Y, Z. Using (1.3), (3.1) and (3.2), we have

$$
\begin{aligned}
&\frac{1}{4}[(c+3)\{g(Y,hZ)X-g(X,hZ)Y-g(Y,Z)hX+g(X,Z)hY\}\\
&+(c+3-4k)\{g(X,hZ)\eta(Y)\xi-g(Y,hZ)\eta(X)\xi-\eta(X)\eta(Z)hY\\
&+\eta(Y)\eta(Z)hX\}+(c-1)\{g(X,\phi hZ)\phi Y-g(Y,\phi hZ)\phi X-g(X,\phi Z)h\phi Y\\
&+g(Y,\phi Z)h\phi X\}-2\{g(hX,hZ)hY-g(hY,hZ)hX+2g(X,hZ)hY\\
&-2g(Y,hZ)hX+2g(hX,hZ)Y-2g(hY,hZ)X+2g(hY,hZ)\eta(X)\xi\\
&-2g(hX,hZ)\eta(Y)\xi g(\phi hX,hZ)\phi hY+g(\phi hY,hZ)\phi hX+g(hX,Z)h^2Y\\
&-g(hY,Z)h^2X+2g(X,Z)h^2Y-2g(Y,Z)h^2X-2\eta(X)\eta(Z)h^2Y+2\eta(Y)\eta(Z)h^2X\\
&-g(\phi hX,Z)h\phi hY+g(\phi hY,Z)h\phi hX+2g(hX,Z)hY-2g(hY,Z)hX\}\\
&+4\mu\{g(hY,hZ)\eta(X)\xi-g(hX,hZ)\eta(Y)\xi+\eta(X)\eta(Z)h^2Y-\eta(Y)\eta(Z)h^2X\}]\\
&-\frac{1}{2n}[S(Y,hZ)X-S(X,hZ)Y-S(Y,Z)hX+S(X,Z)hY]=0. \qquad (3.3)
\end{aligned}
$$

Replacing X by hX and contracting with W, by using (2.8) and symmetry property of h, we get

$$
\begin{aligned}
&\frac{1}{4}[(c+3)\{g(Y,hZ)g(hX,W)-g(hX,hZ)g(Y,W)+(k-1)g(Y,Z)g(X,W)\\
&-(k-1)g(Y,Z)\eta(X)\eta(W)+g(hX,Z)g(hY,W)\}+(c+3-4k)\{g(hX,hZ)\eta(Y)\eta(W)\\
&-(k-1)\eta(Y)\eta(Z)g(X,W)+(k-1)\eta(X)\eta(Z)\eta(Y)\eta(W)\}+(c-1)\{g(hX,\phi hZ)g(\phi Y,W)\\
&-g(Y,\phi hZ)g(\phi hX,W)-g(hX,\phi Z)g(h\phi Y,W)+(k-1)g(Y,\phi Z)g(\phi X,W)\}\\
&-2\{-(k-1)g(X,hZ)g(hY,W)+(k-1)[g(hY,hZ)g(X,W)-g(hY,hZ)\eta(X)\eta(W)\\
&+2g(hX,hZ)g(hY,W)+2(k-1)[g(Y,hZ)g(X,W)-g(Y,hZ)\eta(X)\eta(W)]\\
&-2(k-1)g(X,hZ)g(Y,W)-2g(hY,hZ)g(hX,W)+2(k-1)g(X,hZ)\eta(Y)\eta(W)\\
&+(k-1)g(\phi X,hZ)g(\phi hY,W)-(k-1)g(\phi hY,hZ)g(\phi X,W)+(k-1)^2[g(X,Z)g(Y,W)\\
&+2(k-1)[-g(hX,Z)g(Y,W)+g(hX,Z)\eta(Y)\eta(W)]+2(k-1)g(Y,Z)g(hX,W)\\
&-2(k-1)\eta(Y)\eta(Z)g(hX,W)+(k-1)^2g(\phi X,Z)g(\phi Y,W)+(k-1)g(\phi hY,Z)g(\phi hX,W)\\
&+2(k-1)[-g(X,Z)g(hY,W)+\eta(X)\eta(Z)g(hY,W)]+2(k-1)[g(hY,Z)g(X,W)\\
&-g(hY,Z)\eta(X)\eta(W)]]\}+4\mu\{(k-1)g(X,hZ)\eta(Y)\eta(W)+(k-1)\eta(Y)\eta(Z)g(hX,W)\}]\\
&-\frac{1}{2n}[S(Y,Z)g(hX,W)-S(hX,hZ)g(Y,W)+(k-1)[S(Y,Z)g(X,W)\\
&-S(Y,Z)\eta(X)\eta(W)]+S(hX,Z)g(hY,W)]=0. \qquad (3.4)
\end{aligned}
$$

Let e_i, $i = 1,2,3,\dots,2n+1$ be an orthonormal basis of vector fields in M. If we put $X = W = e_i$ in (3.4) and summing over i, then using (2.9), we obtain

$$\begin{aligned}&\frac{1}{4}\big[[(c+3)(k-1)2n+2(c-1)(k-1)-2(k-1)^2]g(Y,Z)\\&\quad+[-(c+3-4k)2n(k-1)-2(c-1)(k-1)+2(k-1)^2]\eta(Y)\eta(Z)\\&\quad+[(k-1)(-8+10n)]g(Y,hZ)\big]-(k-1)S(Y,Z)=0.\end{aligned}\tag{3.5}$$

Again using (2.9) in (3.5), we obtain

$$g(Y,hZ)=\left[\frac{6k-c-5}{18n-4\mu-16}\right]g(Y,Z)+\left[\frac{c-6k+5}{18n-4\mu-16}\right]\eta(Y)\eta(Z).\tag{3.6}$$

In view of (3.6), (3.5) takes the form

$$S(Y,Z)=A_1g(Y,Z)+B_1\eta(Y)\eta(Z),\tag{3.7}$$

where

$$\begin{aligned}A_1&=2n(c+3)+2(c-1)-2(k-1)+\frac{(8-10n)(c-6k+5)}{-16+18n-4\mu},\\B_1&=-2n(c+3-4k)-2(c-1)+2(k-1)+\frac{(-8+10n)(c-6k+5)}{-16+18n-4\mu}.\end{aligned}$$

Thus M is an η-Einstein manifold.

Taking $Y = Z = e_i$ in (3.7), we obtain

$$r=n\{(n+1)c+3n+k\}.\tag{3.8}$$

A h-projectively semi-symmetric non-Sasakian (k,μ)-space form is an η-Einstein manifold and the scalar curvature in this case is $r = n\{(n+1)c+3n+k\}$.

Comparing r of (3.8) with (2.10), we get $c = 6k-5$. □

Definition 3.2. *A (k,μ)-space form M is said to be ϕ-projectively semi-symmetric if $P(X,Y)\cdot\phi = 0$ holds in M.*

Theorem 3.2. *Let M be a non-Sasakian (k,μ)-space form-space form. If M is ϕ-projectively semi symmetric, then M is an η-Einstein manifold.*

Proof. Let M be an $(2n+1)$-dimensional ϕ-projectively semi-symmetric non-Sasakian (k,μ)-space form-space form. The condition $P(X,Y)\cdot\phi = 0$ turns into

$$(P(X,Y)\cdot\phi)Z=P(X,Y)\phi Z-\phi P(X,Y)Z=0.\tag{3.9}$$

From (1.1), we have

$$\begin{aligned}&R(X,Y)\phi Z-\phi R(X,Y)Z\\&\quad=\frac{1}{4}[(c+3)\{g(Y,\phi Z)X-g(X,\phi Z)Y-g(Y,Z)\phi X+g(X,Z)\phi Y\}\\&\qquad+(c+3-4k)\{g(X,\phi Z)\eta(Y)\xi-g(Y,\phi Z)\eta(X)\xi-\eta(X)\eta(Z)\phi Y\\&\qquad+\eta(Y)\eta(Z)\phi X\}+(c-1)\{g(X,\phi^2Z)\phi Y-g(Y,\phi^2Z)\phi X-g(X,\phi Z)\phi^2Y\end{aligned}$$

$$+g(Y,\phi Z)\phi^2 X\}-2\{g(hX,\phi Z)hY-g(hY,\phi Z)hX+2g(X,\phi Z)hY$$
$$-2g(Y,\phi Z)hX+2g(hX,\phi Z)Y-2g(hY,\phi Z)X+2g(hY,\phi Z)\eta(X)\xi$$
$$-2g(hX,\phi Z)\eta(Y)\xi-g(\phi hX,\phi Z)\phi hY+g(\phi hY,\phi Z)\phi hX$$
$$+g(hX,Z)\phi hY-g(hY,Z)\phi hX+2g(X,Z)\phi hY-2g(Y,Z)\phi hX$$
$$-2\eta(X)\eta(Z)\phi hY+2\eta(Y)\eta(Z)\phi hX-g(\phi hX,Z)\phi^2 hY g(\phi hY,Z)\phi^2 hX$$
$$+2g(hX,Z)\phi Y-2g(hY,Z)\phi X\}+4\mu\{g(hY,\phi Z)\eta(X)\xi$$
$$-g(hX,\phi Z)\eta(Y)\xi+\eta(X)\eta(Z)\phi hY-\eta(Y)\eta(Z)\phi hX\}]. \tag{3.10}$$

for any vector fields X, Y, Z. Using (1.3) and (3.9) in (3.10), we have

$$\frac{1}{4}[(c+3)\{g(Y,\phi Z)X-g(X,\phi Z)Y-g(Y,Z)\phi X+g(X,Z)\phi Y\}$$
$$+(c+3-4k)\{g(X,\phi Z)\eta(Y)\xi-g(Y,\phi Z)\eta(X)\xi-\eta(X)\eta(Z)\phi Y+\eta(Y)\eta(Z)\phi X\}$$
$$+(c-1)\{-g(X,Z)\phi Y+\eta(Z)\eta(X)\phi Y+g(Y,Z)\phi X-\eta(Z)\eta(Y)\phi X+g(X,\phi Z)\phi Y$$
$$-g(X,\phi Z)\eta(Y)\xi-g(Y,\phi Z)X+g(Y,\phi Z)\eta(X)\xi\}-2\{g(hX,\phi Z)hY-g(hY,\phi Z)hX$$
$$+2g(X,\phi Z)hY-2g(Y,\phi Z)hX+2g(hX,\phi Z)Y-2g(hY,\phi Z)X+2g(hY,\phi Z)\eta(X)\xi$$
$$-2g(hX,\phi Z)\eta(Y)\xi-g(\phi hX,\phi Z)\phi hY+g(\phi hY,\phi Z)\phi hX+g(hX,Z)\phi hY-g(hY,Z)\phi hX$$
$$+2g(X,Z)\phi hY-2g(Y,Z)\phi hX-2\eta(X)\eta(Z)\phi hY+2\eta(Y)\eta(Z)\phi hX$$
$$-g(\phi hX,Z)\phi^2 hY+g(\phi hY,Z)\phi^2 hX+2g(hX,Z)\phi Y-2g(hY,Z)\phi X\}$$
$$+4\mu\{g(hY,\phi Z)\eta(X)\xi-g(hX,\phi Z)\eta(Y)\xi+\eta(X)\eta(Z)\phi hY-\eta(Y)\eta(Z)\phi hX\}]$$
$$-\frac{1}{2n}[S(Y,\phi Z)X-S(X,\phi Z)Y-S(Y,Z)\phi X+S(X,Z)\phi Y]=0. \tag{3.11}$$

Replacing X by ϕX and contracting with W in (3.11) from (2.4), we obtain

$$\frac{1}{4}[(c+3)\{g(Y,\phi Z)g(\phi X,W)-g(\phi X,\phi Z)g(Y,W)+g(Y,Z)g(X,W)-g(Y,Z)\eta(X)\eta(W)$$
$$+g(\phi X,Z)g(\phi Y,W)\}+(c+3-4k)\{g(\phi X,\phi Z)\eta(Y)\eta(W)-\eta(Y)\eta(Z)g(X,W)$$
$$+\eta(Y)\eta(Z)\eta(X)\eta(W)\}+(c-1)\{-g(\phi X,Z)g(\phi Y,W)-g(Y,Z)g(X,W)+g(Y,Z)\eta(X)\eta(W)$$
$$+\eta(Z)\eta(Y)g(X,W)+g(X,Z)g(Y,W)-\eta(Z)\eta(X)g(Y,W)-g(X,Z)\eta(Y)\eta(W)$$
$$-g(Y,\phi Z)g(\phi X,W)\}-2\{-g(X,Z)g(hY,W)-g(hY,\phi Z)g(h\phi X,W)+2[-g(X,Z)g(hY,W)$$
$$+\eta(Z)\eta(X)g(hY,W)]-2g(Y,\phi Z)g(h\phi X,W)-2g(hX,Z)g(Y,W)-2g(hY,\phi Z)g(\phi X,W)$$
$$+2g(hX,Z)\eta(Y)\eta(W)-g(h\phi X,Z)g(h\phi Y,W)+g(hY,Z)g(hX,W)+g(h\phi X,Z)g(\phi hY,W)$$
$$-g(hY,Z)g(hX,W)+2g(\phi X,Z)g(\phi hY,W)-2g(Y,Z)g(hX,W)+2\eta(Y)\eta(Z)g(hX,W)$$
$$-g(\phi hY,Z)g(h\phi X,W)\}+4\mu\{g(hX,Z)\eta(Y)\eta(W)-\eta(Y)\eta(Z)g(hX,W)\}]$$
$$-\frac{1}{2n}[S(Y,\phi Z)g(\phi X,W)-S(\phi X,\phi Z)g(Y,W)+S(Y,Z)g(X,W)-S(Y,Z)\eta(X)\eta(W)$$
$$-\frac{1}{2n}[S(Y,\phi Z)g(\phi X,W)-S(\phi X,\phi Z)g(Y,W)+S(Y,Z)g(X,W)-S(Y,Z)\eta(X)\eta(W)$$
$$+S(\phi X,Z)g(\phi Y,W)]=0. \tag{3.12}$$

Let $\{e_i\}$, $i=1,2,\dots,2n+1$ be an orthonormal basis of vector fields in M. If we put $X=W=e_i$ in (3.11) and summing over i, then using (2.2) and (2.8), we obtain

$$\begin{aligned}\frac{1}{4}\Big[&\Big[8n+6k-14+\frac{c(2n+1)+6n+4k-5}{n}\Big]g(Y,Z)\\&+\Big[c+8nk-2n-2k-6-\frac{c(2n+1)+6n+4k-5}{n}\Big]\eta(Y)\eta(Z)\\&+18g(Z,hY)\Big]-S(Y,Z)=0.\end{aligned}\tag{3.13}$$

Using (2.9) in (3.13), we obtain

$$S(Y,Z)=A_2g(Y,Z)+B_2\eta(Y)\eta(Z),\tag{3.14}$$

where

$$A_2=\Big[\Big(8n+6k-14+\frac{c(2n+1)+6n+4k-5}{n}\Big)(c-17)\Big]-(c(2n+1)+6n+4k-5),$$
$$B_2=\Big[\Big(c+8nk-2n-2k-6-\frac{c(2n+1)+6n+4k-5}{n}\Big)(c-17)\Big]-(c(2n+1)+6n+4k-5).$$

Thus M is an η-Einstein manifold.

Taking $Y=Z=e_i$ in (3.13), we obtain

$$r=\frac{1}{4}[16n^2+20nk-16n+12k-14+4nc+2c].\tag{3.15}$$

A ϕ-projectively semi symmetric non-Sasakian (k,μ)-space form is an η-Einstein manifold and the scalar curvature in this case is

$$r=\frac{1}{4}[16n^2+20nk-16n+12k-14+4nc+2c],\tag{3.16}$$

Comparing r of ϕ-projectively semi-symmetric non-Sasakian (k,μ)-contact metric space form with (2.10), we get

$$c=\frac{2n^2+2nk+3n+6k-7}{2n^2-n-1}.$$ □

4. h-Weyl and ϕ-Weyl Semi-symmetric Non-Sasakian (k,μ)-Space Form

Definition 4.1. *A (k,μ)-space form M is said to be h-Weyl semi-symmetric if $C(X,Y)\cdot h=0$ holds on M.*

Theorem 4.1. *Let M be a non-Sasakian (k,μ)-space form-space form. If M is h-Weyl semi symmetric, then M is an η-Einstein manifold.*

Proof. Let M be an $(2n+1)$-dimensional h-Weyl semi symmetric non-Sasakian (k,μ)-space form-space form. The condition $C(X,Y)\cdot h=0$ turns into

$$(C(X,Y)\cdot h)Z=C(X,Y)hZ-hC(X,Y)Z=0,\tag{4.1}$$

for any vector fields X, Y, Z. Using (1.3)and (3.1) in (4.1), we have

$$\begin{aligned}
&\frac{1}{4}[(c+3)\{g(Y,hZ)X-g(X,hZ)Y-g(Y,Z)hX+g(X,Z)hY\}\\
&\quad+(c+3-4k)\{g(X,hZ)\eta(Y)\xi-g(Y,hZ)\eta(X)\xi-\eta(X)\eta(Z)hY\\
&\quad+\eta(Y)\eta(Z)hX\}+(c-1)\{g(X,\phi hZ)\phi Y-g(Y,\phi hZ)\phi X-g(X,\phi Z)h\phi Y\\
&\quad+g(Y,\phi Z)h\phi X\}-2\{g(hX,hZ)hY-g(hY,hZ)hX+2g(X,hZ)hY\\
&\quad-2g(Y,hZ)hX+2g(hX,hZ)Y-2g(hY,hZ)X+2g(hY,hZ)\eta(X)\xi\\
&\quad-2g(hX,hZ)\eta(Y)\xi-g(\phi hX,hZ)\phi hY+g(\phi hY,hZ)\phi hX+g(hX,Z)h^2Y\\
&\quad-g(hY,Z)h^2X+2g(X,Z)h^2Y-2g(Y,Z)h^2X-2\eta(X)\eta(Z)h^2Y\\
&\quad+2\eta(Y)\eta(Z)h^2X-g(\phi hX,Z)h\phi hY+g(\phi hY,Z)h\phi hX+2g(hX,Z)hY\\
&\quad-2g(hY,Z)hX\}+4\mu\{g(hY,hZ)\eta(X)\xi-g(hX,hZ)\eta(Y)\xi+\eta(X)\eta(Z)h^2Y\\
&\quad+\frac{r}{2n(2n-1)}\{g(Y,hZ)X-g(X,hZ)Y-g(Y,Z)hX+g(X,Z)hY\}=0. \qquad (4.2)
\end{aligned}$$

Replacing X by hX, contracting with W and using (4.2) and symmetry property of h, we obtain,

$$\begin{aligned}
&\frac{1}{4}[(c+3)\{g(Y,hZ)g(hX,W)-g(hX,hZ)g(Y,W)+(k-1)g(Y,Z)g(X,W)\\
&-(k-1)g(Y,Z)\eta(X)\eta(W)+g(hX,Z)g(hY,W)\}+(c+3-4k)\{g(hX,hZ)\eta(Y)\eta(W)\\
&-(k-1)\eta(Y)\eta(Z)g(X,W)+(k-1)\eta(X)\eta(Z)\eta(Y)\eta(W)\}+(c-1)\{g(hX,\phi hZ)g(\phi Y,W)\\
&-g(Y,\phi hZ)g(\phi hX,W)-g(hX,\phi Z)g(h\phi Y,W)+(k-1)g(Y,\phi Z)g(\phi X,W)\}\\
&-2\{-(k-1)g(X,hZ)g(hY,W)+(k-1)[g(hY,hZ)g(X,W)-g(hY,hZ)\eta(X)\eta(W)]\\
&+2g(hX,hZ)g(hY,W)+2(k-1)[g(Y,hZ)g(X,W)-g(Y,hZ)\eta(X)\eta(W)]\\
&-2(k-1)g(X,hZ)g(Y,W)-2g(hY,hZ)g(hX,W)+2(k-1)g(X,hZ)\eta(Y)\eta(W)\\
&+(k-1)g(\phi X,hZ)g(\phi hY,W)-(k-1)g(\phi hY,hZ)g(\phi X,W)+(k-1)^2[g(X,Z)g(Y,W)\\
&-g(X,Z)\eta(Y)\eta(W)-\eta(X)\eta(Z)g(Y,W)+\eta(X)\eta(Y)\eta(Z)\eta(W)]+(k-1)g(hY,Z)g(hX,W)\\
&+2(k-1)[-g(hX,Z)g(Y,W)+g(hX,Z)\eta(Y)\eta(W)]+2(k-1)g(Y,Z)g(hX,W)\\
&-g(hY,Z)\eta(X)\eta(W)]\}+4\mu\{(k-1)g(X,hZ)\eta(Y)\eta(W)+(k-1)\eta(Y)\eta(Z)g(hX,W)\}]\\
&-\frac{1}{2n-1}[S(Y,hZ)g(hX,W)+(k-1)[S(X,Z)g(Y,W)-2nk\eta(Z)\eta(X)g(Y,W)]\\
&+g(Y,hZ)g(QhX,W)+g(X,Z)g(QY,W)-\eta(Z)\eta(X)g(QY,W)+(k-1)[S(Y,Z)g(X,W)\\
&-S(Y,Z)\eta(X)\eta(W)]+S(hX,Z)g(hY,W)-g(Y,Z)g(hQhX,W)+g(Y,hZ)g(QhX,W)\\
&+g(X,Z)g(QY,W)-\eta(Z)\eta(X)g(QY,W)+(k-1)[S(Y,Z)g(X,W)-S(Y,Z)\eta(X)\eta(W)]\\
&+S(hX,Z)g(hY,W)-g(Y,Z)g(hQhX,W)+g(hX,Z)g(hQY,W)\}\\
&+\frac{r}{2n(2n-1)}\{g(Y,hZ)g(hX,W)+g(X,Z)g(Y,W)-\eta(X)\eta(Z)g(Y,W)\\
&+(k-1)[g(Y,Z)g(X,W)-g(Y,Z)\eta(X)\eta(W)]+g(hX,Z)g(hY,W)\}=0. \qquad (4.3)
\end{aligned}$$

Taking $Y = W = \xi$ in (4.3), we obtain

$$S(X,Z) = \left[k(2n-1) - \frac{2nk}{k-1} + \frac{r}{2n(k-1)}\right] g(X,Z) + \left[-k(2n-1) + \frac{2nk^2}{k-1} - \frac{r}{2n(k-1)}\right] \eta(X)\eta(Z) + 4\mu(2n-1)g(hX,Z) \tag{4.4}$$

Now using (2.9) in (4.4), we get

$$g(X,hZ) = \frac{1}{2-2n-8n\mu+5\mu}\left[\frac{8nk(2n-1)(k-1)-4n^2k+r}{2n(k-1)} - \frac{c(2n+1)+6n+4k-5}{4}\right] g(X,Z) + \left[\frac{-8nk(2n-1)(k-1)+4n^2k^2-r}{2n(k-1)} + \frac{c(2n+1)+6n+4k-5-8nk}{4}\right]\eta(X)\eta(Z) \tag{4.5}$$

Using (4.5) in (4.4), we obtain

$$S(X,Z) = A_1' g(X,Z) + B_1'\eta(X)\eta(Z), \tag{4.6}$$

where

$$A_1' = \frac{4\mu(2n-1)}{2-2n-8n\mu+5\mu}\left[\frac{8nk(2n-1)(k-1)-4n^2k+r}{2n(k-1)} - \frac{c(2n+1)+6n+4k-5}{4}\right] + \frac{2nk(2n-1)(k-1)++4n^2k^2-r}{2n(k-1)},$$

$$B_1' = \frac{4\mu(2n-1)}{2-2n-8n\mu+5\mu}\left[\frac{-8nk(2n-1)(k-1)+4n^2k^2-r}{2n(k-1)} + \frac{c(2n+1)+6n+4k-5-8nk}{4}\right] + \frac{-2nk(2n-1)(k-1)++4n^2k^2-r}{2n(k-1)}.$$

Thus M is an η-Einstein manifold.

Taking $X = Z = e_i$ in (4.4), we obtain

$$r = 4n^2. \tag{4.7}$$

An h-Weyl semi-symmetric non-Sasakian (k,μ)-space form is an η-Einstein manifold and the scalar curvature in this case is $4n^2$.

Comparing r of (4.7) with (2.10), we obtain

$$c = \frac{5+2n-8k}{2n+1}. \qquad \square$$

Definition 4.2. *A (k,μ)-space form M is said to be ϕ-Weyl semi-symmetric if $C(X,Y)\cdot\phi = 0$ holds on M.*

Theorem 4.2. *Let M be a non-Sasakian (k,μ)-space form. If M is ϕ-Weyl semi-symmetric, then M is an η-Einstein manifold.*

Proof. Let M be an $(2n+1)$-dimensional ϕ-Weyl semi-symmetric non-Sasakian (k,μ)-space form. The condition $C(X,Y)\cdot\phi=0$ turns into

$$(C(X,Y)\cdot\phi)Z = C(X,Y)\phi Z-\phi C(X,Y)Z=0, \tag{4.8}$$

for any vector fields X, Y, Z. Using (1.3)and (3.8) in (4.7), we have

$$\begin{aligned}
&\frac{1}{4}[(c+3)\{g(Y,\phi Z)X-g(X,\phi Z)Y-g(Y,Z)\phi X+g(X,Z)\phi Y\}\\
&\quad+(c+3-4k)\{g(X,\phi Z)\eta(Y)\xi-g(Y,\phi Z)\eta(X)\xi-\eta(X)\eta(Z)\phi Y+\eta(Y)\eta(Z)\phi X\}\\
&\quad+(c-1)\{g(X,\phi^2Z)\phi Y-g(Y,\phi^2Z)\phi X-g(X,\phi Z)\phi^2Y+g(y,\phi Z)\phi^2X\}-2\{g(hX,\phi Z)hY\\
&\quad-g(hY,\phi Z)hX+2g(X,\phi Z)hY-2g(Y,\phi Z)hX+2g(hX,\phi Z)Y-2g(hY,\phi Z)X\\
&\quad+2g(hY,\phi Z)\eta(X)\xi-2g(hX,\phi Z)\eta(Y)\xi-g(\phi hX,\phi Z)\phi hY+g(\phi hY,\phi Z)\phi hX+g(hX,Z)\phi hY\\
&\quad-g(hY,Z)\phi hX+2g(X,Z)\phi hY-2g(Y,Z)\phi hX-2\eta(X)\eta(Z)\phi hY+2\eta(Y)\eta(Z)\phi hX\\
&\quad-g(\phi hX,Z)\phi^2hY+g(\phi hY,Z)\phi^2hX+2g(hX,Z)\phi Y-2g(hY,Z)\phi X\}+4\mu\{g(hY,\phi Z)\eta(X)\xi\\
&\quad-g(hX,\phi Z)\eta(Y)\xi+\eta(X)\eta(Z)\phi hY-\eta(Y)\eta(Z)\phi hX\}]-\frac{1}{2n-1}[S(Y,\phi Z)X-S(X,\phi Z)Y\\
&\quad+g(Y,\phi Z)QX-g(X,\phi Z)QY-S(Y,Z)\phi X+S(X,Z)\phi Y-g(Y,Z)\phi QX+g(X,Z)\phi QY\}\\
&\quad+\frac{r}{2n(2n-1)}[g(Y,\phi Z)X-g(X,\phi Z)Y-g(Y,Z)\phi X+g(X,Z)\phi Y]=0.
\end{aligned}\tag{4.9}$$

Replacing X by ϕX, contracting with W and using (4.9) and symmetry property of h, we obtain,

$$\begin{aligned}
&\frac{1}{4}[(c+3)\{g(Y,\phi Z)g(\phi X,W)-g(\phi X,\phi Z)g(Y,W)+g(Y,Z)g(X,W)-g(Y,Z)\eta(X)\eta(W)\\
&\quad+g(\phi X,Z)g(\phi Y,W)\}+(c+3-4k)\{g(\phi X,\phi Z)\eta(Y)\eta(W)-\eta(Y)\eta(Z)g(X,W)\\
&\quad+\eta(X)\eta(Z)\eta(Y)\eta(W)\}+(c-1)\{-g(\phi X,Z)g(\phi Y,W)-g(Y,Z)g(X,W)+g(Y,Z)\eta(X)\eta(W)\\
&\quad+\eta(Y)\eta(Z)g(X,W)+g(X,Z)g(Y,W)-\eta(X)\eta(Z)g(Y,W)-g(X,Z)\eta(Y)\eta(W)\\
&\quad-g(Y,\phi Z)g(\phi X,W)\}-2\{-g(X,Z)g(hY,W)-g(hy,\phi Z)g(h\phi X,W)+2[-g(X,Z)g(hY,W)\\
&\quad+\eta(X)\eta(Z)g(hY,W)]-2g(Y,\phi Z)g(h\phi X,W)-2g(hX,Z)g(Y,W)-2g(hY,\phi Z)g(\phi X,W)\\
&\quad+2g(hX,Z)\eta(Y)\eta(W)-g(h\phi X,Z)g(h\phi Y,W)+g(hY,Z)g(hX,W)+g(h\phi X,Z)g(h\phi Y,W)\\
&\quad-g(hY,Z)g(hX,W)+2g(h\phi X,Z)g(\phi hY,W)-2g(Y,Z)g(hX,W)+2\eta(Y)\eta(Z)g(hX,W)\\
&\quad+2g(h\phi X,Z)g(\phi Y,W)+2[g(hY,Z)g(X,W)-g(hY,Z)\eta(X)\eta(W)]+g(hX,Z)g(hY,W)\\
&\quad-g(\phi hY,Z)g(hX,W)\}+4\mu\{g(hX,Z)\eta(Y)\eta(W)-\eta(Y)\eta(Z)g(hX,W)\}]\\
&\quad-\frac{1}{2n-1}[S(Y,\phi Z)g(\phi X,W)-S(\phi X,\phi Z)g(Y,W)+g(Y,\phi Z)S(\phi X,W)-g(\phi X,\phi Z)S(Y,W)\\
&\quad-S(Y,Z)g(\phi^2X,W)+S(\phi X,Z)g(\phi Y,W)-g(Y,Z)g(\phi Q\phi X,W)+g(\phi X,Z)g(\phi QY,W)\}\\
&\quad+\frac{r}{2n(2n-1)}\{g(Y,\phi Z)g(\phi X,W)-g(\phi X,\phi Z)g(Y,W)-g(Y,Z)g(\phi^2X,W)\\
&\quad+g(\phi X,Z)g(\phi Y,W)\}=0.
\end{aligned}\tag{4.10}$$

Taking $Y = W = \xi$ in (4.10), we obtain

$$\left[\frac{16nk+2n^2c+nc+6n^2-5n-2r}{4n(2n-1)}\right]g(X,Z) + \left[\frac{-16nk-2n^2c-nc-6n^2+5n+2r}{4n(2n-1)}\right]\eta(X)\eta(Z)+\frac{2\mu n-8n+3\mu+8}{2n-1}g(hX,Z)=0. \tag{4.11}$$

Using (2.9) in (4.11), we get

$$S(X,Z)=A_2'g(X,Z)+B_2'\eta(X)\eta(Z), \tag{4.12}$$

where

$$A_2'=\frac{-(8-8n+\mu)(16nk+2n^2c+nc+6n^2-5n-2r)}{16n(2\mu n-8n+3\mu+8)}+\frac{c(2n+1)+6n+4k-5}{4},$$

$$B_2'=\frac{(8-8n+\mu)(16nk+2n^2c+nc+6n^2-5n-2r)}{16n(2\mu n-8n+3\mu+8)}-\frac{c(2n+1)+6n+4k-5}{4}.$$

Thus M is an η-Einstein manifold.

Taking $X = Z = e_i$ in (4.12), we obtain

$$r=\frac{(8-8n+\mu)(16nk+2n^2c+nc+6n^2-5n)}{48n-16\mu n-22\mu-48}.$$

A ϕ-Weyl non-Sasakian (k,μ)-space form is an η-Einstein manifold and the scalar curvature in this case is given by

$$r=\frac{(8-8n+\mu)(16nk+2n^2c+nc+6n^2-5n)}{48n-16\mu n-22\mu-48}. \tag{4.13}$$

Comparing r of (4.13) with (2.10), we obtain

$$c=\frac{-12\mu n^2+24n^2-5\mu n-64n-16k\mu n-80nk-26\mu k-16k+15\mu+40}{4\mu n^2-16n^2+7\mu n+4n+3\mu+8}. \qquad \square$$

Table 1. Scalar curvature of a non-Sasakian (k,μ)-space form M

S. No.	Curvature tensor	Condition	Scalar curvature
1	Projective curvature tensor	$P(X,Y)\cdot h=0$	$r=n\{(n+1)c+3n+k\}$
2	Projective curvature tensor	$P(X,Y)\cdot\phi=0$	$r=\frac{1}{4}[16n(n-1)+4k(5n+3)]+\frac{1}{4}[2c(2n+1)-14]$
3	Conformal curvature tensor	$C(X,Y)\cdot h=0$	$r=4n^2$
4	Conformal curvature tensor	$C(X,Y)\cdot\phi=0$	$r=\frac{(8-8n+\mu)(16nk+2n^2c+nc+6n^2-5n)}{48n-16\mu n-22\mu-48}$

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] T. Koufogiorgos, Contact Riemannian manifolds with constant ϕ-sectional curvature, *Tokyo J. Math.* **20** (1) (1997), 55–57.

[2] H. G. Nagaraja and C. R. Premalatha, Curvature properties and η-Einstein (k,μ)-contact metric manifolds, *ISRN Geometry* Vol. **2012**, Article ID 970682, 18 pages, doi:10.5402/2012/970682.

[3] D. E. Blair, T. Koufogiorgos and B. J. Papantoaiou, Contact metric manifolds satisfying nullity condition, *Israel J. Math.* **91** (1-3) (1995), 189–214.

[4] E. Boeckx, A full classification of contact metric (k,μ)-spaces, *Illinois J. Math.* **44** (1) (2000), 212–219.

[5] S. Tanno, Ricci curvature of contact Riemannian manifolds, *Tohoku Math. J.* **40** (3) (1988), 441–448.