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On coefficients of edge domination polynomial of a graph

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Abstract

An edge domination polynomial of a graph G is the polynomial $D_e(G,x)=\sum_{t=\gamma_e(G)}^{m} d_e(G,t)x^t$, where $d_e(G,t)$ is the number of edge dominating sets of G of cardinality t. In this paper, we provide tables which contain coefficient of edge domination polynomial of path and cycle. Also, certain properties of edge dominating polynomial are given.

Keywords: *Graph, edge domination number, domination polynomial, edge domination polynomial.*

2010 Mathematics Subject Classication: *05C69, 05C70*

1. Introduction

All the graphs $G = (V, E)$ considered here are simple, finite, nontrivial and undirected, where $|V| = n$ denotes number of vertices and $|E| = m$ denotes number of edges of G. Let $V = V_1 \cup V_2$, where V_1 and V_2 be two partitions of the vertex set of G. The graph G^c is called complement of a graph G, if G and G^c have the same vertex set and two vertices are adjacent in G if and only if they are not adjacent in G^c. The line graph of a graph

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G, denoted by *L*(*G*) is derived graph where the vertices of *L*(*G*) are the lines of *G*, with two vertices of *L*(*G*) adjacent whenever the corresponding lines of *G* are adjacent. A collection of independent edges of a graph *G* is called a matching of *G*. If there is a matching consisting of all vertices of *G* it is called a perfect matching. The number of distinct subsets with *r* vertices that can be selected from a set with *n* vertices is denoted by $\binom{n}{r}$ or $nC_r = \frac{n!}{(n-r)!r!}$. This number $\binom{n}{r}$ is called a binomial coefficient. For any undefined term in this paper, we refer Harary [7].

A set $D \subseteq V$ is a dominating set if every vertex not in *D* is adjacent to one or more vertices in *D*. The minimum cardinality taken over all dominating sets in *G* is called domination number $\gamma(G)$. For a complete review on theory of domination, we follow [8].

A set $S \subseteq E$ is an edge dominating set if every edge not in *S* is adjacent to one or more vertices in *S*. The minimum cardinality taken over all edge dominating sets in *G* is called edge domination number $\gamma_e(G)$. The concept of edge domination was initiated by Mitchell et al.[10] and studied by [5], [9] and [11].

A domination polynomial of a graph *G* is the polynomial $D(G,x) = \sum_{t=\gamma(G)}^{n} d(G,t)x^t$, where $d(G, t)$ is number of dominating sets of *G* of cardinality *t*. Domination polynomial was initiated by Arocha et al. [4] and later studied by Alikhani et al. [1], [2] and [3].

Analogously, edge domination polynomial was studied by Askari et al. [5]. An edge domination polynomial of a graph *G* is the polynomial $D_e(G,x) = \sum_{t=\gamma_e(G)}^{m} d_e(G,t)x^t$, where $d_e(G, t)$ is the number of edge dominating sets of *G* of cardinality *t*. In this paper, we obtain further results on edge domination polynomial.

An element *a* is said to be zero of a polynomial $f(x)$ if $f(a) = 0$. An element *a* is called a zero of a polynomial of multiplicity *p* if $(x-a)^p / f(x)$ and $(x-a)^{p+1}$ is not a divisor of $f(x)$. A polynomial in which coefficient of highest order term is 1 is monic polynomial.

2. Results

Theorem 2.1: *For any nontrivial graph G,*

(i) $d_e(G,t) \neq 0$ for $t = \gamma_e(G)$ to *m*.

(ii) $D_e(G,x)$ does not have a constant term.

(iii) $D_e(G,x)$ is a monic polynomial.

(iv) $x^{\gamma_e(G)}$ is a divisor of $D_e(G,x)$.

(v) $x = 0$ is the zero of $D_e(G,x)$ of multiplicity $\gamma_e(G)$.

(vi) $d_e(G, m) = 1$ and $d_e(G, m - 1) = m$.

Proof: Let G be a graph with n vertices and m edges.

(i) $d_e(G,t)$ denotes number of edge dominating sets with cardinality t. As nontrivial graph will have an edge dominating set of minimum cardinality 1, (i) follows.

(ii) As $D_e(G,x)=\sum_{t=\gamma_e(G)}^{m} d_e(G,t)x^t$ and $\gamma_e(G)\geq 1$, it follows that every term of $D_e(G,x)$ has an x in it. Hence there is no constant term.

(iii) Coefficient of x^m in $D_e(G,x)$ is $d_e(G,m)$ which is the number of edge dominating set with cardinality m. That is $d_e(G,m)=mC_m=1$. Highest power of x in $D_e(G,x)$ is 1 which implies $D_e(G,x)$ is a monic polynomial.

(iv) Since t ranges from $\gamma_e(G)$ to m, least power of x is $\gamma_e(G)$ and highest power of x is m. Also from (ii) $D_e(G,x)$ has no constant term. Hence $\gamma_e(G)$ is a divisor of $D_e(G,x)$.

(v) If $D_e(G,x)=0$, from (iv) it follows that $x = 0$ is zero of $D\ (G,x)$ of multiplicity $\gamma_e(G)$.

(vi) $d_e(G,m)=mC_m=1$ and $d_e(G,m-1)=mC_{m-1}=m$.

Theorem 2.2: *For any graph* $G\cong K_{r,s}$ *with* $1\leq r\leq s$ *vertices,*

$$D_e(G,x)=[(1+x)^s-1]^r.$$

Proof: We shall prove the result for r = 1. As $\gamma_e(G) = \gamma(L(G))$, we shall find domination set of $L(G)$. If $G \cong K_{1,s}, L(G) \cong K_s$, for which $\gamma(L(G)) = 1$. For $d(L(G),1)$: choose any one vertex from s vertices of $L(G)$, which can be done in sC_1 ways. For $d(L(G),2)$ choose 2 vertices out of s vertices which can be done in sC_2 ways. Continuing this procedure till s terms, $d(L(G),s) = sC_s$. The edge domination polynomial for r = 1 is $D_e(G,x) = D(L(G),x) = \sum_{t=1}^{s} d(L(G),t) = sC_1x + sC_2x^2 + ... + sC_sx^s = (1+x)^s - 1$.

For $r = 2$, that is $|V_1| = 2$ and $|V_2| = s \geq 2$. Thus the number of edges of G are $2s$ edges. Let E_1 be set of edges incident to a vertex of V_1 and E_2 be set of edges incident to another vertex of V_1. An edge of E_1 dominates remaining $(s - 1)$ edges of E_1. Similarly an edge of E_2 dominates remaining $(s - 1)$ edges of E_2. Thus $\gamma_e(G) = 2$ and $t \in \{2, 3, ..., 2s\}$. The edges of E_1 along with vertices on which they are incident forms graph $K_{1,s}$ and similarly edges of E_2 along with vertices on which they are incident forms graph $K_{1,s}$. For an edge dominating set E of cardinality t, all t vertices cannot be selected only from E_1 or only from E_2 as it leads to contradiction of E being edge dominating set. Hence j edges are selected from E_1 and $(t - j)$ edges are selected from E_2. The total number of ways of doing this is the coefficient of x^t in $D_e(K_{1,s},x)D_e(K_{1,s},x)$. Hence $D_e(G,x) = [(1+x)^s - 1]^2$.

The above method can be followed to prove the result for a graph G with $|V_1| = r$.

To prove next result, we use the following definition:

The corona of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2.

Theorem 2.3: *Let $H = K_n^c \circ K_1$, be corona of graphs K_n^c and K_1 with $n \geq 1$ vertices. Then,*

$$D_e(H,x) = x^n,$$

where K_n^c is complement of complete graph.

Proof: Since K_n^c has n vertices, H has $2n$ vertices connected by n edges. There is a perfect matching. The edge dominating set of H consist of all the edges of H. Hence $D_e(H,x) = x^n$.

To prove next result, we use Theorem stated and proved in [3].

Theorem 2.4: *For any path P_n with $n \geq 4$ vertices,*

(i) $d(P_n, t) = d(P_{n-1}, t-1) + d(P_{n-2}, t-1) + d(P_{n-3}, t-1).$

(ii) $D(P_n, x) = x[D(P_{n-1}, x) + D(P_{n-2}, x) + D(P_{n-3}, x)].$

Theorem 2.5: *For any path p_n with $n \geq 5$ vertices,*

(i) $d_e(P_n, t) = d_e(P_{n-1}, t-1) + d_e(P_{n-2}, t-1) + d_e(P_{n-3}, t-1).$

(ii) $D_e(P_n, x) = x[D_e(P_{n-1}, x) + D_e(P_{n-2}, x) + D_e(P_{n-3}, x)].$

Proof: For any graph G, edge domination of G is same as vertex domination of line graph of G and for a path P_n, $L(P_n) \cong P_{n-1}$. Hence in the above theorem replace n by $n - 1$, that is $d_e(P_n, t) = d(P_{n-1}, t)$ and $D_e(P_n, x) = D(P_{n-1}, x)$. Hence the proof.

With this theorem we form a table for $d_e(P_n, t)$.

	t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
n																
1		-														
2		1														
3		2	1													
4		1	3	1												
5		0	4	4	1											
6		0	3	8	5	1										
7		0	1	10	13	6	1									
8		0	0	8	22	19	7	1								
9		0	0	4	26	40	26	8	1							
10		0	0	1	22	61	65	34	9	1						
11		0	0	0	13	70	20	98	43	10	1					
12		0	0	0	5	61	71	211	140	53	11	1				
13		0	0	0	1	40	92	356	343	192	64	12	1			
14		0	0	0	0	19	71	483	665	526	255	76	13	1		
15		0	0	0	0	6	20	534	050	1148	71	330	89	14	1	

Theorem 2.6. *From the above table, we get following properties:*

(i) $d_e(P_n, n-1) = 1.$

(ii) $d_e(P_n, n-2) = n-1.$

(iii) $d_e(P_n, t) = 0$ for $t \leq \left\lceil \frac{n-1}{3} \right\rceil.$

(iv) $d_e(P_{3n+1}, n) = 1.$

(v) $d_e(P_n, n-3) = (n-1)C_2 - 2$

(vi) $d_e(P_{3n}, n) = n+1$

(vii) $d_e(P_{3n-1}, n) = \frac{(n+1)(n+2)}{2} - 2.$

(viii) For $n \in N$ and $k = 0, 1, 2, \ldots, n-1$,

$$d_e(P_{2n+2+k}, n) = d_e(P_{2n-k}, n).$$

Proof: Let $G \cong P_n$ with $n \geq 2$ vertices. Then G has $m = (n-1)$ edges.

(i) For $d_e(P_n, n-1)$, choose $(n-1)$ edges from $(n-1)$ edges of G which can be done in $(n-1)C_{(n-1)}$ ways.

(ii) For $d_e(P_n, n-2)$, choose $(n-2)$ edges from $(n-1)$ edges of G which can be done in $(n-1)C_{n-2} = n-1$ ways.

(iii) For a path with $m = n-1$ edges, for every three edges there is one dominating edge belonging to the edge dominating set. For $m \geq 4$ there is no edge dominating set with cardinality one. Also for $m \geq 7$ there is no edge dominating set with cardinality one and two. Hence $d_e(G, t) = 0$ for $t \leq \left\lceil \frac{m}{3} \right\rceil.$

(iv) From Theorem 2.5, $d_e(P_{3n+1}, n) = d_e(P_{3n}, n-1) + d_e(P_{3n-1}, n-1) + d_e(P_{3n-2}, n-1).$ Since $d_e(P_4, 1) = 1$ and from (i), (ii) and (iii) result follows.

(v) We shall prove this result by induction hypothesis on $n - 1$ with the condition $d_e(P_4, 1) = 1$. Assume the result to be true for a graph G with $(n - 2)$ edges. We shall prove the result for a graph with $n-1$ edges. From Theorem 2.5, $d_e(P_n, n-3) = d_e(P_{n-1}, n-4) + d_e(P_{n-2}, n-4) + d_e(P_{n-3}, n-4)$. From (i) and (ii)

$$\begin{aligned} d_e(P_n, n-3) &= (n-2)C_2 - 2 + n - 3 + 1, \\ &= (n-1)C_2 - 2. \end{aligned}$$

(vi) We shall prove this result by induction hypothesis on n with the condition $d_e(P_2, 1) = 2$. Assume the result to be true for a graph G with $(n - 1)$ edges. We shall prove the result for a graph with n edges. From Theorem 2.5, $d_e(P_{3n}, n) = d_e(P_{3n-1}, n-1) + d_e(P_{3n-2}, n-1) + d_e(P_{3n-3}, n-1)$. From (iii)

$$\begin{aligned} d_e(P_{3n}, n) &= d_e(P_{3(n-1)+2}, n-1) + d_e(P_{3(n-1)+1}, n-1) + d_e(P_{3(n-1)}, n-1), \\ &= 1 + n. \end{aligned}$$

(vii) We shall use induction hypothesis to prove the result. Assume the result is true for a graph G with $(n - 1)$ edges. We shall prove the result for a graph with n edges. From theorem 2.5,

$$\begin{aligned} d_e(P_{3n-1}, n) &= d_e(P_{3n-2}, n-1) + d_e(P_{3n-3}, n-1) + d_e(P_{3n-4}, n-1), \\ &= d_e(P_{3(n-1)+1}, n-1) + d_e(P_{3(n-1)}, n-1) + d_e(P_{3(n-1)-1}, n-1). \end{aligned}$$

Using results of (iii) and (iv), we have

$$\begin{aligned} d_e(P_{3n-1}, n) &= 1 + n - 1 + 1 + \frac{n(n+1)}{2} - 2, \\ &= \frac{(n+1)(n+2)}{2} - 2. \end{aligned}$$

(viii) The proof is by induction hypothesis on n. Since $d_e(P_2, 1) = d_e(p_4, 1)$, the result is true for $n = 1$. Assume the result to be true for $n - 1$ edges. We shall prove the result in a graph with n edges. By Theorem 2.5,

$$d_e(P_{2n-k}, n) = d_e(P_{2n-1-k}, n-1) + d_e(P_{2n-2-k}, n-1) + d_e(P_{2n-3-k}, n-1),$$

$$= d_e(P_{2(n-1)+1-k}, n-1) + d_e(P_{2(n-1)-k}, n-1) + d_e(P_{2(n-1)-1-k}, n-1),$$
$$= d_e(P_{2(n-1)+2+k-1}, n-1) + d_e(P_{2(n-1)+2+k}, n-1) + d_e(P_{2(n-1)+2+k+1}, n-1),$$
$$= d_e(P_{2n-1+k}, n-1) + d_e(P_{2n+k}, n-1) + d_e(P_{2n+1+k}, n-1),$$
$$= d_e(P_{2n+2+k}, n).$$

To prove our next result, we make use of the following Theorem [2].

Theorem 2.7: *For any cycle C_n with $n \geq 4$ vertices,*

(i) $d(C_n, t) = d(C_{n-1}, t-1) + d(C_{n-2}, t-1) + d(C_{n-3}, t-1)$.

(ii) $D(C_n, x) = x[D(C_{n-1}, x) + D(C_{n-2}, x) + D(C_{n-3}, x)]$.

Theorem 2.8: *For any cycle C_n with $n \geq 5$ vertices,*

(i) $d_e(C_n, t) = d(C_{n-1}, t-1) + d(C_{n-2}, t-1) + d(C_{n-3}, t-1)$.

(ii) $D_e(C_n, x) = x[D(C_{n-1}, x) + D(C_{n-2}, x) + D(C_{n-3}, x)]$.

Proof: As the edge domination of G is same as the vertex domination of line graph $L(G)$ of G. For C_n, $L(C_n) \cong C_n$. Hence, from above Theorem the result follows.

From the above Theorem, we form a table 2 for $d_e(C_n, t)$

n \ t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	3	3	1												
4	0	6	4	1											
5	0	5	10	5	1										
6	0	3	14	15	6	1									
7	0	0	14	28	21	7	1								
8	0	0	8	38	48	28	8	1							
9	0	0	3	36	81	75	36	9	1						
10	0	0	0	25	102	150	110	45	10	1					
11	0	0	0	11	99	231	253	154	55	11	1				
12	0	0	0	3	72	282	456	399	208	66	12	1			
13	0	0	0	0	39	273	663	819	598	273	78	13	1		
14	0	0	0	0	14	210	786	1372	1372	861	350	91	14	1	
15	0	0	0	0	3	125	765	1905	2590	2178	1200	440	105	15	1

Theorem 2.9: *From the above table, we get following properties:*

(i) $d_e(C_n, m) = 1.$

(ii) $d_e(C_n, m-1) = m.$

(iii) $d_e(C_n, t) = 0$ for $t \leq \left\lceil \frac{m}{3} \right\rceil.$

(iv) $d_e(C_n, m-2) = mC_2.$

(v) $d_e(C_{3n}, m) = 3.$

(vi) $d_e(C_{3n-1}, m) = 3m - 1.$

(vii) $D_e(C_{3n+1}, m+1) = \frac{m(3m+7)+2}{2}.$

Proof: Let $G \cong C_n$ with $n \geq 3$ vertices.

(i) For $d_e(C_n, m)$, from m edges of G choose m edges which can be done in mC_m ways.

(ii) For $d_e(C_n, m-1)$, choose $(m - 1)$ edges from m edges of G which can be done in mC_{m-1} ways.

(iii) For a cycle with m edges, for every three edges there is one dominating edge belonging to the edge dominating set. For $m \geq 4$ there is no edge dominating set with cardinality one. Also for $m \geq 7$ there is no edge dominating set with cardinality one and two. Hence $d_e(G, t) = 0$ for $t \leq \left\lceil \frac{m}{3} \right\rceil.$

(iv) For $d_e(C_n, m-2)$, choose $(m - 2)$ edges from m edges which can be done in $mC_{m-2} = mC_2$ ways.

(v) To prove this result, we shall use mathematical induction with $d_e(C_3, 1) = 3.$ Assume the result to be true for a graph G with $(m - 1)$ edges. From Theorem 2.8, $d_e(C_{3n}, m) = d_e(C_{3n-1}, m-1) + d_e(C_{3n-2}, m-1) + d_e(C_{3n-3}, m-1).$ Using (iii), the result follows.

(vi) We shall prove this result by induction hypothesis. Assume the result to be true for a graph G with $(m - 1)$ edges. We shall prove the result for a graph with m edges. From Theorem 2.8, $d_e(C_{3n-1}, m) = d_e(C_{3n-2}, m-1) + d_e(C_{3n-3}, m-1) + d_e(C_{3n-4}, m-1)$. Also from (iii) and (v),

$$\begin{aligned} d_e(C_{3n-1}, m) &= d_e(C_{3n-2}, m-1) + d_e(C_{3(n-1)}, m-1) + d_e(C_{3(n-1)-1}, m-1), \\ &= 3 + 3(m-1) - 1 = 3m - 1. \end{aligned}$$

(vii) We use mathematical induction to prove the result. Assume the result to be true for a graph G with $(m-1)$ edges. From Theorem 2.8, $d_e(C_{3n+1}, m+1) = d_e(C_{3n}, m) + d_e(C_{3n-1}, m) + d_e(C_{3n-2}, m)$. Using (v) and (vi),

$$\begin{aligned} d_e(C_{3n+1}, m+1) &= 3 + 3m - 1 + \frac{(m-1)(3m+4)+2}{2}, \\ &= \frac{m(3m+7)+2}{2}. \end{aligned}$$

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