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On Pathos Litact Graph of a Tree

Venkanagouda M. Goudar*
Department of Mathematics
Sri Siddhartha Institute of Technology
Sri Gouthama Research Center, Tumkur
Karnataka
India

Abstract

In this paper, the concept of pathos litact graph of a tree is introduced. We present a characterization of those graphs whose pathos litact graphs are planar, outerplanar, maximal nonouterplanar, crossing number, eulerian and hamiltonian.

Keywords: *Line graph/lict graph/litact graph/pathos lict graph/pathos litact graph/outerplanar/crossing number.*

Mathematics Subject Classification: *05C10, 30.*

1. Introduction

The concept of pathos of a graph G was introduced by Harary [1] as a collection of minimum number of edge disjoint open paths whose union is G. The path number of a graph G is the number of paths in the pathos. The path number of a tree T is equal to k where $2k$ is the number of odd degree vertices of T. Also the end vertices of each path of any pathos of a tree are of odd degree.

The edge degree of an edge $\{u, v\}$ of a tree T is the sum of the degrees of u and v. The pathoslength is the number of edges which lie on a particular path P_i of pathos of T. A pendent pathos is a path P_i of pathos having unit length which corresponds to a pendent edge in T. A pathosvertex is a vertex in $P_m(T)$ corresponding to the boundary of the exterior region in any embedding of G in the plane. A graph is said to be minimally nonouterplanar if $i(G) = 1$.

The lict graph [4] $n(G)$ of a graph G is the graph whose vertex set is the union of the set of edges and the set of cutvertices of G in which two

*E-mail: vmgouda@gmail.com

vertices are adjacent if and only if the corresponding edges of G are adjacent or the corresponding members of G are incident.

The pathos line graph of a tree T denoted by $PL(T)$ is defined as the graph whose vertex set is the union of the set of edges and paths of pathos of T in which two vertices are adjacent if and only if the corresponding edges of T are adjacent and the edge lies on the corresponding path P_i of pathos.

The pathos lict graph [5] of a tree T denoted by $P_n(T)$ is defined as the graph whose vertex set is the union of the set of edges, set of paths of pathos and set of cutvertices of T in which two vertices are adjacent if and only if the corresponding edges of T are adjacent and the edge lies on the path of pathos and the edges are incident to the cutvertex.

The litact graph [4] $m\ (T)$ of a graph G is the graph whose vertex set is the union of edges and the set of cut vertices of G in which two vertices are adjacent if and only if the corresponding edges of G are adjacent or the corresponding members of G are incident or the cutvertices are adjacent.

The pathos litact graph of a tree T denoted by $P_m(T)$ is defined as the graph whose vertex set is the union of the set of edges, set of paths of pathos and set of cutvertices of T, in which two vertices are adjacent if and only if the corresponding edges of T are adjacent, the edge lies on the path of pathos, edges are incident to the cutvertices and the cutvertices are adjacent. Since the system of path of pathos for a tree T is not unique, the corresponding pathos litact graph is also not unique. The pathos litact graph is defined for a tree having at least two cutvertices

All the undefined terms may be referred to Harary [2]. All graphs considered here are finite, undirected and without loops or multiple edges.

2. Preliminary results

We need the following Theorems for the proof of our further results.

Theorem 2.1 [Ref.3]. *If G is a (p. q) graph whose vertices have degree d_i then its pathos line graph PL(T) has (q + k) vertices & ½ $\sum d_i^2$ edges, where k is the path number.*

Theorem 2.2 [Ref.3]. *The pathos line graph PL (T) is planar if and only if $\Delta(T)<4$ for every vertex v of T.*

Theorem 2.3 [Ref.4]. *For any graph G, the litact graph m (G) is minimally outerplanar if and only if G is a path.*

Theorem 2.4 [Ref.4]. *The litact graph m (G) of a graph G is planar if and only if G is planar and the degree of each vertex is at most 3.*

Theorem 2.5 [Ref.4]. *For any graph G, the litact graph m (G) is maximal outerplanar if and only is G is a path.*

3. Pathos litact graphs

Remark 3.1. For any tree T with $p \geq 3$ vertices $L(T) \subset PL(T) \subset P_m(T)$ and $L(T) \subset n(T) \subset m(T) \subset P_n(T) \subset P_m(T)$.

Remark 3.2. If the edge degree of an edge uv in a tree T is even (or odd) and u and v are cutvertices then the corresponding vertex in P_m (T) is of odd (or even) degree.

Remark 3.3. If the edge degree of an edge in a tree T is even (or odd) and the edge is a pendent edge then the corresponding vertex in $P_m(T)$ is of even (or odd) degree.

Remark 3.4. The degree of pathosvertex in $P_m(T)$ is equal to the pathoslength of the corresponding path p_i of pathos of T.

Remark 3.5. Every pendent pathos in a tree T adds one cutvertex to $P_m(T)$ and the number of cutvertices in $P_m(T)$ is the number of pendent pathos in T.

In the following Theorem we obtain the number of vertices and edges in a pathos litact graph.

Theorem 3.6. *For any (p, q) graph T whose vertices have degree d_i and cutvertices c have degree c_j, the pathos litact graph P_m (T) has (q+k+c) vertices and $\frac{1}{2}\sum d_i^{\,2} + \sum c_j$ +(c-1) edges, where k is the path number.*

Proof. By Theorem 1, $PL(T)$ has $(q + k)$ vertices and $\frac{1}{2}\sum d_i^2$ edges. The number of vertices of $P_m(T)$ is the sum of number of vertices of $PL(T)$ and the number of cutvertices c in T. Hence $P_m(T)$ has $(q + k + c)$ vertices.

The number of edges in $P_m(T)$ is the sum of edges in $PL(T)$, the number of edges incident with cutvertices in T and the number of edges in $c(G)$. By Theorem 2.1, the number of edges in $PL(T)$ is $\sum d_i^2$, the number of edges incident to cutvertices is $\sum c_j$ and the $c(G)$ has $\sum \frac{c_j(c_j-1)}{2}$ edges.

Since T contains at most two adjacent vertices, $\sum \frac{c_j(c_j-1)}{2}$ +(c – 1). Hence $P_m(T)$ has $\frac{1}{2}\sum d_i^2 + \sum c_j + (c-1)$ edges.

The following Lemma gives the number of regions in pathos litact graph.

Lemma 3.7. For any (p. q) graph T, the number of regions in pathos litact graph $P_m(T)$ is $(p + q + c - 3)$.

In the following theorems we obtain the condition for planarity of pathos litact graph.

Theorem 3.8. *The pathos litact graph $p_m(T)$ of a tree T is planar if and only if $\Delta(T) < 3$, for every vertex v of T.*

Proof. Suppose $P_m(T)$ is planar. Assume $\Delta(T) < 4$. If there exists a vertex v of degree 4, then by Theorem 2.2, $PL(T)$ is planar and contains K_4 as induced subgraph. In $P_m(T)$, the vertex v having degree 4 is adjacent to all the vertices of $\langle K_4 \rangle$ and it gives $\langle K_5 \rangle$ as subgraph. Clearly $P_m(T)$ is nonplanar, a contradiction.

Conversely, suppose $\Delta(T) < 3$. By Theorem 2.4, $m(T)$ is planar. Each block of $m(T)$ is either K_3 or K_4. The pathosvertex is adjacent to almost two vertices of each block of $m(T)$. This gives a planar $Pm(T)$.

We now present a characterization of trees whose pathos litact graphs are non outer planar and minimally nonouterplanar.

Theorem 3.9. *For any tree T, the pathos litact graph $P_m(T)$ is nonouterplanar.*

Proof. Suppose $P_m(T)$ of a tree T is outerplanar. We use contradiction for the proof of the theorem. Now we consider the following cases:

Case 1. If T is a path P_t of length $t \geq 3$. By Theorem 2.5, $m(T)$ is maximal outer planar and the pathosvertex is adjacent to (t-1) vertices. Hence $P_m(T)$ has inner vertices, which is nonouterplanar.

Case 2. If $\Delta(T) \leq 3$, for any vertex v of T. By Theorem 2.4, $m(T)$ is planar. The vertex v having degree 3 is adjacent to all the vertices to form $\langle K_3 \rangle$ as a subgraph. Also the pathosvertex is adjacent to all vertices of $\langle K_3 \rangle$ to form $\langle K_4 \rangle$ as a subgraph, which is nonouterplanar, a contradiction. Hence $P_m(T)$ is nonouterplanar.

Theorem 3.10. *The pathos litact graph P_m (T) of a tree T is minimally nonouterplanar if and only if T is a path of length 3.*

Proof. Suppose P_m (*T*) is minimally nonouterplanar. Assume that *T* is a path P_t for $t > 3$. By Theorem 3.8, P_m (*T*) has at least (*t*–2) inner vertices, i.e. $i\, [P_m(T)] > 1$, a contradiction. Thus $t = 3$.

Conversely, suppose *T* is a path of length 3. Then $P_m(T)$ has (3-2) =1, inner vertex. Hence $i\, [P_m(T)] = 1$, which is minimally nonouterplanar.

The noneulerian property of P_m (*T*) is given by the following Theorem.

Theorem 3.11. *For any tree T, the pathos litact graph P_m (T) is noneulerian.*

Proof. Suppose *T* is a tree with $p<3$ vertices, then $P_m(T) = PL(T) = K_1$ or K_2 and the result is obvious.

Suppose *T* is a tree with $p = 3$ vertices, then $P_m(T) = K_4 - x$ which is noneulerian. Suppose *T* is a tree with $p \geq 4$ vertices, then there exists at least two cutvertices v_1 and v_2 such that an edge $q = v_1v_2$ between these two vertices. We consider the following cases:

Case 1. If edge degree of q is even, then by the Remark 2, the corresponding vertex in P_m (*T*) is of odd degree vertex. Hence P_m (*T*) is noneulerian.

Case 2. If edge degree of q is odd, then it contains one pendent edge. By definition, P_m (*T*) contains an odd degree vertex. Hence P_m (*T*) is noneulerian.

From the above two cases, P_m (*T*) is noneulerian.

In the following Theorem we obtain a condition for P_m (*T*) to be a block.

Theorem 3.12 *The pathos litact graph of a tree is a block if and only if every cutvertex of T is of even degree.*

Proof. Suppose the pathos litact graph of a tree *T* is a block. Assume that *T* has one cutvertex of odd degree ≥ 3. Since every path of pathos of *T* starts and ends with an odd degree vertex, there is a pendent pathos in *T* coinciding with the odd degree vertex . By the Remark 3.5, P_m (*T*) contains at least one cutvertex. Thus P_m (*T*) is not a block, a contradiction.

Conversely, suppose T has all its cutvertices of even degree $2n$. Then n number of paths of pathos passes through these cutvertices and the pathoslength of every path is at least two. Since there is no pendent pathos in T and none of the non end edges coincide with an end edge of path of pathos. By the Remark 3.5, P_m (T) has no cutvertex. Thus P_m (T) is a block.

The next Theorem Characterizes P_m (T) in terms of crossing number.

Theorem 3.13: *The pathos litact graph of a tree T has crossing number two if and only if* $\Delta (T) \leq 4$ *for every vertex v of T and has a unique vertex of degree 4.*

Proof. Suppose P_m (T) has crossing number two. We have the following cases:

Case 1. Assume that T has vertex of degree 5. Then the edges incident to this vertex together with the cut vertex form $\langle K_6 \rangle$ as subgraph in m (T). The pathos vertex is adjacent to at least two vertices of $\langle K_6 \rangle$ in P_m (T). Clearly C $[P_m$ $(T)] > 2$, a contradiction.

Case 2. Assume that T contains two vertices v_1 and v_2 of degree 4. Then T has two $K_{1,4}$ as subgraphs. Then the cutvertices v_1 and v_2 together with their corresponding incident edges form two subgraphs as K_5 in P_m (T). Cleary C $[P_m(T)] > 2$, a contradiction. Thus T has exactly one cutvertex of degree 4.

Conversely, suppose T holds both the conditions of the Theorem. Then by necessity the result follows.

In the next Theorem, we characterize hamiltonian P_m (T).

Theorem 3.14. *The pathos litact graph* P_m (T) *of a tree T is hamiltonian if and only if every pathos of path length is at least 2.*

Proof. Suppose P_m (T) is hamiltonian. Assume that T has at least one path such that its pathoslength is one. By the remark 3.5, P_m (T) contains a cut-vertex, a contradiction. Hence, every pathos of path length is at least 2.

Conversely, suppose every pathos of path length at least 2. We now consider the following cases:

Case 1. If T is a path then it has exactly one path of pathos. Let V $[m$ $(T)] =$ $\{e_1, e_2 \ldots\ldots e_n\} \cup \{c_1, c_2 \ldots\ldots c_{n-2}\}$ where $c_{1,}$ $c_{2\ldots}$ c_{n-2} are the cutvertices of T. Since each block is a triangle and each block consist vertices as $B_1 = \{e_1, c_1, e_2\}$, B_2 $= \{e_2, c_2, e_3\}$ .... $B_m = \{e_{n-1,} c_{n-2}, e_n\}$. In P_m (T) the pathos the pathosvertex is adjacent to $[e_1, e_2 \ldots e_n$]. Hence V $[P_m$ $(T)] = \{e_1, e_2 \ldots e_n\} \cup \{c_1, c_2 \ldots c_n\text{-}_2\} \cup$ w form a closed path w $e_1c_1e_2c_2\ldots.$ e_{n-1} c_{n-2} e_n w containing all the vertices of P_m (T), clearly P_m (T) is hamiltonian.

Case 2. If T is not a path, and every pathos of path Length is at least two, then by the Remark 3.4, degree of every pathos vertex is at least two. Let $V[m(T)] = \{e_1, e_2 .. e_n\} \cup \{c_1, c_2 .. c_k\}$ which are cutvertices of T. since T has p_i pathosvertices $i \geq 1$, each having path length at least two, then $V[P_m(T)]$ contains $\{e_1, e_2 .. e_n\} \cup \{c_1, c_2 .. c_k\} \cup \{p_1, p_2 .. p_i\}$, vertices. But each p_i is adjacent to at least two vertices of e_n and each c_i are adjacent to form a cycle containing all the vertices $p_1, e_1, e_2, e_3, c_1, c_2, e_4 \ldots p_i$ of $P_m(T)$. Hence, $P_m(T)$ is hamiltonian.

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