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Comput. Methods Appl. Mech. Engrg. 181 (2000) 295–344

Computer methods in applied mechanics and engineering

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General complete Lagrange family for the cube in finite element interpolations

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Received 15 September 1998; accepted 01 January 1999

Abstract

In this paper, we have first derived the interpolation polynomials for the General Serendipity elements which allow arbitrarily placed nodes along the edges. We have then presented a method to determine the interpolation functions for the General Complete Lagrange elements which allow arbitrarily placed nodes. Explicit expressions for interpolation functions of the Serendipity and Complete Lagrange family elements which allow uniform spacing of nodes over the element domain are derived for elements of orders 4–10. We have also modified the Shape functions of Complete Lagrange family so that they can correctly interpolate the complete polynomial in the global space for *angular distortions*. © 2000 Elsevier Science S.A. All rights reserved.

1. Introduction

In Finite Element Analysis, interpolation theories based on Lagrangian and Hermitian interpolation polynomials have produced a variety of useful Finite Elements [1]. For Rectangular Finite Elements in two- and three-dimensions, Argyris et al. [2,3] have presented the Regular Lagrange family for which the nodes are regularly placed everywhere on the grid. Lagrange Rectangular elements are conceptually simple but are of limited use because of the large number of internal nodes. Nevertheless they provide a natural introduction to interpolation functions for Serendipity Rectangular elements which have no internal nodes. Zalmal [4] presented a general formula for the derivation of interpolation functions for Serendipity elements with uniformly spaced nodes. Ball [5] derived an explicit expression for interpolation function in the form of matrix triple product for the most general Serendipity element. Zienkiewicz [6] intended to define a Serendipity family so that polynomial completeness is realised with necessary minimum nodes and presented the few lower order elements viz. Linear, Quadratic and Cubic elements which have equal number of nodes along each side which are uniformly spaced. It is obvious that the Basis functions for Serendipity elements with nodes placed only along the edges cannot generate complete polynomials beyond cubic.

For this reason, Zienkiewicz [6] has suggested a central node for the next Quartic member of this family and remarks that progression to yet higher members is difficult and requires some ingenuity. Taylor [7] has suggested Serendipity elements composed of vertices and side nodes placed at regular intervals with some internal nodeless variables. The interpolation bases are given in his paper mainly by geometrical consideration of Shape functions and does not result in an explicit formula but it provides insight into the form of interpolation functions and is applicable to all Serendipity Rectangular elements.

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PII: S0045-7825(99)00080-8

Okabe et al. [8] rephrase Taylor's Serendipity definition in the conventional nodal concept and according to which an mth Serendipity 2-cube ($m > 4$) should contain apparently the $(m-4)$th complete triangle composed of $(m-3)(m-2)/2$ internal nodes. However Okabe et al. [8] noticed that the Taylor's Serendipity element is lacking in nodal symmetry with respect to co-ordinate axes. In some application areas, such nodal symmetry may be important.

From this viewpoint, Serendipity family is classified into two independent families:

- the Complete Lagrange family and
- the mixed Complete Lagrange family.

In Complete Lagrange family, the polynomial completeness is realised with necessary minimum nodes without destroying the nodal symmetry whereas in mixed complete Lagrange family, the nodal symmetry is not maintained.

Okabe [9] has revealed the interpolation Bases for the Complete Lagrange family which realise the polynomial completeness with necessary minimum nodes without destroying the nodal symmetry.

This paper concentrates on the derivation of interpolation functions for the general Serendipity and General Complete Lagrange elements which allow arbitrarily placed nodes over the domain of Element Geometry. We have obtained explicit formulae for the interpolation functions of both the families viz. the General Serendipity and General Complete Lagrange. We have then illustrated these formulae by obtaining explicit interpolation functions for both of these families when the nodes are uniformly spaced over the element domains for Quartic to tenth-order members.

It has also been observed that [16] the predictive capabilities of the Lagrangian isoparametric elements are not affected by angular distortions. The Serendipity elements are on the other hand very sensitive to such distortions [17–19]. For straight-edged Quadrilateral elements, x is bilinear in ξ, η i.e.

$$x = a_1 + a_2\xi + a_3\eta + a_4\xi\eta.$$

Then x^p will include

$$\xi^p\eta^p$$

term which is in the Basis of a pth-order Lagrange element but not either in the pth-order Serendipity or Complete Lagrange element [15]. This, then, is the undisputed *advantage* of the pth-order Lagrange element. The one clear advantage of a pth-order Lagrange element is that it can correctly represent pth-order displacement fields when its shape is bilinear. If pth-order Complete Lagrange element could be constructed, which also had this property, it might well compete in accuracy with pth-order Lagrange element while avoiding the need to supress spurious mechanisms. Following [15], we have developed a method for the modification of Shape functions for the Complete Lagrange elements up to tenth-order proposed in this paper.

2. A Fundamental Lemma

Let

$$f(x) = \sum_{i=0}^{p} f_i x^i$$

be a polynomial function of degree p in the variable x and defined over the interval

$$x_0 < x_1 < x_2 < \cdots < x_l < \cdots < x_p$$

as

$$f(x_k) = \begin{cases} 1, & k = l, \\ 0, & k \neq l, \end{cases} \tag{1}$$

$$k = 0(1)p.$$

Then the sequence of unknowns

$$\{f_i\}, \quad i = 0(1)p$$

is a solution of the following set of linear equations:

$$-1 = \sum_{i=1}^{p} (x_k^i - x_l^i) f_i \quad (k \neq l) \tag{2}$$

and the function $f(x)$ can be uniquely determined by the relation

$$f(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_{l-1})(x - x_{l+1})\cdots(x - x_p)}{(x_l - x_0)(x_l - x_1)\cdots(x_l - x_{l-1})(x_l - x_{l+1})\cdots(x_l - x_p)}. \tag{3}$$

Proof. Follows from the Lagrangian interpolation theory. □

3. General Serendipity elements over a 2-cube

Serendipity elements have no internal nodes. The three simplest such elements of Serendipity family were found by inspection [6] (see Fig. 1).

We consider the Serendipity Rectangular elements in its most general form which has different number of nodes along each side as shown in Fig. 2.

Nodal co-ordinates of node k for the general Serendipity element of Fig. 2 are assumed as

$$(\xi_k, \eta_k), \quad k = 1(1)(n + m + p + q).$$

When

$$n = m = p = q,$$

we shall obtain the General Serendipity element over 2-cube having $(n + 1)$ nodes along each side.

We shall assume the following Monomial Bases for a General Serendipity element along ξ and η directions as shown in Fig. 3:

(i) $r = n,\ s = q$ along sides $\overline{1\,2}$ and $\overline{4\,1}$,
(ii) $r = n,\ s = m$ along sides $\overline{1\,2}$ and $\overline{2\,3}$,
(iii) $r = p,\ s = m$ along sides $\overline{3\,4}$ and $\overline{2\,3}$,
(iv) $r = p,\ s = q$ along sides $\overline{3\,4}$ and $\overline{4\,1}$.

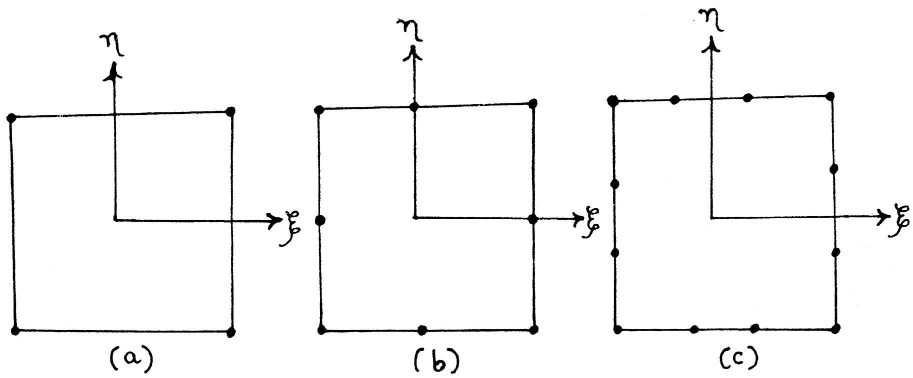

Fig. 1. Serendipity elements over a 2-cube: (a) Linear, (b) Quadratic (Parabolic), (c) Cubic.

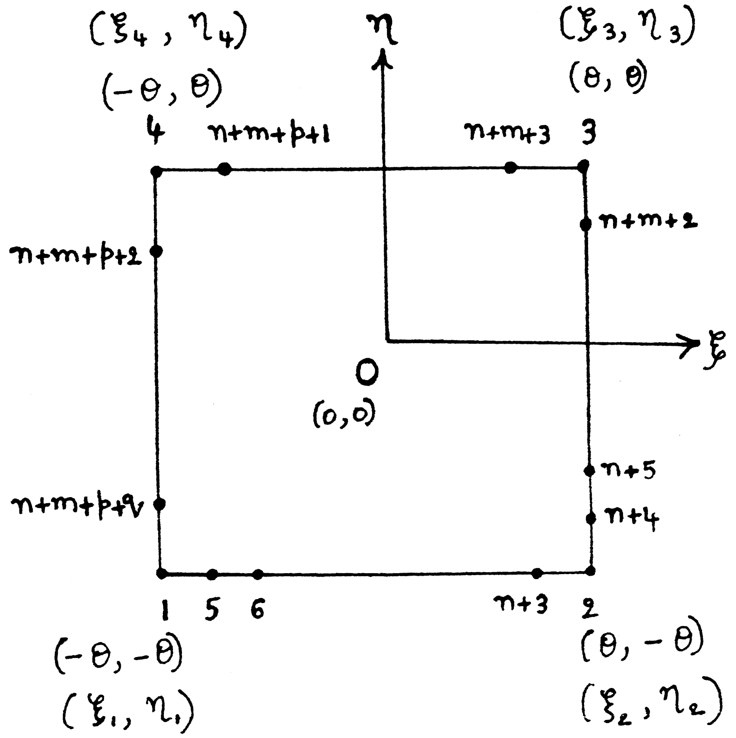

Fig. 2. General Serendipity element over a 2-cube $-\theta \leqslant \xi, \eta \leqslant \theta$ with $n+1, m+1, p+1, q+1$ nodes along sides $\overline{12}, \overline{23}, \overline{34}, \overline{45}$.

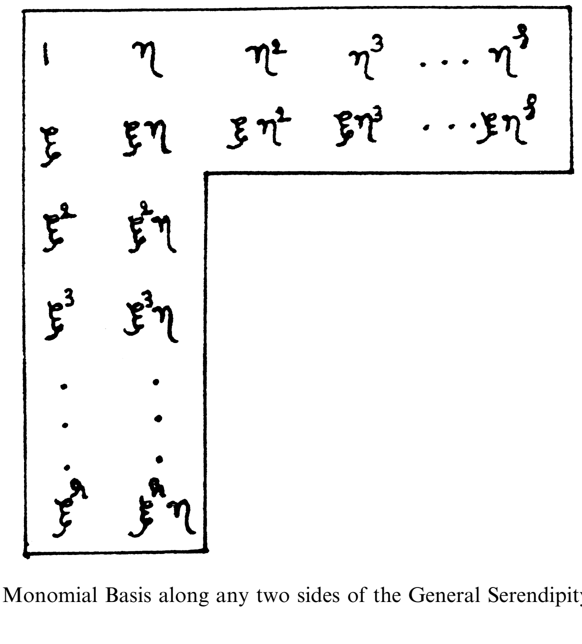

Fig. 3. Monomial Basis along any two sides of the General Serendipity element.

4. Determination of Shape functions for the General Serendipity element

Consider the General Serendipity element shown in Fig. 2. We assume the following expression for Shape function at the *corner node* 1 as

$$N_1(\xi,\eta) = \frac{(\xi-\xi_2)(\eta-\eta_2)}{(\xi_1-\xi_2)(\eta_1-\eta_2)}\left[a_{00} + \sum_{k=1}^{n-1} a_{k0}\xi^k + \sum_{k=1}^{q-1} a_{0k}\eta^k\right]. \tag{4}$$

By the properties of Shape functions, we have

$$N_1(\xi_i,\eta_i) = 0, \quad i = 5,6,\ldots,n+3 \tag{5}$$

and

$$N_1(\xi_1,\eta_1) = 1. \tag{6}$$

From Eqs. (5) and (6), we obtain, by using Eq. (4), the following relations:

$$a_{00} + \sum_{k=1}^{n-1} a_{k0}\xi_i^k + \sum_{k=1}^{q-1} a_{0k}\eta_i^k = 0, \tag{7}$$

$$a_{00} + \sum_{k=1}^{n-1} a_{k0}\xi_1^k + \sum_{k=1}^{q-1} a_{0k}\eta_1^k = 1. \tag{8}$$

Now subtracting Eq. (8) from Eq. (7), we obtain

$$\sum_{k=1}^{n-1} a_{k0}(\xi_i^k - \xi_1^k) = -1, \quad i \neq 1. \tag{9}$$

Using the Fundamental Lemma and comparing Eqs. (4) and (9) with Eqs. (1) and (2), we infer that the solution to the set of Eqs. (7) and (8) can be immediately written as

$$\left(a_{00} + \sum_{k=1}^{q-1} a_{0k}\eta_1^k\right) + \sum_{k=1}^{n-1} a_{k0}\xi^k = \frac{(\xi-\xi_5)(\xi-\xi_6)\cdots(\xi-\xi_{n+2})(\xi-\xi_{n+3})}{(\xi_1-\xi_5)(\xi_1-\xi_6)\cdots(\xi_1-\xi_{n+2})(\xi_1-\xi_{n+3})}. \tag{10}$$

We also have along the edges of the 2-cube

$$\xi = \xi_1 \quad \text{and} \quad \xi_1 = \xi_{n+m+p+i}, \quad i = 2(1)q,$$

so that we can write

$$N_1(\xi_{n+m+p+i},\eta_{n+m+p+i}) = N_1(\xi_1,\eta_{n+m+p+i}) = 0, \quad i = 2(1)q. \tag{11}$$

Eq. (11) is further equivalent to

$$a_{00} + \sum_{k=1}^{n-1} a_{k0}\xi_1^k + \sum_{k=1}^{q-1} a_{0k}\eta_{n+m+p+i}^k = 0, \quad i = 2(1)q. \tag{12}$$

Again, referring the Fundamental Lemma, the solution to the set of Eqs. (8) and (12) is given by the relation

$$a_{00} + \sum_{k=1}^{n-1} a_{k0}\xi_1^k + \sum_{k=1}^{q-1} a_{0k}\eta^k = \frac{(\eta-\eta_{n+m+p+2})(\eta-\eta_{n+m+p+3})\cdots(\eta-\eta_{n+m+p+q})}{(\eta_1-\eta_{n+m+p+2})(\eta_1-\eta_{n+m+p+3})\cdots(\eta_1-\eta_{n+m+p+q})}. \tag{13}$$

Adding Eqs. (10) and (13) and also using Eq. (8), we obtain

$$a_{00} + \sum_{k=1}^{n-1} a_{k0}\xi^k + \sum_{k=1}^{q-1} a_{0k}\eta^k$$
$$= -1 + \frac{(\xi-\xi_5)(\xi-\xi_6)\cdots(\xi-\xi_{n+2})(\xi-\xi_{n+3})}{(\xi_1-\xi_5)(\xi_1-\xi_6)\cdots(\xi_1-\xi_{n+2})(\xi_1-\xi_{n+3})}$$
$$+ \frac{(\eta-\eta_{n+m+p+2})(\eta-\eta_{n+m+p+3})\cdots(\eta-\eta_{n+m+p+q-1})(\eta-\eta_{n+m+p+q})}{(\eta_1-\eta_{n+m+p+2})(\eta_1-\eta_{n+m+p+3})\cdots(\eta_1-\eta_{n+m+p+q-1})(\eta_1-\eta_{n+m+p+q})}. \tag{14}$$

Substituting Eq. (14) in Eq. (4), we obtain

$$N_1(\xi,\eta) = \frac{(\xi-\xi_2)(\eta-\eta_2)}{(\xi_1-\xi_2)(\eta_1-\eta_2)}\Bigg[-1 + \frac{(\xi-\xi_5)(\xi-\xi_6)\cdots(\xi-\xi_{n+2})(\xi-\xi_{n+3})}{(\xi_1-\xi_5)(\xi_1-\xi_6)\cdots(\xi_1-\xi_{n+2})(\xi_1-\xi_{n+3})}$$
$$+ \frac{(\eta-\eta_{n+m+p+2})(\eta-\eta_{n+m+p+3})\cdots(\eta-\eta_{n+m+p+q-1})(\eta-\eta_{n+m+p+q})}{(\eta_1-\eta_{n+m+p+2})(\eta_1-\eta_{n+m+p+3})\ldots(\eta_1-\eta_{n+m+p+q-1})(\eta_1-\eta_{n+m+p+q})}\Bigg]$$
$$n+3>4, \qquad n+m+p+q>n+m+p+q+1. \tag{15}$$

The expressions for element Shape functions can be derived in a similar manner when

$\{n+3=4,\ n+m+p+q>n+m+p+1\}$,
$\{n+3>4,\ n+m+p+q=n+m+p+1\}$,
$\{n+3=4,\ n+m+p+q=n+m+p+1\}$.

These are

(i) For $\{n+3=4,\ n+m+p+q>n+m+p+1\}$

$$N_1(\xi,\eta) = \frac{(\xi-\xi_2)(\eta-\eta_3)}{(\xi_1-\xi_2)(\eta_1-\eta_3)}, \tag{16}$$

(ii) For $\{n+3>4,\ n+m+p+q=n+m+p+1\}$

$$N_1(\xi,\eta) = \frac{(\xi-\xi_2)(\eta-\eta_3)}{(\xi_1-\xi_2)(\eta_1-\eta_3)} \times \frac{(\xi-\xi_5)(\xi-\xi_6)\cdots(\xi-\xi_{n+2})(\xi-\xi_{n+3})}{(\xi_1-\xi_5)(\xi_1-\xi_6)\cdots(\xi_1-\xi_{n+2})(\xi_1-\xi_{n+3})}, \tag{17}$$

(iii) For $\{n+3=4, n+m+p+q>n+m+p+1\}$

$$N_1(\xi,\eta) = \frac{(\xi-\xi_2)(\eta-\eta_3)}{(\xi_1-\xi_2)(\eta_1-\eta_3)}\frac{(\eta-\eta_{n+m+p+2})(\eta-\eta_{n+m+p+3})\cdots(\eta-\eta_{n+m+p+q-1})(\eta-\eta_{n+m+p+q})}{(\eta_1-\eta_{n+m+p+2})(\eta_1-\eta_{n+m+p+3})\cdots(\eta_1-\eta_{n+m+p+q-1})(\eta_1-\eta_{n+m+p+q})}. \tag{18}$$

The Shape functions for the mid-side nodes can be immediately written as

$$N_k(\xi,\eta) = \frac{(\xi-\xi_1)(\xi-\xi_2)(\eta-\eta_3)}{(\xi_k-\xi_1)(\xi_k-\xi_2)(\eta_k-\eta_3)}$$
$$\times \frac{(\xi-\xi_5)(\xi-\xi_6)\cdots(\xi-\xi_{k-1})(\xi-\xi_{k+1})\cdots(\xi-\xi_{n+3})}{(\xi_k-\xi_5)(\xi_k-\xi_6)\cdots(\xi_k-\xi_{k-1})(\xi_k-\xi_{k+1})\cdots(\xi_k-\xi_{n+3})}. \tag{19}$$

The expressions for element Shape functions of the remaining corner nodes can be derived similarly by following the above procedure.

5. Determination of Shape functions for the Serendipity element of conventional type

Consider the General Serendipity element over the 2-cube, $-\theta \leqslant \xi, \eta \leqslant \theta$ having equal number of nodes along each side as shown in Fig. 4.

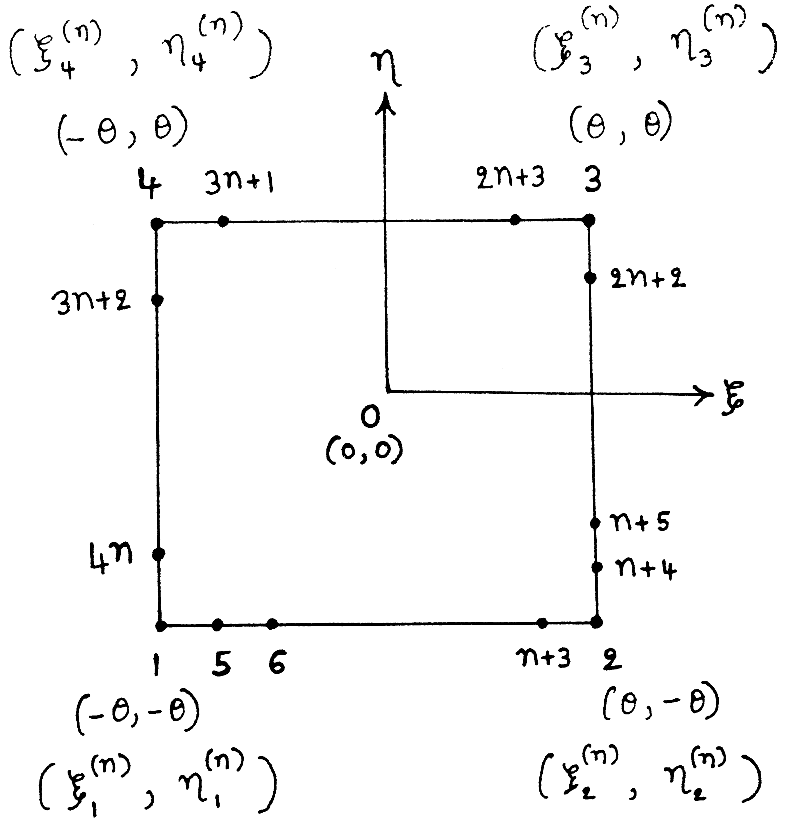

Fig. 4. A General Serendipity element over $-\theta \leqslant \xi, \eta \leqslant \theta$ with $(n+1)$ nodes along each side.

We can define the conventional Serendipity element of nth-order by choosing uniform spacing over the 2θ-square, $-\theta \leqslant \xi, \eta \leqslant \theta$ of Fig. 4 as

$$
\begin{aligned}
&\xi^{(n)}_{4+i} = \eta^{(n)}_{n+2+i} = -\theta + 2i\theta/n, && i = 1, 2, 3, \ldots, n-1,\\
&\xi^{(n)}_{3n+i} = \eta^{(n)}_{3+i} = -\theta, && i = 2, 3, \ldots, n,\\
&\eta^{(n)}_{2n+i+1} = \xi^{(n)}_{n+2+i} = \theta, && i = 2, 3, \ldots, n,\\
&\xi^{(n)}_{3n+2-i} = \xi^{(n)}_{4+i}, && i = 2, 3, \ldots, n,\\
&\eta^{(n)}_{4n+2-i} = \eta^{(n)}_{n+2+i}, && i = 2, 3, \ldots, n.
\end{aligned}
\tag{20}
$$

On using the formulae of Eqs. (15)–(19) and the definition of conventional Serendipity element of Eq. (20), the element Shape functions for a conventional Serendipity element of any order can be immediately obtained.

For $\theta = 1$ and $n = 1, 2, 3$, we obtain the conventional Shape functions reported in Zienkiewicz [6] for Linear, Quadratic (parabolic) and Cubic elements.

We have determined the Shape functions for the higher-order Serendipity elements – Quartic, Quintic, Sextic, Septic, Octic, Nineth and Tenth-orders. These are listed in Tables 1–14, respectively.

Node numbering sequence for these elements is described in Fig. 5 and the Monomial Bases used for an nth-order element can be described by choosing $r = s = n$ in Fig. 3.

6. General Complete Lagrange elements over a 2-cube

Zienkiewicz [6] intended to define a Lagrange family for the 2-cube namely the Serendipity family so that polynomial completeness is realised with necessary minimum nodes and presented a few lower-order

Table 1
Quartic General Serendipity element Shape functions

Node i	Co-ordinates (ξ_i, η_i)	$N_i^{(4)}$	Shape function
1	$(-1,-1)$	$N_1^{(4)}$	$(1/12)(1-\xi)(1-\eta)[-4(\xi^3+\eta^3)+(\xi+\eta)-3]$
2	$(1,-1)$	$N_2^{(4)}$	$(1/12)(1+\xi)(1-\eta)[4(\xi^3-\eta^3)-(\xi-\eta)-3]$
3	$(1,1)$	$N_3^{(4)}$	$(1/12)(1+\xi)(1+\eta)[4(\xi^3+\eta^3)-(\xi+\eta)-3]$
4	$(-1,1)$	$N_4^{(4)}$	$(1/12)(1-\xi)(1+\eta)[-4(\xi^3-\eta^3)+(\xi-\eta)-3]$
5	$(-1/2,-1)$	$N_5^{(4)}$	$(2/3)(1-\xi^2)(1-\eta)\xi(2\xi-1)$
6	$(0,-1)$	$N_6^{(4)}$	$-(1/2)(1-\xi^2)(1-\eta)(4\xi^2-1)$
7	$(1/2,-1)$	$N_7^{(4)}$	$(2/3)(1-\xi^2)(1-\eta)\xi(2\xi+1)$
8	$(1,-1/2)$	$N_8^{(4)}$	$(2/3)(1+\xi)(1-\eta^2)\eta(2\eta-1)$
9	$(1,0)$	$N_9^{(4)}$	$-(1/2)(1+\xi)(1-\eta^2)(4\eta^2-1)$
10	$(1,1/2)$	$N_{10}^{(4)}$	$(2/3)(1+\xi)(1-\eta^2)\eta(2\eta+1)$
11	$(1/2,1)$	$N_{11}^{(4)}$	$(2/3)(1-\xi^2)(1+\eta)\xi(2\xi+1)$
12	$(0,1)$	$N_{12}^{(4)}$	$-(1/2)(1-\xi^2)(1+\eta)(4\xi^2-1)$
13	$(-1/2,1)$	$N_{13}^{(4)}$	$(2/3)(1-\xi^2)(1+\eta)\xi(2\xi-1)$
14	$(-1,1/2)$	$N_{14}^{(4)}$	$(2/3)(1-\xi)(1-\eta^2)\eta(2\eta+1)$
15	$(-1,0)$	$N_{15}^{(4)}$	$-(1/2)(1-\xi)(1-\eta^2)(4\eta^2-1)$
16	$(-1,1/2)$	$N_{16}^{(4)}$	$(2/3)(1-\xi)(1-\eta^2)\eta(2\eta-1)$

elements – Linear, Quadratic (parabolic) and Cubic elements of Serendipity family which depend on nodal co-ordinate values uniformly spaced on the element boundary over the domain $-1 \leqslant \xi, \eta \leqslant 1$.

However, Zienkiewicz [6] observed that the interior nodes are necessary for elements of orders higher than the Cubic Serendipity element to realise the polynomial completeness.

Okabe [8,9] proposed the Complete Lagrange family for the 2-cube which realised polynomial completeness.

In this section, we wish to determine the element Shape functions for the General Complete Lagrange family in an explicit form, as far as possible, for the Quartic, Quintic, Sextic, Septic, Octic, Nineth and Tenth-order elements. Following Okabe [8,9] the Monomial Bases for the General Complete Lagrange elements and the respective element geometry are shown in Figs. 6–12.

We shall now outline the method of obtaining element Shape functions for the General Complete Lagrange elements of fourth- to tenth-order. For the sake of illustration, let us consider the nth-order General Complete Langrange element which allows arbitrary placement of nodes in all its orbits.

Let

$$N_i^{(n)}(\xi,\eta), \quad i = 1, 2, \ldots, 4n,$$

refer to Shape functions of nth-order General Serendipity element.

Then following Okabe [9], the Shape functions

$$\hat{N}_i^{(n)}(\xi,\eta), \quad i = 1, 2, \ldots, 4n,$$

on the zeroth orbit of the nth-order General Complete Lagrange element can be determined by the formula

$$\hat{N}_i^{(n)}(\xi,\eta) = N_i^{(n)}(\xi,\eta) - \sum_{k=4n+1}^{4n+i_n} N_i^{(n)}\left(\xi_k^{(n)}, \eta_k^{(n)}\right)\hat{N}_k^{(n)}(\xi,\eta), \quad i = 1(1)4n, \tag{21}$$

Table 2
Quintic General Serendipity element Shape functions

Node i	Co-ordinates (ξ_i, η_i)	$N_i^{(5)}$	Shape function
1	$(-1,-1)$	$N_1^{(5)}$	$(1/1536)(1-\xi)(1-\eta)[625(\xi^4+\eta^4)-250(\xi^2+\eta^2)-366]$
2	$(1,-1)$	$N_2^{(5)}$	$(1/1536)(1+\xi)(1-\eta)[625(\xi^4+\eta^4)-250(\xi^2+\eta^2)-366]$
3	$(1,1)$	$N_3^{(5)}$	$(1/1536)(1+\xi)(1+\eta)[625(\xi^4+\eta^4)-250(\xi^2+\eta^2)-366]$
4	$(-1,1)$	$N_4^{(5)}$	$(1/1536)(1-\xi)(1+\eta)[625(\xi^4+\eta^4)-250(\xi^2+\eta^2)-366]$
5	$(-3/5,-1)$	$N_5^{(5)}$	$-(25/1536)(1-\xi^2)(1-\eta)(25\xi^2-1)(5\xi-3)$
6	$(-1/5,-1)$	$N_6^{(5)}$	$(25/768)(1-\xi^2)(1-\eta)(25\xi^2-9)(5\xi-1)$
7	$(1/5,-1)$	$N_7^{(5)}$	$-(25/768)(1-\xi^2)(1-\eta)(25\xi^2-9)(5\xi+1)$
8	$(3/5,-1)$	$N_8^{(5)}$	$(25/1536)(1-\xi^2)(1-\eta)(25\xi^2-1)(5\xi+3)$
9	$(1,-3/5)$	$N_9^{(5)}$	$-(25/1536)(1+\xi)(1-\eta^2)(25\eta^2-1)(5\eta-3)$
10	$(1,-1/5)$	$N_{10}^{(5)}$	$(25/2304)(1+\xi)(1-\eta^2)(25\eta^2-9)(5\eta-1)$
11	$(1,1/5)$	$N_{11}^{(5)}$	$-(25/2304)(1+\xi)(1-\eta^2)(25\eta^2-9)(5\eta+1)$
12	$(1,3/5)$	$N_{12}^{(5)}$	$(25/1536)(1+\xi)(1-\eta^2)(25\eta^2-1)(5\eta+3)$
13	$(3/5,1)$	$N_{13}^{(5)}$	$(25/1536)(1-\xi^2)(1+\eta)(25\xi^2-1)(5\xi+3)$
14	$(1/5,1)$	$N_{14}^{(5)}$	$-(25/2304)(1-\xi^2)(1+\eta)(25\xi^2-9)(5\xi+1)$
15	$(-1/5,1)$	$N_{15}^{(5)}$	$(25/2304)(1-\xi^2)(1+\eta)(25\xi^2-9)(5\xi-1)$
16	$(-3/5,1)$	$N_{16}^{(5)}$	$-(25/1536)(1-\xi^2)(1+\eta)(25\xi^2-1)(5\xi-3)$
17	$(-1,3/5)$	$N_{17}^{(5)}$	$(25/1536)(1-\xi)(1-\eta^2)(25\eta^2-1)(5\eta+3)$
18	$(-1,1/5)$	$N_{18}^{(5)}$	$-(25/2304)(1-\xi)(1-\eta^2)(25\eta^2-9)(5\eta+1)$
19	$(-1,-1/5)$	$N_{19}^{(5)}$	$(25/2304)(1-\xi)(1-\eta^2)(25\eta^2-9)(5\eta-1)$
20	$(-1,-3/5)$	$N_{20}^{(5)}$	$-(25/1536)(1-\xi)(1-\eta^2)(25\eta^2-1)(5\eta-3)$

where i_n is the number of interior nodes required for the nth-order General Complete Lagrange element, $\hat{N}_k^{(n)}(\xi,\eta),\ k=4n+1,\ldots,4n+i_n$ are the Shape functions of interior nodes and hence they belong to higher orbits, say 1, 2, 3, etc., and $(\xi_k^{(n)},\eta_k^{(n)})$ is the co-ordinates of node k for the nth-order General Serendipity element.

Without loss of generality, we can omit the double-rank formalism, nodal rank and orbital rank, introduced by Okabe [8,9] to determine the explicit expressions of Shape functions of interior nodes.

Referring Okabe [8,9] the Shape functions of interior nodes can be obtained by multiplying the Shape functions of General Serendipity element over the 2-cubes

$$-\theta \leqslant \xi,\eta \leqslant \theta \quad \text{and} \quad -\varepsilon \leqslant \xi,\eta \leqslant \varepsilon$$

by a multiplication factor called *orbital modifier*. On using this procedure, we can assume that the explicit expressions for Shape functions

$$\hat{N}_k^{(n)}(\xi,\eta), \quad k=4n+1,\ldots,4n+i_n, \qquad n \geqslant 4,$$

are known.

The determination of Shape functions over the edges of zeroth orbit, i.e., on the boundary of $-1 \leqslant \xi,\eta \leqslant 1$ follows upon application of Eq. (21). We shall now explain the application of Eq. (21) to determine

$$\hat{N}_i^{(n)}(\xi,\eta), \quad i=1,2,\ldots,4n,$$

Table 3
Sextic General Serendipity element Shape functions

Node i	Co-ordinates (ξ_i, η_i)	$N_i^{(6)}$	Shape function
1	$(-1,-1)$	$N_1^{(6)}$	$(1/160)(1-\xi)(1-\eta)[-81(\xi^5+\eta^5)+45(\xi^3+\eta^3)-4(\xi+\eta)-40]$
2	$(1,-1)$	$N_2^{(6)}$	$(1/160)(1+\xi)(1-\eta)[81(\xi^5-\eta^5)-45(\xi^3-\eta^3)+4(\xi-\eta)-40]$
3	$(1,1)$	$N_3^{(6)}$	$(1/160)(1+\xi)(1+\eta)[81(\xi^5+\eta^5)-45(\xi^3+\eta^3)+4(\xi+\eta)-40]$
4	$(-1,1)$	$N_4^{(6)}$	$(1/160)(1-\xi)(1+\eta)[-81(\xi^5-\eta^5)+45(\xi^3+\eta^3)-4(\xi-\eta)-40]$
5	$(-2/3,-1)$	$N_5^{(6)}$	$(9/80)(1-\xi^2)(1-\eta)(9\xi^2-1)\xi(3\xi-2)$
6	$(-1/3,-1)$	$N_6^{(6)}$	$-(9/32)(1-\xi^2)(1-\eta)(9\xi^2-4)\xi(3\xi-1)$
7	$(0,-1)$	$N_7^{(6)}$	$(1/8)(1-\xi^2)(1-\eta)(9\xi^2-4)(9\xi^2-1)$
8	$(1/3,-1)$	$N_8^{(6)}$	$-(9/32)(1-\xi^2)(1-\eta)(9\xi^2-4)\xi(3\xi+1)$
9	$(2/3,-1)$	$N_9^{(6)}$	$(9/80)(1-\xi^2)(1-\eta)(9\xi^2-1)\xi(3\xi+2)$
10	$(1,-2/3)$	$N_{10}^{(6)}$	$(9/80)(1+\xi)(1-\eta^2)(9\eta^2-1)\eta(3\eta-2)$
11	$(1,-1/3)$	$N_{11}^{(6)}$	$-(9/32)(1+\xi)(1-\eta^2)(9\eta^2-4)\eta(3\eta-1)$
12	$(1,0)$	$N_{12}^{(6)}$	$(1/8)(1+\xi)(1-\eta^2)(9\eta^2-1)(9\eta^2-1)$
13	$(1,1/3)$	$N_{13}^{(6)}$	$-(9/32)(1+\xi)(1-\eta^2)(9\eta^2-4)\eta(3\eta+1)$
14	$(1,2/3)$	$N_{14}^{(6)}$	$(9/80)(1+\xi)(1-\eta^2)(9\eta^2-1)\eta(3\eta+2)$
15	$(2/3,1)$	$N_{15}^{(6)}$	$(9/80)(1-\xi^2)(1+\eta)(9\xi^2-1)\xi(3\xi+2)$
16	$(1/3,1)$	$N_{16}^{(6)}$	$-(9/32)(1-\xi^2)(1+\eta)(9\xi^2-4)\xi(3\xi+1)$
17	$(0,1)$	$N_{17}^{(6)}$	$(1/8)(1-\xi^2)(1+\eta)(9\xi^2-4)(9\xi^2-1)$
18	$(-1/3,1)$	$N_{18}^{(6)}$	$-(9/32)(1-\xi^2)(1+\eta)(9\xi^2-4)\xi(3\xi-1)$
19	$(-2/3,-1)$	$N_{19}^{(6)}$	$(9/80)(1-\xi^2)(1+\eta)(9\xi^2-1)\xi(3\xi-2)$
20	$(-1,2/3)$	$N_{20}^{(6)}$	$(9/80)(1+\xi)(1-\eta^2)(9\eta^2-1)\eta(3\eta+2)$
21	$(-1,1/3)$	$N_{21}^{(6)}$	$-(9/32)(1+\xi)(1-\eta^2)(9\eta^2-4)\eta(3\eta+1)$
22	$(-1,0)$	$N_{22}^{(6)}$	$(1/8)(1+\xi)(1-\eta^2)(9\eta^2-1)(9\eta^2-1)$
23	$(-1,-1/3)$	$N_{23}^{(6)}$	$-(9/32)(1+\xi)(1-\eta^2)(9\eta^2-4)\eta(3\eta-1)$
24	$(-1,-2/3)$	$N_{24}^{(6)}$	$(9/80)(1+\xi)(1-\eta^2)(9\eta^2-1)\eta(3\eta-2)$

for the General Complete Lagrange elements of orders $n = 4(1)10$ which are Shape functions defined along the boundary of the zeroth orbit of $-1 \leqslant \xi, \eta \leqslant 1$. Eq. (21) now assumes the following forms for the General Complete Lagrange elements.

1. *Quartic General Complete Lagrange element* $(n = 4,\ i_4 = 1)$. From Fig. 6a,b and Eq. (21), we can write

$$\hat{N}_i^{(4)}(\xi,\eta) = N_i^{(4)}(\xi,\eta) - N_i^{(4)}\left(\xi_{17}^{(4)}, \eta_{17}^{(4)}\right)\hat{N}_{17}^{(4)}(\xi,\eta), \quad i = 1(1)16. \tag{22}$$

2. *Quintic General Complete Lagrange element* $(n = 5,\ i_5 = 4)$. From Fig. 7a,b and Eq. (21), we obtain

$$\hat{N}_i^{(5)}(\xi,\eta) = N_i^{(5)}(\xi,\eta) - \sum_{k=21}^{24} N_i^{(5)}\left(\xi_k^{(5)}, \eta_k^{(5)}\right)\hat{N}_k^{(5)}(\xi,\eta), \quad i = 1(1)20. \tag{23}$$

3. *Sextic General Complete Lagrange element* $(n = 6,\ i_6 = 8)$. From Fig. 8a,b and Eq. (21), we can write

$$\hat{N}_i^{(6)}(\xi,\eta) = N_i^{(6)}(\xi,\eta) - \sum_{K=25}^{32} N_i^{(6)}\left(\xi_k^{(6)}, \eta_k^{(6)}\right)\hat{N}_k^{(6)}(\xi,\eta), \quad i = 1(1)24. \tag{24}$$

Table 4
Septic General Serendipity element Shape functions

Node i	Co-ordinates (ξ_i, η_i)	$N_i^{(7)}$	Shape function
1	$(-1,-1)$	$N_1^{(7)}$	$(1/184320)(1-\xi)(1-\eta)[117649(\xi^6+\eta^6)-84035(\xi^4+\eta^4)$ $+12691(\xi^2+\eta^2)-46530]$
2	$(1,-1)$	$N_2^{(7)}$	$(1/184320)(1-\xi)(1-\eta)[117649(\xi^6+\eta^6)-84035(\xi^4+\eta^4)$ $+12691(\xi^2+\eta^2)-46530]$
3	$(1,1)$	$N_3^{(7)}$	$(1/184320)(1-\xi)(1-\eta)[117649(\xi^6+\eta^6)-84035(\xi^4+\eta^4)$ $+12691(\xi^2+\eta^2)-46530]$
4	$(-1,1)$	$N_4^{(7)}$	$(1/184320)(1-\xi)(1-\eta)[117649(\xi^6+\eta^6)-84035(\xi^4+\eta^4)$ $+12691(\xi^2+\eta^2)-46530]$
5	$(-5/7,-1)$	$N_5^{(7)}$	$-(49/184320)(1-\xi^2)(1-\eta)(49\xi^2-9)(49\xi^2-1)(7\xi-5)$
6	$(-3/7,-1)$	$N_6^{(7)}$	$(49/61440)(1-\xi^2)(1-\eta)(49\xi^2-25)(49\xi^2-1)(7\xi-3)$
7	$(-1/7,-1)$	$N_7^{(7)}$	$-(49/36864)(1-\xi^2)(1-\eta)(49\xi^2-25)(49\xi^2-9)(7\xi-1)$
8	$(1/7,-1)$	$N_8^{(7)}$	$(49/36864)(1-\xi^2)(1-\eta)(49\xi^2-25)(49\xi^2-9)(7\xi+1)$
9	$(3/7,-1)$	$N_9^{(7)}$	$-(49/61440)(1-\xi^2)(1-\eta)(49\xi^2-25)(49\xi^2-1)(7\xi+3)$
10	$(5/7,-1)$	$N_{10}^{(7)}$	$-(49/184320)(1-\xi^2)(1-\eta)(49\xi^2-9)(49\xi^2-1)(7\xi+5)$
11	$(1,-5/7)$	$N_{11}^{(7)}$	$-(49/184320)(1+\xi)(1-\eta^2)(49\eta^2-9)(49\eta^2-1)(7\eta-5)$
12	$(1,-3/7)$	$N_{12}^{(7)}$	$(49/61440)(1+\xi)(1-\eta^2)(49\eta^2-25)(49\eta^2-1)(7\eta-3)$
13	$(1,-1/7)$	$N_{13}^{(7)}$	$-(49/36864)(1+\xi)(1-\eta^2)(49\eta^2-25)(49\eta^2-9)(7\eta-1)$
14	$(1,1/7)$	$N_{14}^{(7)}$	$-(49/36864)(1+\xi)(1-\eta^2)(49\eta^2-25)(49\eta^2-9)(7\eta+1)$
15	$(1,3/7)$	$N_{15}^{(7)}$	$(49/61440)(1+\xi)(1-\eta^2)(49\eta^2-25)(49\eta^2-1)(7\eta+3)$
16	$(1,-5/7)$	$N_{16}^{(7)}$	$-(49/184320)(1+\xi)(1-\eta^2)(49\eta^2-9)(49\eta^2-1)(7\eta+5)$
17	$(5/7,1)$	$N_{17}^{(7)}$	$(49/184320)(1-\xi^2)(1+\eta)(49\xi^2-9)(49\xi^2-1)(7\xi+5)$
18	$(3/7,1)$	$N_{18}^{(7)}$	$-(49/61440)(1-\xi^2)(1+\eta)(49\xi^2-25)(49\xi^2-1)(7\xi+3)$
19	$(1/7,1)$	$N_{19}^{(7)}$	$(49/36864)(1-\xi^2)(1+\eta)(49\xi^2-25)(49\xi^2-9)(7\xi+1)$
20	$(-1/7,1)$	$N_{20}^{(7)}$	$-(49/36864)(1-\xi^2)(1+\eta)(49\xi^2-25)(49\xi^2-9)(7\xi-1)$
21	$(-3/7,-1)$	$N_{21}^{(7)}$	$(49/61440)(1-\xi^2)(1+\eta)(49\xi^2-25)(49\xi^2-1)(7\xi-3)$
22	$(-5/7,1)$	$N_{22}^{(7)}$	$-(49/184320)(1-\xi^2)(1+\eta)(49\xi^2-9)(49\xi^2-1)(7\xi-5)$
23	$(-1,5/7)$	$N_{23}^{(7)}$	$-(49/184320)(1-\xi)(1-\eta^2)(49\eta^2-9)(49\eta^2-1)(7\eta+5)$
24	$(-1,3/7)$	$N_{24}^{(7)}$	$(49/61440)(1-\xi)(1-\eta^2)(49\eta^2-25)(49\eta^2-1)(7\eta+3)$
25	$(-1,1/7)$	$N_{25}^{(7)}$	$-(49/36864)(1-\xi)(1-\eta^2)(49\eta^2-25)(49\eta^2-9)(7\eta+1)$
26	$(-1,-1/7)$	$N_{26}^{(7)}$	$-(49/36864)(1-\xi)(1-\eta^2)(49\eta^2-25)(49\eta^2-9)(7\eta-1)$
27	$(-1,-3/7)$	$N_{27}^{(7)}$	$(49/61440)(1-\xi)(1-\eta^2)(49\eta^2-25)(49\eta^2-1)(7\eta-3)$
28	$(-1,-5/7)$	$N_{28}^{(7)}$	$-(49/184320)(1-\xi)(1-\eta^2)(49\eta^2-9)(49\eta^2-1)(7\eta-5)$

4. *Septic General Complete Lagrange element* ($n=7,\ i_7=12$). From Fig. 9a,b and Eq. (25), we obtain

$$\hat{N}_i^{(7)}(\xi,\eta)=N_i^{(7)}(\xi,\eta)-\sum_{k=29}^{32}N_i^{(7)}\left(\xi_k^{(7)},\eta_k^{(7)}\right)\hat{N}_k^{(7)}(\xi,\eta),\quad i=1(1)28. \tag{25}$$

5. *Octic General Complete Lagrange element* ($n=8,\ i_8=17$). From Fig. 10a,b and Eq. (21), we obtain

$$\hat{N}_i^{(8)}(\xi,\eta)=N_i^{(8)}(\xi,\eta)-\sum_{k=33}^{49}N_i^{(8)}\left(\xi_k^{(8)},\eta_k^{(8)}\right)\hat{N}_k^{(8)}(\xi,\eta),\quad i=1(1)32. \tag{26}$$

Table 5
Octic General Serendipity element Shape functions

Node i	Co-ordinates (ξ_i, η_i)	$N_i^{(8)}$	Shape function
1	$(-1, -1)$	$N_1^{(8)}$	$(1/44100)(1-\xi)(1-\eta)[-35840(\xi^7+\eta^7)+31360(\xi^5+\eta^5)$ $-6860(\xi^3+\eta^3)+315(\xi+\eta)-11025]$
2	$(1, -1)$	$N_2^{(8)}$	$(1/44100)(1+\xi)(1-\eta)[35840(\xi^7-\eta^7)-31360(\xi^5-\eta^5)$ $+6860(\xi^3-\eta^3)-315(\xi-\eta)-11025]$
3	$(-1, -1)$	$N_3^{(8)}$	$(1/44100)(1+\xi)(1+\eta)[35840(\xi^7+\eta^7)-31360(\xi^5+\eta^5)$ $+6860(\xi^3+\eta^3)-315(\xi+\eta)-11025]$
4	$(-1, 1)$	$N_4^{(8)}$	$(1/44100)(1-\xi)(1+\eta)[-35840(\xi^7-\eta^7)+31360(\xi^5-\eta^5)$ $-6860(\xi^3-\eta^3)+315(\xi-\eta)-11025]$
5	$(-3/4, -1)$	$N_5^{(8)}$	$(2/315)(1-\xi^2)(1-\eta)(16\xi^2-4)(16\xi^2-1)\xi(4\xi-3)$
6	$(-2/4, -1)$	$N_6^{(8)}$	$-(1/45)(1-\xi^2)(1-\eta)(16\xi^2-9)(16\xi^2-1)\xi(4\xi-2)$
7	$(-1/4, -1)$	$N_7^{(8)}$	$(2/45)(1-\xi^2)(1-\eta)(16\xi^2-9)(16\xi^2-4)\xi(4\xi-1)$
8	$(0, -1)$	$N_8^{(8)}$	$-(1/72)(1-\xi^2)(1-\eta)(16\xi^2-9)(16\xi^2-4)(16\xi^2-1)$
9	$(1/4, -1)$	$N_9^{(8)}$	$(2/45)(1-\xi^2)(1-\eta)(16\xi^2-9)(16\xi^2-4)\xi(4\xi+1)$
10	$(2/4, -1)$	$N_{10}^{(8)}$	$-(1/45)(1-\xi^2)(1-\eta)(16\xi^2-9)(16\xi^2-1)\xi(4\xi+2)$
11	$(3/4, -1)$	$N_{11}^{(8)}$	$(2/315)(1-\xi^2)(1-\eta)(16\xi^2-4)(16\xi^2-1)\xi(4\xi+3)$
12	$(1, -3/4)$	$N_{12}^{(8)}$	$(2/315)(1+\xi)(1-\eta^2)(16\eta^2-4)(16\eta^2-1)\eta(4\eta-3)$
13	$(1, -2/4)$	$N_{13}^{(8)}$	$-(1/45)(1+\xi)(1-\eta^2)(16\eta^2-9)(16\eta^2-1)\eta(4\eta-2)$
14	$(1, -1/4)$	$N_{14}^{(8)}$	$(2/45)(1+\xi)(1-\eta^2)(16\eta^2-9)(16\eta^2-4)\eta(4\eta-1)$
15	$(1, 0)$	$N_{15}^{(8)}$	$-(1/72)(1+\xi)(1-\eta^2)(16\eta^2-9)(16\eta^2-4)(16\eta^2-1)$
16	$(1, 1/4)$	$N_{16}^{(8)}$	$(2/45)(1+\xi)(1-\eta^2)(16\eta^2-9)(16\eta^2-4)\eta(4\eta+1)$
17	$(1, 2/4)$	$N_{17}^{(8)}$	$-(1/45)(1+\xi)(1-\eta^2)(16\eta^2-9)(16\eta^2-1)\eta(4\eta+2)$
18	$(1, 3/4)$	$N_{18}^{(8)}$	$(2/315)(1+\xi)(1-\eta^2)(16\eta^2-4)(16\eta^2-1)\eta(4\eta+3)$
19	$(3/4, 1)$	$N_{19}^{(8)}$	$(2/315)(1-\xi^2)(1+\eta)(16\xi^2-4)(16\xi^2-1)\xi(4\xi+3)$
20	$(2/4, 1)$	$N_{20}^{(8)}$	$-(1/45)(1-\xi^2)(1+\eta)(16\xi^2-9)(16\xi^2-1)\xi(4\xi+2)$
21	$(1/4, 1)$	$N_{21}^{(8)}$	$(2/45)(1-\xi^2)(1+\eta)(16\xi^2-9)(16\xi^2-4)\xi(4\xi+1)$
22	$(0, 1)$	$N_{22}^{(8)}$	$-(1/72)(1-\xi^2)(1+\eta)(16\xi^2-9)(16\xi^2-4)(16\xi^2-1)$
23	$(-1/4, 1)$	$N_{23}^{(8)}$	$(2/45)(1-\xi^2)(1+\eta)(16\xi^2-9)(16\xi^2-4)\xi(4\xi-1)$
24	$(-2/4, 1)$	$N_{24}^{(8)}$	$-(1/45)(1-\xi^2)(1+\eta)(16\xi^2-9)(16\xi^2-1)\xi(4\xi-2)$
25	$(-3/4, 1)$	$N_{25}^{(8)}$	$(2/315)(1-\xi^2)(1+\eta)(16\xi^2-4)(16\xi^2-1)\xi(4\xi-3)$
26	$(-1, 3/4)$	$N_{26}^{(8)}$	$(2/315)(1-\xi)(1-\eta^2)(16\eta^2-4)(16\eta^2-1)\eta(4\eta+3)$
27	$(-1, 2/4)$	$N_{27}^{(8)}$	$-(1/45)(1-\xi)(1-\eta^2)(16\eta^2-9)(16\eta^2-1)\eta(4\eta+2)$
28	$(-1, 1/4)$	$N_{28}^{(8)}$	$(2/45)(1-\xi)(1-\eta^2)(16\eta^2-9)(16\eta^2-4)\eta(4\eta+1)$
29	$(-1, 0)$	$N_{29}^{(8)}$	$-(1/72)(1-\xi)(1-\eta^2)(16\eta^2-9)(16\eta^2-4)(16\eta^2-1)$
30	$(-1, -1/4)$	$N_{30}^{(8)}$	$(2/45)(1-\xi)(1-\eta^2)(16\eta^2-9)(16\eta^2-4)\eta(4\eta-1)$
31	$(-1, -2/4)$	$N_{31}^{(8)}$	$-(1/45)(1-\xi)(1-\eta^2)(16\eta^2-9)(16\eta^2-1)\eta(4\eta-2)$
32	$(-1, -3/4)$	$N_{32}^{(8)}$	$(2/315)(1-\xi)(1-\eta^2)(16\eta^2-4)(16\eta^2-1)\eta(4\eta-3)$

Table 6
Nineth-order General Serendipity element Shape functions

Node i	Co-ordinates (ξ_i, η_i)	$N_i^{(9)}$	Shape function
1	$(-1,-1)$	$N_1^{(9)}$	$(1/41287680)(1-\xi)(1-\eta)[43046721(\xi^8+\eta^8)-44641044(\xi^6+\eta^6)$ $+12951414(\xi^4+\eta^4)-1046196(\xi^2+\eta^2)-10299870]$
2	$(1,-1)$	$N_2^{(9)}$	$(1/41287680)(1+\xi)(1-\eta)[43046721(\xi^8+\eta^8)-44641044(\xi^6+\eta^6)$ $+12951414(\xi^4+\eta^4)-1046196(\xi^2+\eta^2)-10299870]$
3	$(1,1)$	$N_3^{(9)}$	$(1/41287680)(1+\xi)(1+\eta)[43046721(\xi^8+\eta^8)-44641044(\xi^6+\eta^6)$ $+12951414(\xi^4+\eta^4)-1046196(\xi^2+\eta^2)-10299870]$
4	$(-1,1)$	$N_4^{(9)}$	$(1/41287680)(1-\xi)(1+\eta)[43046721(\xi^8+\eta^8)-44641044(\xi^6+\eta^6)$ $+12951414(\xi^4+\eta^4)-1046196(\xi^2+\eta^2)-10299870]$
5	$(-7/9,-1)$	$N_5^{(9)}$	$-(81/41287680)(1-\xi^2)(1+\eta)(81\xi^2-25)(81\xi^2-9)(81\xi^2-1)(9\xi-7)$
6	$(-5/9,-1)$	$N_6^{(9)}$	$(81/10321920)(1-\xi^2)(1-\eta)(81\xi^2-49)(81\xi^2-9)(81\xi^2-1)(9\xi-5)$
7	$(-3/9,-1)$	$N_7^{(9)}$	$-(81/4423680)(1-\xi^2)(1-\eta)(81\xi^2-49)(81\xi^2-25)(81\xi^2-1)(9\xi-3)$
8	$(-1/9,-1)$	$N_8^{(9)}$	$(81/2949120)(1-\xi^2)(1-\eta)(81\xi^2-49)(81\xi^2-25)(81\xi^2-9)(9\xi-1)$
9	$(1/9,-1)$	$N_9^{(9)}$	$-(81/2949120)(1-\xi^2)(1-\eta)(81\xi^2-49)(81\xi^2-25)(81\xi^2-9)(9\xi+1)$
10	$(3/9,-1)$	$N_{10}^{(9)}$	$(81/4423680)(1-\xi^2)(1-\eta)(81\xi^2-49)(81\xi^2-25)(81\xi^2-1)(9\xi+3)$
11	$(5/9,-1)$	$N_{11}^{(9)}$	$-(81/10321920)(1-\xi^2)(1-\eta)(81\xi^2-49)(81\xi^2-9)(81\xi^2-1)(9\xi+5)$
12	$(7/9,-1)$	$N_{12}^{(9)}$	$(81/41287680)(1-\xi^2)(1-\eta)(81\xi^2-25)(81\xi^2-9)(81\xi^2-1)(9\xi+7)$
13	$(1,-7/9)$	$N_{13}^{(9)}$	$-(81/41287680)(1+\xi)(1-\eta^2)(81\eta^2-25)(81\eta^2-9)(81\eta^2-1)(9\eta-7)$
14	$(1,-5/9)$	$N_{14}^{(9)}$	$(81/10321920)(1+\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-9)(81\eta^2-1)(9\eta-5)$
15	$(1,-3/9)$	$N_{15}^{(9)}$	$-(81/4423680)(1+\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-25)(81\eta^2-1)(9\eta-3)$
16	$(1,-1/9)$	$N_{16}^{(9)}$	$(81/2949120)(1+\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-25)(81\eta^2-9)(9\eta-1)$
17	$(1,1/9)$	$N_{17}^{(9)}$	$-(81/2949120)(1+\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-25)(81\eta^2-9)(9\eta+1)$
18	$(1,3/9)$	$N_{18}^{(9)}$	$(81/4423680)(1+\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-25)(81\eta^2-1)(9\eta+3)$
19	$(1,5/9)$	$N_{19}^{(9)}$	$-(81/10321920)(1+\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-9)(81\eta^2-1)(9\eta+5)$
20	$(1,7/9)$	$N_{20}^{(9)}$	$(81/41287680)(1+\xi)(1-\eta^2)(81\eta^2-25)(81\eta^2-9)(81\eta^2-1)(9\eta+7)$
21	$(7/9,1)$	$N_{21}^{(9)}$	$(81/41287680)(1-\xi^2)(1+\eta)(81\xi^2-25)(81\xi^2-9)(81\xi^2-1)(9\xi+7)$
22	$(5/9,1)$	$N_{22}^{(9)}$	$-(81/10321920)(1-\xi^2)(1+\eta)(81\xi^2-49)(81\xi^2-9)(81\xi^2-1)(9\xi+5)$
23	$(3/9,1)$	$N_{23}^{(9)}$	$(81/4423680)(1-\xi^2)(1+\eta)(81\xi^2-49)(81\xi^2-25)(81\xi^2-1)(9\xi+3)$
24	$(1/9,1)$	$N_{24}^{(9)}$	$-(81/2949120)(1-\xi^2)(1+\eta)(81\xi^2-49)(81\xi^2-25)(81\xi^2-9)(9\xi+1)$
25	$(-1/9,1)$	$N_{25}^{(9)}$	$(81/2949120)(1-\xi^2)(1+\eta)(81\xi^2-49)(81\xi^2-25)(81\xi^2-9)(9\xi-1)$
26	$(-3/9,1)$	$N_{26}^{(9)}$	$-(81/4423680)(1-\xi^2)(1+\eta)(81\xi^2-49)(81\xi^2-25)(81\xi^2-1)(9\xi-3)$
27	$(-5/9,1)$	$N_{27}^{(9)}$	$(81/10321920)(1-\xi^2)(1+\eta)(81\xi^2-49)(81\xi^2-9)(81\xi^2-1)(9\xi-5)$
28	$(-7/9,1)$	$N_{28}^{(9)}$	$-(81/41287680)(1-\xi^2)(1+\eta)(81\xi^2-25)(81\xi^2-9)(81\xi^2-1)(9\xi-7)$
29	$(-1,7/9)$	$N_{29}^{(9)}$	$(81/41287680)(1-\xi)(1-\eta^2)(81\eta^2-25)(81\eta^2-9)(81\eta^2-1)(9\eta+7)$
30	$(-1,5/9)$	$N_{30}^{(9)}$	$-(81/10321920)(1-\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-9)(81\eta^2-1)(9\eta+5)$
31	$(-1,3/9)$	$N_{31}^{(9)}$	$(81/4423680)(1-\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-25)(81\eta^2-1)(9\eta+3)$
32	$(-1,1/9)$	$N_{32}^{(9)}$	$-(81/2949120)(1-\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-25)(81\eta^2-9)(9\eta+1)$
33	$(-1,-1/9)$	$N_{33}^{(9)}$	$(81/2949120)(1-\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-25)(81\eta^2-9)(9\eta-1)$

(*continued overleaf*)

Table 6 (Continued)

Node i	Co-ordinates (ξ_i, η_i)	$N_i^{(9)}$	Shape function
34	$(-1, -3/9)$	$N_{34}^{(9)}$	$-(81/4423680)(1-\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-25)(81\eta^2-1)(9\eta-3)$
35	$(-1, -5/9)$	$N_{35}^{(9)}$	$(81/10321920)(1-\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-9)(81\eta^2-1)(9\eta-5)$
36	$(-1, -7/9)$	$N_{36}^{(9)}$	$-(81/41287680)(1-\xi)(1-\eta^2)(81\eta^2-25)(81\eta^2-9)(81\eta^2-1)(9\eta-7)$

Table 7
Tenth-order General Serendipity element Shape functions

Node i	Co-ordinates (ξ_i, η_i)	$Ni^{(10)}$	Shape function
1	$(-1, -1)$	$N_1^{(10)}$	$(1/290304)(1-\xi)(1-\eta)[-390625(\xi^9+\eta^9)+468750(\xi^7+\eta^7)$ $-170625(\xi^5+\eta^5)+20500(\xi^3+\eta^3)-576(\xi+\eta)-72576]$
2	$(1, -1)$	$N_2^{(10)}$	$(1/290304)(1+\xi)(1-\eta)[-390625(\xi^9-\eta^9)-468750(\xi^7-\eta^7)$ $+170625(\xi^5-\eta^5)-20500(\xi^3-\eta^3)+576(\xi-\eta)-72576]$
3	(1,1)	$N_3^{(10)}$	$(1/290304)(1+\xi)(1+\eta)[390625(\xi^9+\eta^9)-468750(\xi^7+\eta^7)$ $+70625(\xi^5+\eta^5)-20500(\xi^3+\eta^3)+576(\xi+\eta)-72576]$
4	$(-1, 1)$	$N_4^{(10)}$	$(1/290304)(1-\xi)(1+\eta)[-390625(\xi^9-\eta^9)+468750(\xi^7-\eta^7)$ $-170625(\xi^5-\eta^5)+20500(\xi^3-\eta^3)-576(\xi-\eta)-72576]$
5	$(-4/5, -1)$	$N_5^{(10)}$	$(25/145152)(1-\xi^2)(1-\eta)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)\xi(5\xi-4)$
6	$(-3/5, -1)$	$N_6^{(10)}$	$(-25/32256)(1-\xi^2)(1-\eta)(25\xi^2-16)(25\xi^2-4)(25\xi^2-1)\xi(5\xi-3)$
7	$(-2/5, -1)$	$N_7^{(10)}$	$(25/12096)(1-\xi^2)(1-\eta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-1)\xi(5\xi-2)$
8	$(-4/5, -1)$	$N_8^{(10)}$	$(-25/6912)(1-\xi^2)(1-\eta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)\xi(5\xi-1)$
9	$(-4/5, -1)$	$N_9^{(10)}$	$(1/1152)(1-\xi^2)(1-\eta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)(25\xi-1)$
10	$(1/5, -1)$	$N_{10}^{(10)}$	$(-25/6912)(1-\xi^2)(1-\eta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)\xi(5\xi+1)$
11	$(2/5, -1)$	$N_{11}^{(10)}$	$(25/12096)(1-\xi^2)(1-\eta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-1)\xi(5\xi+2)$
12	$(3/5, -1)$	$N_{12}^{(10)}$	$(-25/32256)(1-\xi^2)(1-\eta)(25\xi^2-16)(25\xi^2-4)(25\xi^2-1)\xi(5\xi+3)$
13	$(4/5, -1)$	$N_{13}^{(10)}$	$(25/145152)(1-\xi^2)(1-\eta)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)\xi(5\xi+4)$
14	$(1, -4/5)$	$N_{14}^{(10)}$	$(25/145152)(1+\xi)(1-\eta^2)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)\eta(5\eta-4)$
15	$(1, -3/5)$	$N_{15}^{(10)}$	$(-25/32256)(1+\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-4)(25\eta^2-1)\eta(5\eta-3)$
16	$(1, -2/5)$	$N_{16}^{(10)}$	$(25/12096)(1+\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-9)(25\eta^2-1)\eta(5\eta-2)$
17	$(1, -1/5)$	$N_{17}^{(10)}$	$(-25/6912)(1+\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)\eta(5\eta-1)$
18	$(1, 0)$	$N_{18}^{(10)}$	$(1/1152)(1+\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)$
19	$(1, 1/5)$	$N_{19}^{(10)}$	$(-25/6912)(1+\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)\eta(5\eta+1)$
20	$(1, 2/5)$	$N_{20}^{(10)}$	$(25/12096)(1+\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-9)(25\eta^2-1)\eta(5\eta+2)$
21	$(1, 3/5)$	$N_{21}^{(10)}$	$(-25/32256)(1+\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-4)(25\eta^2-1)\eta(5\eta+3)$
22	$(1, 4/5)$	$N_{22}^{(10)}$	$(25/145152)(1+\xi)(1-\eta^2)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)\eta(5\eta+4)$
23	$(4/5, 1)$	$N_{23}^{(10)}$	$(25/145152)(1-\xi^2)(1+\eta)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)\xi(5\xi+4)$
24	$(3/5, 1)$	$N_{24}^{(10)}$	$(-25/32256)(1-\xi^2)(1+\eta)(25\xi^2-16)(25\xi^2-4)(25\xi^2-1)\xi(5\xi+3)$
25	$(2/5, 1)$	$N_{25}^{(10)}$	$(25/12096)(1-\xi^2)(1+\eta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-1)\xi(5\xi+2)$
26	$(1/5, 1)$	$N_{26}^{(10)}$	$(-25/6912)(1-\xi^2)(1+\eta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)\xi(5\xi+1)$

Table 7 (Continued)

Node i	Co-ordinates (ξ_i, η_i)	$Ni^{(10)}$	Shape function
27	$(0, 1)$	$N_{27}^{(10)}$	$(1/1152)(1-\xi^2)(1+\eta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)$
28	$(-1/5, 1)$	$N_{28}^{(10)}$	$(-25/6912)(1-\xi^2)(1+\eta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)\xi(5\xi-1)$
29	$(-2/5, 1)$	$N_{29}^{(10)}$	$(25/12096)(1-\xi^2)(1+\eta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-1)\xi(5\xi-2)$
30	$(-3/5, 1)$	$N_{30}^{(10)}$	$(-25/32256)(1-\xi^2)(1+\eta)(25\xi^2-16)(25\xi^2-4)(25\xi^2-1)\xi(5\xi-3)$
31	$(-4/5, 1)$	$N_{31}^{(10)}$	$(25/145152)(1-\xi^2)(1+\eta)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)\xi(5\xi-4)$
32	$(-1, 4/5)$	$N_{32}^{(10)}$	$(25/145152)(1-\xi)(1-\eta^2)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)\eta(5\eta+4)$
33	$(-1, 3/5)$	$N_{33}^{(10)}$	$(-25/32256)(1-\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-4)(25\eta^2-1)\eta(5\eta+3)$
34	$(-1, 2/5)$	$N_{34}^{(10)}$	$(25/12096)(1-\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-9)(25\eta^2-1)\eta(5\eta+2)$
35	$(-1, 1/5)$	$N_{35}^{(10)}$	$(-25/6912)(1-\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)\eta(5\eta+1)$
36	$(-1, 0)$	$N_{36}^{(10)}$	$(1/1152)(1-\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)$
37	$(-1, -1/5)$	$N_{37}^{(10)}$	$(-25/6912)(1-\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)\eta(5\eta-1)$
38	$(-1, -2/5)$	$N_{38}^{(10)}$	$(25/12096)(1-\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-9)(25\eta^2-1)\eta(5\eta-2)$
39	$(-1, -3/5)$	$N_{39}^{(10)}$	$(-25/32256)(1-\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-4)(25\eta^2-1)\eta(5\eta-3)$
40	$(-1, -4/5)$	$N_{40}^{(10)}$	$(25/145152)(1-\xi)(1-\eta^2)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)\eta(5\eta-4)$

Table 8
Quartic General Complete Lagrange element Shape functions

Node i	Co-ordinates (ξ_i, η_i)	$\hat{N}_i^{(4)}$	Shape function
1	$(-1, -1)$	$\hat{N}_1^{(4)}$	$(1/12)(1-\xi)(1-\eta)[-4(\xi^3+\eta^3)+4(\xi+\eta)+3\xi\eta]$
2	$(1, -1)$	$\hat{N}_2^{(4)}$	$(1/12)(1+\xi)(1-\eta)[4(\xi^3-\eta^3)-4(\xi-\eta)-3\xi\eta]$
3	$(1, 1)$	$\hat{N}_3^{(4)}$	$(1/12)(1+\xi)(1+\eta)[4(\xi^3+\eta^3)-4(\xi+\eta)+3\xi\eta]$
4	$(-1, 1)$	$\hat{N}_4^{(4)}$	$(1/12)(1-\xi)(1+\eta)[-4(\xi^3-\eta^3)+4(\xi-\eta)-3\xi\eta]$
5	$(-1/2, -1)$	$\hat{N}_5{}^{(4)}$	$(2/3)(1-\xi^2)(1-\eta)\xi(2\xi-1)$
6	$(0, -1)$	$\hat{N}_6^{(4)}$	$-(1/2)(1-\xi^2)(1-\eta)(4\xi^2+\eta)$
7	$(1/2, -1)$	$\hat{N}_7^{(4)}$	$(2/3)(1-\xi^2)(1-\eta)\xi(2\xi+1)$
8	$(1, -1/2)$	$\hat{N}_8^{(4)}$	$(2/3)(1+\xi)(1-\eta^2)\eta(2\eta-1)$
9	$(1, 0)$	$\hat{N}_9^{(4)}$	$-(1/2)(1+\xi)(1-\eta^2)(4\eta^2-\xi)$
10	$(1, 1/2)$	$\hat{N}_{10}^{(4)}$	$(2/3)(1+\xi)(1-\eta^2)\eta(2\eta+1)$
11	$(1/2, 1)$	$\hat{N}_{11}^{(4)}$	$(2/3)(1-\xi^2)(1+\eta)\xi(2\xi+1)$
12	$(0, 1)$	$\hat{N}_{12}^{(4)}$	$-(1/2)(1-\xi^2)(1+\eta)(4\xi^2-\eta)$
13	$(-1/2, 1)$	$\hat{N}_{13}^{(4)}$	$(2/3)(1-\xi^2)(1+\eta)\xi(2\xi-1)$
14	$(-1, 1/2)$	$\hat{N}_{14}^{(4)}$	$(2/3)(1-\xi)(1-\eta^2)\eta(2\eta+1)$
15	$(-1, 0)$	$\hat{N}_{15}^{(4)}$	$-(1/2)(1-\xi)(1-\eta^2)(4\eta^2+\xi)$
16	$(-1, -1/2)$	$\hat{N}_{16}^{(4)}$	$(2/3)(1-\xi)(1-\eta^2)\eta(2\eta-1)$
17	$(0, 0)$	$\hat{N}_{17}^{(4)}$	$(1-\xi^2)(1-\eta^2)$

Table 9
Quintic General Complete Lagrange element Shape functions

Node i	Co-ordinates (ξ_i, η_i)	$\hat{N}_i^{(5)}$	Shape function
1	$(-1,-1)$	$\hat{N}_1^{(5)}$	$(1/1536)(1-\xi)(1-\eta)[625(\xi^4+\eta^4)-984(\xi^2+\eta^2)+734\xi^2\eta^2+368]$
2	$(1,-1)$	$\hat{N}_2^{(5)}$	$(1/1536)(1+\xi)(1-\eta)[625(\xi^4+\eta^4)-984(\xi^2+\eta^2)+734\xi^2\eta^2+368]$
3	$(1,1)$	$\hat{N}_3^{(5)}$	$(1/1536)(1+\xi)(1+\eta)[625(\xi^4+\eta^4)-984(\xi^2+\eta^2)+734\xi^2\eta^2+368]$
4	$(-1,1)$	$\hat{N}_4^{(5)}$	$(1/1536)(1-\xi)(1+\eta)[625(\xi^4+\eta^4)-984(\xi^2+\eta^2)+734\xi^2\eta^2+368]$
5	$(-3/5,-1)$	$\hat{N}_5^{(5)}$	$-(25/1536)(1-\xi^2)(1-\eta)(5\xi-3)(25\xi^2+7\eta^2-8)$
6	$(-1/5,-1)$	$\hat{N}_6^{(5)}$	$(25/2304)(1-\xi^2)(1-\eta)(5\xi-1)(75\xi^2-11\eta^2-16)$
7	$(1/5,-1)$	$\hat{N}_7^{(5)}$	$-(25/2304)(1-\xi^2)(1-\eta)(5\xi+1)(75\xi^2-11\eta^2-16)$
8	$(3/5,-1)$	$\hat{N}_8^{(5)}$	$(25/1536)(1-\xi^2)(1-\eta)(5\xi+3)(25\xi^2+7\eta^2-8)$
9	$(1,-3/5)$	$\hat{N}_9^{(5)}$	$-(25/1536)(1+\xi)(1-\eta^2)(5\eta-3)(7\xi^2+25\eta^2-8)$
10	$(1,-1/5)$	$\hat{N}_{10}^{(5)}$	$(25/2304)(1+\xi)(1-\eta^2)(5\eta-1)(-11\xi^2+75\eta^2-16)$
11	$(1,1/5)$	$\hat{N}_{11}^{(5)}$	$-(25/2304)(1+\xi)(1-\eta^2)(5\eta+1)(-11\xi^2+75\eta^2-16)$
12	$(1,3/5)$	$\hat{N}_{12}^{(5)}$	$(25/1536)(1+\xi)(1-\eta^2)(5\eta+3)(7\xi^2+25\eta^2-8)$
13	$(3/5,1)$	$\hat{N}_{13}^{(5)}$	$(25/1536)(1-\xi^2)(1+\eta)(5\xi+3)(25\xi^2+7\eta^2-8)$
14	$(1/5,1)$	$\hat{N}_{14}^{(5)}$	$-(25/2304)(1-\xi^2)(1+\eta)(5\xi+1)(75\xi^2-11\eta^2-16)$
15	$(-1/5,1)$	$\hat{N}_{15}^{(5)}$	$(25/2304)(1-\xi^2)(1+\eta)(5\xi-1)(75\xi^2-11\eta^2-16)$
16	$(-3/5,1)$	$\hat{N}_{16}^{(5)}$	$-(25/1536)(1-\xi^2)(1+\eta)(5\xi-3)(25\xi^2+7\eta^2-8)$
17	$(-1,3/5)$	$\hat{N}_{17}^{(5)}$	$(25/1536)(1-\xi)(1-\eta^2)(5\eta+3)(7\xi^2+25\eta^2-8)$
18	$(-1,1/5)$	$\hat{N}_{18}^{(5)}$	$-(25/2304)(1-\xi)(1-\eta^2)(5\eta+1)(-11\xi^2+75\eta^2-16)$
19	$(-1,-1/5)$	$\hat{N}_{19}^{(5)}$	$(25/2304)(1-\xi)(1-\eta^2)(5\eta-1)(-11\xi^2+75\eta^2-16)$
20	$(-1,-3/5)$	$\hat{N}_{20}^{(5)}$	$-(25/1536)(1-\xi)(1-\eta^2)(5\eta-3)(7\xi^2+25\eta^2-8)$
21	$(\mathbf{-1/2,-1/2})$	$\hat{\mathbf{N}}_{\mathbf{21}}^{\mathbf{(5)}}$	$\mathbf{(4/9)(1-\xi^2)(1-\eta^2)(1-2\xi)(1-2\eta)}$
22	$(\mathbf{1/2,-1/2})$	$\hat{\mathbf{N}}_{\mathbf{22}}^{\mathbf{(5)}}$	$\mathbf{(4/9)(1-\xi^2)(1-\eta^2)(1+2\xi)(1-2\eta)}$
23	$(\mathbf{1/2,1/2})$	$\hat{\mathbf{N}}_{\mathbf{23}}^{\mathbf{(5)}}$	$\mathbf{(4/9)(1-\xi^2)(1-\eta^2)(1+2\xi)(1+2\eta)}$
24	$(\mathbf{-1/2,1/2})$	$\hat{\mathbf{N}}_{\mathbf{24}}^{\mathbf{(5)}}$	$\mathbf{(4/9)(1-\xi^2)(1-\eta^2)(1-2\xi)(1+2\eta)}$

6. *Nineth-order General Complete Lagrange element* ($n = 9,\ i_9 = 24$). From Fig. 11a,b and Eq. (21), we obtain

$$\hat{N}_i^{(9)}(\xi,\eta) = N_i^{(9)}(\xi,\eta) - \sum_{k=37}^{60} N_i^{(9)}\left(\xi_k^{(9)},\eta_k^{(9)}\right)\hat{N}_k^{(9)}(\xi,\eta), \quad i = 1(1)36. \tag{27}$$

7. *Tenth-order General Complete Lagrange element* ($n = 10,\ i_{10} = 32$). From Fig. 12a,b and Eq. (21), we obtain

$$\hat{N}_i^{(10)}(\xi,\eta) = N_i^{(10)}(\xi,\eta) - \sum_{k=41}^{72} N_i^{(10)}\left(\xi_k^{(10)},\eta_k^{(10)}\right)\hat{N}_k^{(10)}(\xi,\eta), \quad i = 1(1)40. \tag{28}$$

In Eqs. (21)–(28), we have used the notations for internal Shape functions as

$$\hat{N}_k^{(n)}(\xi,\eta), \quad k = 4n+1, 4n+2, \ldots, 4n+i_n.$$

Table 10
Sextic General Complete Lagrange element Shape functions

Node i	Co-ordinates (ξ_i, η_i)	$\hat{N}_i^{(6)}$	Shape function
1	$(-1,-1)$	$N_1^{(6)}$	$(1/160)(1-\xi)(1-\eta)[-81(\xi^5+\eta^5)+45(\xi^3+\eta^3)-4(\xi+\eta)-40]$
2	$(1,-1)$	$N_2^{(6)}$	$(1/160)(1+\xi)(1-\eta)[81(\xi^5-\eta^5)-45(\xi^3-\eta^3)+4(\xi-\eta)-40]$
3	$(1,1)$	$N_3^{(6)}$	$(1/160)(1+\xi)(1+\eta)[81(\xi^5+\eta^5)-45(\xi^3+\eta^3)+4(\xi+\eta)-40]$
4	$(-1,1)$	$N_4^{(6)}$	$(1/160)(1-\xi)(1+\eta)[-81(\xi^5-\eta^5)+45(\xi^3+\eta^3)-4(\xi-\eta)-40]$
5	$(-2/3,-1)$	$N_5^{(6)}$	$(9/80)(1-\xi^2)(1-\eta)(9\xi^2-1)\xi(3\xi-2)$
6	$(-1/3,-1)$	$N_6^{(6)}$	$-(9/32)(1-\xi^2)(1-\eta)(9\xi^2-4)\xi(3\xi-1)$
7	$(0,-1)$	$N_7^{(6)}$	$(1/8)(1-\xi^2)(1-\eta)(9\xi^2-4)(9\xi^2-1)$
8	$(1/3,-1)$	$N_8^{(6)}$	$-(9/32)(1-\xi^2)(1-\eta)\xi(9\xi^2-4)\xi(3\xi+1)$
9	$(2/3,-1)$	$N_9^{(6)}$	$(9/80)(1-\xi^2)(1-\eta)(9\xi^2-1)\xi(3\xi+2)$
10	$(1,-2/3)$	$N_{10}^{(6)}$	$(9/80)(1+\xi)(1-\eta^2)(9\eta^2-1)\eta(3\eta-2)$
11	$(1,-1/3)$	$N_{11}^{(6)}$	$-(9/32)(1+\xi)(1-\eta^2)(9\eta^2-4)\eta(3\eta-1)$
12	$(1,0)$	$N_{12}^{(6)}$	$(1/8)(1+\xi)(1-\eta^2)(9\eta^2-1)(9\eta^2-1)$
13	$(1,1/3)$	$N_{13}^{(6)}$	$-(9/32)(1+\xi)(1-\eta^2)(9\eta^2-4)\eta(3\eta+1)$
14	$(1,2/3)$	$N_{14}^{(6)}$	$(9/80)(1+\xi)(1-\eta^2)(9\eta^2-1)\eta(3\eta+2)$
15	$(2/3,1)$	$N_{15}^{(6)}$	$(9/80)(1-\xi^2)(1+\eta)(9\xi^2-1)\xi(3\xi+2)$
16	$(1/3,1)$	$N_{16}^{(6)}$	$-(9/32)(1-\xi^2)(1+\eta)(9\xi^2-4)\xi(3\xi+1)$
17	$(0,1)$	$N_{17}^{(6)}$	$(1/8)(1-\xi^2)(1+\eta)(9\xi^2-4)(9\xi^2-1)$
18	$(-1/3,1)$	$N_{18}^{(6)}$	$-(9/32)(1-\xi^2)(1+\eta)(9\xi^2-4)\xi(3\xi-1)$
19	$(-2/3,-1)$	$N_{19}^{(6)}$	$(9/80)(1-\xi^2)(1+\eta)(9\xi^2-1)\xi(3\xi-2)$
20	$(-1,2/3)$	$N_{20}^{(6)}$	$(9/80)(1+\xi)(1-\eta^2)(9\eta^2-1)\eta(3\eta+2)$
21	$(-1,1/3)$	$N_{21}^{(6)}$	$-(9/32)(1+\xi)(1-\eta^2)(9\eta^2-4)\eta(3\eta+1)$
22	$(-1,0)$	$N_{22}^{(6)}$	$(1/8)(1+\xi)(1-\eta^2)(9\eta^2-1)(9\eta^2-1)$
23	$(-1,-1/3)$	$N_{23}^{(6)}$	$-(9/32)(1+\xi)(1-\eta^2)(9\eta^2-4)\eta(3\eta-1)$
24	$(-1,-2/3)$	$N_{24}^{(6)}$	$(9/80)(1+\xi)(1-\eta^2)(9\eta^2-1)\eta(3\eta-2)$
25	$(-1/2,-1/2)$	$\hat{N}_{25}^{(6)}$	$-(4/9)(1-\xi^2)(1-\eta^2)\ (1-2\xi)(1-2\eta)\{2(\xi+\eta)+1\}$
26	$(1/2,-1/2)$	$\hat{N}_{26}^{(6)}$	$(4/9)(1-\xi^2)(1-\eta^2)(1+2\xi)(1-2\eta)\{2(\xi-\eta)-1\}$
27	$(1/2,1/2)$	$\hat{N}_{27}^{(6)}$	$(4/9)(1-\xi^2)\ (1-\eta^2)(1+2\xi)\ (1+2\eta)\{2(\xi+\eta)-1\}$
28	$(-1/2,1/2)$	$\hat{N}_{28}^{(6)}$	$-(4/9)(1-\xi^2)(1-\eta^2)(1-2\xi)(1+2\eta)\{2(\xi-\eta)+1\}$
29	$(0,-1/2)$	$\hat{N}_{29}^{(6)}$	$(2/3)(1-\xi^2)(1-\eta^2)\ (1-4\xi^2)(1-2\eta)$
30	$(1/2,0)$	$\hat{N}_{30}^{(6)}$	$(2/3)(1-\xi^2)(1-\eta^2)\ (1+2\xi)(1-4\eta^2)$
31	$(0,1/2)$	$\hat{N}_{31}^{(6)}$	$(2/3)(1-\xi^2)(1-\eta^2)\ (1-4\xi^2)(1+2\eta)$
32	$(-1/2,0)$	$\hat{N}_{32}^{(6)}$	$(2/3)(1-\xi^2)(1-\eta^2)\ (1-2\xi)(1-4\eta^2)$

Table 11
Septic General Complete Lagrange element Shape functions

Node i	Co-ordinates (ξ_i, η_i)	$\hat{N}_i^{(7)}$	Shape function
1	$(-1,-1)$	$N_1^{(7)}$	$(1/184320)(1-\xi)(1-\eta)[117649(\xi^6+\eta^6)-84035(\xi^4+\eta^4)$ $+12691(\xi^2+\eta^2)-46530]$
2	$(1,-1)$	$N_2^{(7)}$	$(1/184320)(1-\xi)(1-\eta)[117649(\xi^6+\eta^6)-84035(\xi^4+\eta^4)$ $+12691(\xi^2+\eta^2)-46530]$
3	$(1,1)$	$N_3^{(7)}$	$(1/184320)(1-\xi)(1-\eta)[117649(\xi^6+\eta^6)-84035(\xi^4+\eta^4)$ $+12691(\xi^2+\eta^2)-46530]$
4	$(-1,1)$	$N_4^{(7)}$	$(1/184320)(1-\xi)(1-\eta)[117649(\xi^6+\eta^6)-84035(\xi^4+\eta^4)$ $+12691(\xi^2+\eta^2)-46530]$
5	$(-5/7,-1)$	$N_5^{(7)}$	$-(49/184320)(1-\xi^2)(1-\eta)(49\xi^2-9)(49\xi^2-1)(7\xi-5)$
6	$(-3/7,-1)$	$N_6^{(7)}$	$(49/61440)(1-\xi^2)(1-\eta)(49\xi^2-25)(49\xi^2-1)(7\xi-3)$
7	$(-1/7,-1)$	$N_7^{(7)}$	$-(49/36864)(1-\xi^2)(1-\eta)(49\xi^2-25)(49\xi^2-9)(7\xi-1)$
8	$(1/7,-1)$	$N_8^{(7)}$	$(49/36864)(1-\xi^2)(1-\eta)(49\xi^2-25)(49\xi^2-9)(7\xi+1)$
9	$(3/7,-1)$	$N_9^{(7)}$	$-(49/61440)(1-\xi^2)(1-\eta)(49\xi^2-25)(49\xi^2-1)(7\xi+3)$
10	$(5/7,-1)$	$N_{10}^{(7)}$	$-(49/184320)(1-\xi^2)(1-\eta)(49\xi^2-9)(49\xi^2-1)(7\xi+5)$
11	$(1,-5/7)$	$N_{11}^{(7)}$	$-(49/184320)(1+\xi)(1-\eta^2)(49\eta^2-9)(49\eta^2-1)(7\eta-5)$
12	$(1,-3/7)$	$N_{12}^{(7)}$	$(49/61440)(1+\xi)(1-\eta^2)(49\eta^2-25)(49\eta^2-1)(7\eta-3)$
13	$(1,-1/7)$	$N_{13}^{(7)}$	$-(49/36864)(1+\xi)(1-\eta^2)(49\eta^2-25)(49\eta^2-9)(7\eta-1)$
14	$(1,1/7)$	$N_{14}^{(7)}$	$-(49/36864)(1+\xi)(1-\eta^2)(49\eta^2-25)(49\eta^2-9)(7\eta+1)$
15	$(1,3/7)$	$N_{15}^{(7)}$	$(49/61440)(1+\xi)(1-\eta^2)(49\eta^2-25)(49\eta^2-1)(7\eta+3)$
16	$(1,-5/7)$	$N_{16}^{(7)}$	$-(49/184320)(1+\xi)(1-\eta^2)(49\eta^2-9)(49\eta^2-1)(7\eta+5)$
17	$(5/7,1)$	$N_{17}^{(7)}$	$(49/184320)(1-\xi^2)(1+\eta)(49\xi^2-9)(49\xi^2-1)(7\xi+5)$
18	$(3/7,1)$	$N_{18}^{(7)}$	$-(49/61440)(1-\xi^2)(1+\eta)(49\xi^2-25)(49\xi^2-1)(7\xi+3)$
19	$(1/7,1)$	$N_{19}^{(7)}$	$(49/36864)(1-\xi^2)(1+\eta)(49\xi^2-25)(49\xi^2-9)(7\xi+1)$
20	$(-1/7,1)$	$N_{20}^{(7)}$	$-(49/36864)(1-\xi^2)(1+\eta)(49\xi^2-25)(49\xi^2-9)(7\xi-1)$
21	$(-3/7,-1)$	$N_{21}^{(7)}$	$(49/61440)(1-\xi^2)(1+\eta)(49\xi^2-25)(49\xi^2-1)(7\xi-3)$
22	$(-5/7,1)$	$N_{22}^{(7)}$	$-(49/184320)(1-\xi^2)(1+\eta)(49\xi^2-9)(49\xi^2-1)(7\xi-5)$
23	$(-1,5/7)$	$N_{23}^{(7)}$	$-(49/184320)(1-\xi)(1-\eta^2)(49\eta^2-9)(49\eta^2-1)(7\eta+5)$
24	$(-1,3/7)$	$N_{24}^{(7)}$	$(49/61440)(1-\xi)(1-\eta^2)(49\eta^2-25)(49\eta^2-1)(7\eta+3)$
25	$(-1,1/7)$	$N_{25}^{(7)}$	$-(49/36864)(1-\xi)(1-\eta^2)(49\eta^2-25)(49\eta^2-9)(7\eta+1)$
26	$(-1,-1/7)$	$N_{26}^{(7)}$	$-(49/36864)(1-\xi)(1-\eta^2)(49\eta^2-25)(49\eta^2-9)(7\eta-1)$
27	$(-1,-3/7)$	$N_{27}^{(7)}$	$(49/61440)(1-\xi)(1-\eta^2)(49\eta^2-25)(49\eta^2-1)(7\eta-3)$
28	$(-1,-5/7)$	$N_{28}^{(7)}$	$-(49/184320)(1-\xi)(1+\eta^2)(49\eta^2-9)(49\eta^2-1)(7\eta-5)$
29	$(-1/2,-1/2)$	$\hat{N}_{29}^{(7)}$	$(1/9)(1-\xi^2)(1-\eta^2)(1-2\xi)(1-2\eta)\{18(\xi^2+\eta^2)-5\}$
30	$(1/2,-1/2)$	$\hat{N}_{30}^{(7)}$	$(1/9)(1-\xi^2)(1-\eta^2)(1+2\xi)(1-2\eta)\{18(\xi^2+\eta^2)-5\}$
31	$(1/2,1/2)$	$\hat{N}_{31}^{(7)}$	$(1/9)(1-\xi^2)(1-\eta^2)(1+2\xi)(1+2\eta)\{18(\xi^2+\eta^2)-5\}$
32	$(-1/2,1/2)$	$\hat{N}_{32}^{(7)}$	$(1/9)(1-\xi^2)(1-\eta^2)(1-2\xi)(1+2\eta)\{18(\xi^2+\eta^2)-5\}$
33	$(-1/6,-1/2)$	$\hat{N}_{33}^{(7)}$	$(27/70)(1-\xi^2)(1-\eta^2)(1-4\xi^2)(1-2\eta)(1-6\xi)$

Table 11 (Continued)

Node i	Co-ordinates (ξ_i, η_i)	$\hat{N}_i^{(7)}$	Shape function
34	$(1/6, -1/2)$	$\hat{N}_{34}^{(7)}$	$(27/70)(1-\xi^2)(1-\eta^2)(1-4\xi^2)(1-2\eta)(1+6\xi)$
35	$(1/2, -1/6)$	$\hat{N}_{35}^{(7)}$	$(27/70)(1-\xi^2)(1-\eta^2)(1-4\xi^2)(1-2\eta)(1+6\xi)$
36	$(1/2, 1/6)$	$\hat{N}_{36}^{(7)}$	$(27/70)(1-\xi^2)(1-\eta^2)(1+2\xi)(1-4\eta^2)(1+6\eta)$
37	$(1/6, 1/2)$	$\hat{N}_{37}^{(7)}$	$(27/70)(1-\xi^2)(1-\eta^2)(1-4\xi^2)(1+2\eta)(1+6\xi)$
38	$(-1/6, 1/2)$	$\hat{N}_{38}^{(7)}$	$(27/70)(1-\xi^2)(1-\eta^2)(1-4\xi^2)(1+2\eta)(1-6\xi)$
39	$(-1/2, 1/6)$	$\hat{N}_{39}^{(7)}$	$(27/70)(1-\xi^2)(1-\eta^2)(1-2\xi)(1-4\eta^2)(1+6\eta)$
40	$(-1/2, -1/6)$	$\hat{N}_{40}^{(7)}$	$(27/70)(1-\xi^2)(1-\eta^2)(1-2\xi)(1-4\eta^2)(1-6\eta)$

They can be obtained by the following relations:

$$\hat{N}_{17}^{(4)}(\xi,\eta) = \frac{(1-\xi^2)(1-\eta^2)}{(1-\xi_{17}^2)(1-\eta_{17}^2)}, \quad -1 < \xi_{17}, \eta_{17} < 1 \tag{29}$$

and

$$\hat{N}_{j_i}^{(5)}(\xi,\eta) = \frac{(1-\xi^2)(1-\eta^2)}{(1-\theta^2)^2} N_{J_i}^{(1)}(s,t), \tag{30}$$

where

$$J_i = i + 20, \quad i = 1(1)4, \quad s = \frac{\xi}{\theta}, \quad t = \frac{\eta}{\theta}$$

and

$$N_{J_i}^{(1)}(s,t)$$

are the Shape functions of Linear element over the 2-square $-1 \leqslant s,\ t \leqslant 1$.

$$\hat{N}_{J_i}^{(6)}(\xi,\eta) = \frac{(1-\xi^2)(1-\eta^2)}{\left[1-\left\{\xi_{J_i}^{(6)}\right\}^2\right]\left[1-\left\{\eta_{J_i}^{(6)}\right\}^2\right]} N_{J_i}^{(2)}(s,t) \tag{31}$$

where

$$J_i = i + 24, \quad i = 1(1)8, \quad s = \frac{\xi}{\theta}, \quad t = \frac{\eta}{\theta}$$

and

$$N_{J_i}^{(2)}(s,t)$$

are the Shape functions of Quadratic element over the 2-square $-1 \leqslant s,\ t \leqslant 1$.

$$\hat{N}_{J_i}^{(7)} = \frac{(1-\xi^2)(1-\eta^2)}{\left[1-\left\{\xi_{J_i}^{(7)}\right\}^2\right]\left[1-\left\{\eta_{J_i}^{(7)}\right\}^2\right]} N_{J_i}^{(3)}(s,t), \tag{32}$$

where

$$J_i = i + 28, \quad i = 1(1)12, \quad s = \frac{\xi}{\theta}, \quad t = \frac{\eta}{\theta}$$

Table 12
Octic General Complete Lagrange element Shape functions

Node i	Co-ordinates (ξ_i, η_i)	$\hat{N}_i^{(8)}$	Shape function
1	$(-1,-1)$	$N_1^{(8)}$	$(1/44100)(1-\xi)(1-\eta)[35840(\xi^7+\eta^7)+31360(\xi^5+\eta^5)-6860(\xi^3+\eta^3)$ $+315(\xi+\eta)-11025]$
2	$(1,-1)$	$N_2^{(8)}$	$(1/44100)(1+\xi)(1-\eta)[35840(\xi^7-\eta^7)-31360(\xi^5-\eta^5)+6860(\xi^3-\eta^3)$ $-315(\xi-\eta)-11025]$
3	$(-1,-1)$	$N_3^{(8)}$	$(1/44100)(1+\xi)(1+\eta)[35840(\xi^7+\eta^7)-31360(\xi^5+\eta^5)+6860(\xi^3+\eta^3)$ $-315(\xi+\eta)-11025]$
4	$(-1,1)$	$N_4^{(8)}$	$(1/44100)(1-\xi)(1+\eta)[-35840(\xi^7-\eta^7)+31360(\xi^5-\eta^5)-6860(\xi^3-\eta^3)$ $+315(\xi-\eta)-11025]$
5	$(-3/4,-1)$	$N_5^{(8)}$	$(2/315)(1-\xi^2)(1-\eta)\ (16\xi^2-4)(16\xi^2-1)\xi(4\xi-3)$
6	$(-2/4,-1)$	$N_6^{(8)}$	$-(1/45)\ (1-\xi^2)(1-\eta)\ (16\xi^2-9)(16\xi^2-1)\xi(4\xi-2)$
7	$(-1/4,-1)$	$N_7^{(8)}$	$(2/45)\ (1-\xi^2)\ (1-\eta)\ (16\xi^2-9)(16\xi^2-4)\xi(4\xi-1)$
8	$(0,-1)$	$N_8^{(8)}$	$-(1/72)(1-\xi^2)(1-\eta)(16\xi^2-9)(16\xi^2-4)(16\xi^2-1)$
9	$(1/4,-1)$	$N_9^{(8)}$	$(2/45)(1-\xi^2)(1-\eta)(16\xi^2-9)(16\xi^2-4)(4\xi+1)$
10	$(2/4,-1)$	$N_{10}^{(8)}$	$-(1/45)(1-\xi^2)(1-\eta)(16\xi^2-9)(16\xi^2-1)\xi(4\xi+2)$
11	$(3/4,-1)$	$N_{11}^{(8)}$	$(2/315)(1-\xi^2)(1-\eta)(16\xi^2-4)(16\xi^2-1)\ \xi\ (4\xi+3)$
12	$(1,-3/4)$	$N_{12}^{(8)}$	$(2/315)(1+\xi)(1-\eta^2)(16\eta^2-4)(16\eta^2-1)\eta\ (4\eta-3)$
13	$(1,-2/4)$	$N_{13}^{(8)}$	$-(1/45)(1+\xi)(1-\eta^2)(16\eta^2-9)(16\eta^2-1)\eta\ (4\eta-2)$
14	$(1,-1/4)$	$N_{14}^{(8)}$	$(2/45)(1+\xi)(1-\eta^2)(16\eta^2-9)(16\eta^2-4)\eta\ (4\eta-1)$
15	$(1,0)$	$N_{15}^{(8)}$	$-(1/72)(1+\xi)(1-\eta^2)(16\eta^2-9)(16\eta^2-4)(16\eta^2-1)$
16	$(1,1/4)$	$N_{16}^{(8)}$	$(2/45)(1+\xi)(1-\eta^2)(16\eta^2-9)(16\eta^2-4)\eta(4\eta+1)$
17	$(1,2/4)$	$N_{17}^{(8)}$	$-(1/45)(1+\xi)(1-\eta^2)(16\eta^2-9)(16\eta^2-1)\eta(4\eta+2)$
18	$(1,3/4)$	$N_{18}^{(8)}$	$(2/315)(1+\xi)(1-\eta^2)(16\eta^2-4)(16\eta^2-1)\eta(4\eta+3)$
19	$(3/4,1)$	$N_{19}^{(8)}$	$(2/315)(1-\xi^2)(1+\eta)(16\xi^2-4)(16\xi^2-1)\xi(4\xi+3)$
20	$(2/4,1)$	$N_{20}^{(8)}$	$-(1/45)(1-\xi^2)(1+\eta)(16\xi^2-9)(16\xi^2-1)\xi(4\xi+2)$
21	$(1/4,1)$	$N_{21}^{(8)}$	$(2/45)(1-\xi^2)(1+\eta)(16\xi^2-9)(16\xi^2-4)\xi(4\xi+1)$
22	$(0,1)$	$N_{22}^{(8)}$	$-(1/72)(1-\xi^2)(1+\eta)(16\xi^2-9)(16\xi^2-4)(16\xi^2-1)$
23	$(-1/4,1)$	$N_{23}^{(8)}$	$(2/45)(1-\xi^2)(1+\eta)(16\xi^2-9)(16\xi^2-4)\xi(4\xi-1)$
24	$(-2/4,1)$	$N_{24}^{(8)}$	$-(1/45)(1-\xi^2)(1+\eta)(16\xi^2-9)(16\xi^2-1)\xi(4\xi-2)$
25	$(-3/4,1)$	$N_{25}^{(8)}$	$(2/315)(1-\xi^2)(1+\eta)(16\xi^2-4)(16\xi^2-1)\xi(4\xi-3)$
26	$(-1,3/4)$	$N_{26}^{(8)}$	$(2/315)(1-\xi)(1-\eta^2)(16\eta^2-4)(16\eta^2-1)\eta(4\eta+3)$
27	$(-1,2/4)$	$N_{27}^{(8)}$	$-(1/45)(1-\xi)(1-\eta^2)(16\eta^2-9)(16\eta^2-1)\eta(4\eta+2)$
28	$(-1,1/4)$	$N_{28}^{(8)}$	$(2/45)(1-\xi)(1-\eta^2)(16\eta^2-9)(16\eta^2-4)\eta(4\eta+1)$
29	$(-1,0,)$	$N_{29}^{(8)}$	$-(1/72)(1-\xi)(1-\eta^2)(16\eta^2-9)(16\eta^2-4)(16\eta^2-1)$
30	$(-1,-1/4)$	$N_{30}^{(8)}$	$(2/45)(1-\xi)(1-\eta^2)(16\eta^2-9)(16\eta^2-4)\eta(4\eta-1)$
31	$(-1,-2/4)$	$N_{31}^{(8)}$	$-(1/45)(1-\xi)(1-\eta^2)(16\eta^2-9)(16\eta^2-1)\eta(4\eta-2)$
32	$(-1,-3/4)$	$N_{32}^{(8)}$	$(2/315)(1-\xi)(1-\eta^2)(16\eta^2-4)(16\eta^2-1)\eta(4\eta-3)$
33	$(-1/2,-1/2)$	$\hat{N}_{33}^{(8)}$	$(4/27)(1-\xi^2)(1-\eta^2)(1-2\xi)(1-2\eta)[-32(\xi^3+\eta^3)+8(\xi+\eta)+12\xi\eta]$
34	$(1/2,-1/2)$	$\hat{N}_{34}^{(8)}$	$(4/27)(1-\xi^2)(1-\eta^2)(1+2\xi)(1-2\eta)[32(\xi^3-\eta^3)-8(\xi-\eta)-12\xi\eta]$

Table 12 (Continued)

Node i	Co-ordinates (ξ_i, η_i)	$\hat{N}_i^{(8)}$	Shape function
35	$(1/2, 1/2)$	$\hat{N}_{35}^{(8)}$	$(4/27)(1-\xi^2)(1-\eta^2)(1+2\xi)(1+2\eta)[32(\xi^3+\eta^3)-8(\xi+\eta)+12\xi\eta]$
36	$(-1/2, 1/2)$	$\hat{N}_{36}^{(8)}$	$(4/27)(1-\xi^2)(1-\eta^2)(1-2\xi)(1+2\eta)[-32(\xi^3-\eta^3)+8(\xi-\eta)-12\xi\eta]$
37	$(-1/4, -1/2)$	$\hat{N}_{37}^{(8)}$	$(256/135)(1-\xi^2)(1-\eta^2)(1-4\xi^2)(1-2\eta)\xi(4\xi-1)$
38	$(0, -1/2)$	$\hat{N}_{38}^{(8)}$	$-(2/3)(1-\xi^2)(1-\eta^2)(1-4\xi^2)(1-2\eta)(16\xi^2-1)$
39	$(1/4, -1/2)$	$\hat{N}_{39}^{(8)}$	$(256/135)(1-\xi^2)\ (1-\eta^2)(1-4\xi^2)(1-2\eta)\xi(4\xi+1)$
40	$(1/2, -1/4)$	$\hat{N}_{40}^{(8)}$	$(256/135)(1-\xi^2)\ (1-\eta^2)(1+2\xi)(1-4\eta^2)\eta(4\eta-1)$
41	$(1/2, 0)$	$\hat{N}_{41}^{(8)}$	$-(2/3)(1-\xi^2)(1-\eta^2)(1+2\xi)(1-4\eta^2)\ (16\eta^2-1)$
42	$(1/2, 1/4)$	$\hat{N}_{42}^{(8)}$	$(256/135)(1-\xi^2)(1-\eta^2)(1+2\xi)(1-4\eta^2)\eta(4\eta+1)$
43	$(1/4, 1/2)$	$\hat{N}_{43}^{(8)}$	$(256/135)(1-\xi^2)(1-\eta^2)(1-4\xi^2)(1+2\eta)\xi(4\xi+1)$
44	$(0, 1/2)$	$\hat{N}_{44}^{(8)}$	$-(2/3)(1-\xi^2)(1-\eta^2)(1-4\xi^2)(1+2\eta)\xi(16\xi^2-1)$
45	$(-1/4, 1/2)$	$\hat{N}_{45}^{(8)}$	$(256/135)(1-\xi^2)(1-\eta^2)(1-4\xi^2)(1+2\eta)\xi(4\xi-1)$
46	$(-1/2, 1/4)$	$\hat{N}_{46}^{(8)}$	$(256/135)(1-\xi^2)(1-\eta^2)(1-2\xi)(1-4\eta^2)\eta(4\eta+1)$
47	$(-1/2, 0)$	$\hat{N}_{47}^{(8)}$	$-(2/3)(1-\xi^2)(1-\eta^2)(1-2\xi)(1-4\eta^2)\ (16\eta^2-1)$
48	$(-1/2, -1/4)$	$\hat{N}_{48}^{(8)}$	$(256/135)(1-\xi^2)(1-\eta^2)(1-2\xi)(1-4\eta^2)\eta(4\eta-1)$
49	$(0, 0)$	$\hat{N}_{49}^{(8)}$	$(1-\xi)(1-\eta)(1-4\xi^2)(1-4\eta^2)$

and

$$N_{J_i}^{(3)}(s,t)$$

are the Shape functions of Cubic element over the 2-square $-1 \leqslant s,\ t \leqslant 1$.

$$\hat{N}_{J_i}^{(8)} = \frac{(1-\xi^2)(1-\eta^2)}{\left[1-\left\{\xi_{J_i}^{(8)}\right\}^2\right]\left[1-\left\{\eta_{J_i}^{(8)}\right\}^2\right]} N_{J_i}^{(4)}(s,t), \tag{33}$$

where

$$J_i = i+32, \quad i = 1(1)17, \quad s = \frac{\xi}{\theta}, \quad t = \frac{\eta}{\theta}$$

and

$$N_{J_i}^{(4)}(s,t)$$

are the Shape functions of Quartic Complete Lagrange element over the 2-square $-1 \leqslant s,\ t \leqslant 1$.

$$\hat{N}_{J_i}^{(9)}(\xi,\eta) = \frac{(1-\xi^2)(1-\eta^2)}{\left[1-\left\{\xi_{J_i}^{(9)}\right\}^2\right]\left[1-\left\{\eta_{J_i}^{(9)}\right\}^2\right]} N_{J_i}^{(5)}(s,t), \tag{34}$$

where

$$J_i = i+36, \quad i = 1(1)24, \quad s = \frac{\xi}{\theta}, \quad t = \frac{\eta}{\theta}$$

and

$$N_{J_i}^{(5)}(s,t)$$

are the Shape functions of Quintic Complete Lagrange element over the 2-square $-1 \leqslant s,\ t \leqslant 1$.

Table 13
Nineth-order General Complete Lagrange element Shape functions

Node i	Co-ordinates (ξ_i, η_i)	$\hat{N}_i^{(9)}$	Shape function
1	$(-1,-1)$	$N_1^{(9)}$	$(1/41287680)(1-\xi)(1-\eta)[43046721(\xi^8+\eta^8) - 44641044(\xi^6+\eta^6) +12951414(\xi^4+\eta^4) - 1046196(\xi^2+\eta^2) - 10299870]$
2	$(1,-1)$	$N_2^{(9)}$	$(1/41287680)(1+\xi)(1-\eta)[43046721(\xi^8+\eta^8) - 44641044(\xi^6+\eta^6) +12951414(\xi^4+\eta^4) - 1046196(\xi^2+\eta^2) - 10299870]$
3	$(1,1)$	$N_3^{(9)}$	$(1/41287680)(1+\xi)(1+\eta)[43046721(\xi^8+\eta^8) - 44641044(\xi^6+\eta^6) +12951414(\xi^4+\eta^4) - 1046196(\xi^2+\eta^2) - 10299870]$
4	$(-1,1)$	$N_4^{(9)}$	$(1/41287680)(1-\xi)(1+\eta)[43046721(\xi^8+\eta^8) - 44641044(\xi^6+\eta^6) +12951414(\xi^4+\eta^4) - 1046196(\xi^2+\eta^2) - 10299870]$
5	$(-7/9,-1)$	$N_5^{(9)}$	$-(81/41287680)(1-\xi^2)(1-\eta)(81\xi^2-25)(81\xi^2-9)(81\xi^2-1)(9\xi-7)$
6	$(-5/9,-1)$	$N_6^{(9)}$	$(81/10321920)(1-\xi^2)(1-\eta)(81\xi^2-49)(81\xi^2-9)(81\xi^2-1)(9\xi-5)$
7	$(-3/9,-1)$	$N_7^{(9)}$	$-(81/4423680)(1-\xi^2)(1-\eta)(81\xi^2-49)(81\xi^2-25)(81\xi^2-1)(9\xi-3)$
8	$(-1/9,-1)$	$N_8^{(9)}$	$(81/2949120)(1-\xi^2)(1-\eta)(81\xi^2-49)(81\xi^2-25)(81\xi^2-9)(9\xi-1)$
9	$(1/9,-1)$	$N_9^{(9)}$	$-(81/2949120)(1-\xi^2)(1-\eta)(81\xi^2-49)(81\xi^2-25)(81\xi^2-9)(9\xi+1)$
10	$(3/9,-1)$	$N_{10}^{(9)}$	$(81/4423680)(1-\xi^2)(1-\eta)(81\xi^2-49)(81\xi^2-25)(81\xi^2-1)(9\xi+3)$
11	$(5/9,-1)$	$N_{11}^{(9)}$	$-(81/10321920)(1-\xi^2)(1-\eta)(81\xi^2-49)(81\xi^2-9)(81\xi^2-1)(9\xi+5)$
12	$(7/9,-1)$	$N_{12}^{(9)}$	$(81/41287680)(1-\xi^2)(1-\eta)(81\xi^2-25)(81\xi^2-9)(81\xi^2-1)(9\xi+7)$
13	$(1,-7/9)$	$N_{13}^{(9)}$	$-(81/41287680)(1+\xi)(1-\eta^2)(81\eta^2-25)(81\eta^2-9)(81\eta^2-1)(9\eta-7)$
14	$(1,-5/9)$	$N_{14}^{(9)}$	$(81/10321920)(1+\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-9)(81\eta^2-1)(9\eta-5)$
15	$(1,-3/9)$	$N_{15}^{(9)}$	$-(81/4423680)(1+\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-25)(81\eta^2-1)(9\eta-3)$
16	$(1,-1/9)$	$N_{16}^{(9)}$	$(81/2949120)(1+\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-25)(81\eta^2-9)(9\eta-1)$
17	$(1,1/9)$	$N_{17}^{(9)}$	$-(81/2949120)(1+\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-25)(81\eta^2-9)(9\eta+1)$
18	$(1,3/9)$	$N_{18}^{(9)}$	$(81/4423680)(1+\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-25)(81\eta^2-1)(9\eta+3)$
19	$(1,5/9)$	$N_{19}^{(9)}$	$-(81/10321920)(1+\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-9)(81\eta^2-1)(9\eta+5)$
20	$(1,7/9)$	$N_{20}^{(9)}$	$(81/41287680)(1+\xi)(1-\eta^2)(81\eta^2-25)(81\eta^2-9)(81\eta^2-1)(9\eta+7)$
21	$(7/9,1)$	$N_{21}^{(9)}$	$(81/41287680)(1-\xi^2)(1+\eta)(81\xi^2-25)(81\xi^2-9)(81\xi^2-1)(9\xi+7)$
22	$(5/9,1)$	$N_{22}^{(9)}$	$-(81/10321920)(1-\xi^2)(1+\eta)(81\xi^2-49)(81\xi^2-9)(81\xi^2-1)(9\xi+5)$
23	$(3/9,1)$	$N_{23}^{(9)}$	$(81/4423680)(1-\xi^2)(1+\eta)(81\xi^2-49)(81\xi^2-25)(81\xi^2-1)(9\xi+3)$
24	$(1/9,1)$	$N_{24}^{(9)}$	$-(81/2949120)(1-\xi^2)(1+\eta)(81\xi^2-49)(81\xi^2-25)(81\xi^2-9)(9\xi+1)$
25	$(-1/9,1)$	$N_{25}^{(9)}$	$(81/2949120)(1-\xi^2)(1+\eta)(81\xi^2-49)(81\xi^2-25)(81\xi^2-9)(9\xi-1)$
26	$(-3/9,1)$	$N_{26}^{(9)}$	$-(81/4423680)(1-\xi^2)(1+\eta)(81\xi^2-49)(81\xi^2-25)(81\xi^2-1)(9\xi-3)$
27	$(-5/9,1)$	$N_{27}^{(9)}$	$(81/10321920)(1-\xi^2)(1+\eta)(81\xi^2-49)(81\xi^2-9)(81\xi^2-1)(9\xi-5)$
28	$(-7/9,1)$	$N_{28}^{(9)}$	$-(81/41287680)(1-\xi^2)(1+\eta)(81\xi^2-25)(81\xi^2-9)(81\xi^2-1)(9\xi-7)$
29	$(-1,7/9)$	$N_{29}^{(9)}$	$(81/41287680)(1-\xi)(1-\eta^2)(81\eta^2-25)(81\eta^2-9)(81\eta^2-1)(9\eta+7)$
30	$(-1,5/9)$	$N_{30}^{(9)}$	$-(81/10321920)(1-\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-9)(81\eta^2-1)(9\eta+5)$
31	$(-1,3/9)$	$N_{31}^{(9)}$	$(81/4423680)(1-\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-25)(81\eta^2-1)(9\eta+3)$
32	$(-1,1/9)$	$N_{32}^{(9)}$	$-(81/2949120)(1-\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-25)(81\eta^2-9)(9\eta+1)$
33	$(-1,-1/9)$	$N_{33}^{(9)}$	$(81/2949120)(1-\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-25)(81\eta^2-9)(9\eta-1)$

Table 13 (Continued)

Node i	Co-ordinates (ξ_i, η_i)	$\hat{N}_i^{(9)}$	Shape function
34	$(-1, -3/9)$	$N_{34}^{(9)}$	$-(81/4423680)(1-\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-25)(81\eta^2-1)(9\eta-3)$
35	$(-1, -5/9)$	$N_{35}^{(9)}$	$(81/10321920)(1-\xi)(1-\eta^2)(81\eta^2-49)(81\eta^2-9)(81\eta^2-1)(9\eta-5)$
36	$(-1, -7/9)$	$N_{36}^{(9)}$	$-(81/41287680)(1-\xi)(1-\eta^2)(81\eta^2-25)(81\eta^2-9)(81\eta^2-1)(9\eta-7)$
37	$(-1/2, -1/2)$	$\hat{N}_{37}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_1^{(5)}(\xi/2, \eta/2)$
38	$(1/2, -1/2)$	$\hat{N}_{38}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_2^{(5)}(\xi/2, \eta/2)$
39	$(1/2, 1/2)$	$\hat{N}_{39}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_3^{(5)}(\xi/2, \eta/2)$
40	$(-1/2, 1/2)$	$\hat{N}_{40}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_4^{(5)}(\xi/2, \eta/2)$
41	$(-3/10, -1/2)$	$\hat{N}_{41}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_5^{(5)}(\xi/2, \eta/2)$
42	$(-1/10, -1/2)$	$\hat{N}_{42}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_6^{(5)}(\xi/2, \eta/2)$
43	$(1/10, -1/2)$	$\hat{N}_{43}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_7^{(5)}(\xi/2, \eta/2)$
44	$(3/10, -1/2)$	$\hat{N}_{44}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_8^{(5)}(\xi/2, \eta/2)$
45	$(1/2, -3/10)$	$\hat{N}_{45}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_9^{(5)}(\xi/2, \eta/2)$
46	$(1/2, -1/10)$	$\hat{N}_{46}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_{10}^{(5)}(\xi/2, \eta/2)$
47	$(1/2, 1/10)$	$\hat{N}_{47}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_{11}^{(5)}(\xi/2, \eta/2)$
48	$(1/2, 3/10)$	$\hat{N}_{48}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_{12}^{(5)}(\xi/2, \eta/2)$
49	$(3/10, 1/2)$	$\hat{N}_{49}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_{13}^{(5)}(\xi/2, \eta/2)$
50	$(1/10, 1/2)$	$\hat{N}_{50}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_{14}^{(5)}(\xi/2, \eta/2)$
51	$(-1/10, 1/2)$	$\hat{N}_{51}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_{15}^{(5)}(\xi/2, \eta/2)$
52	$(-3/10, 1/2)$	$\hat{N}_{52}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_{16}^{(5)}(\xi/2, \eta/2)$
53	$(-1/2, 3/10)$	$\hat{N}_{53}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_{17}^{(5)}(\xi/2, \eta/2)$
54	$(-1/2, 1/10)$	$\hat{N}_{54}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_{18}^{(5)}(\xi/2, \eta/2)$
55	$(-1/2, -1/10)$	$\hat{N}_{55}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_{19}^{(5)}(\xi/2, \eta/2)$
56	$(-1/2, -3/10)$	$\hat{N}_{56}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_{20}^{(5)}(\xi/2, \eta/2)$
57	$(-1/4, -1/4)$	$\hat{N}_{57}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_{21}^{(5)}(\xi/2, \eta/2)$
58	$(1/4, -1/4)$	$\hat{N}_{58}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_{22}^{(5)}(\xi/2, \eta/2)$
59	$(1/4, 1/4)$	$\hat{N}_{59}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_{23}^{(5)}(\xi/2, \eta/2)$
60	$(-1/4, 1/4)$	$\hat{N}_{60}^{(9)}$	$(1-\xi^2)(1-\eta^2)N_{24}^{(5)}(\xi/2, \eta/2)$

Further

$$\hat{N}_{J_{i+}}^{(5)}(s,t) = N_{J_i}^{(5)}(s,t) - \sum_{k=21}^{24} N_{J_i}^{(5)}\left(\xi_{J_k}^{(5)}, \eta_{J_K}^{(5)}\right)\hat{N}_{J_k}^{(5)}, \quad i = 1(1)20$$

and

$$\hat{N}_{J_{i+20}}^{(5)}(s,t) = \frac{(1-s^2)(1-t^2)}{(1-\varepsilon^2)(1-\varepsilon^2)} N_{J_{i+20}}^{(1)}(p,q), \tag{35}$$

$$i = 1(1)4, \quad p = \frac{s}{\varepsilon}, \quad q = \frac{\eta}{\varepsilon}.$$

Table 14
Tenth-order General Complete Lagrange element Shape functions

Node i	Co-ordinates (ξ_i, η_i)	$\hat{N}_i^{(10)}$	Shape function
1	$(-1,-1)$	$N_1^{(10)}$	$(1/290304)(1-\xi)(1-\eta)[-390625(\xi^9+\eta^9)+468750(\xi^7+\eta^7)$ $-170625(\xi^5+\eta^5)+20500(\xi^3+\eta^3)-576(\xi+\eta)-72576]$
2	$(1,-1)$	$N_2^{(10)}$	$(1/290304)(1+\xi)(1-\eta)[390625(\xi^9-\eta^9)-468750(\xi^7-\eta^7)$ $+170625(\xi^5-\eta^5)-20500(\xi^3-\eta^3)+576(\xi-\eta)-72576]$
3	$(1,1)$	$N_3^{(10)}$	$(1/290304)(1+\xi)(1+\eta)[390625(\xi^9+\eta^9)-468750(\xi^7+\eta^7)$ $+170625(\xi^5+\eta^5)-20500(\xi^3+\eta^3)+576(\xi+\eta)-72576]$
4	$(-1,1)$	$N_4^{(10)}$	$(1/290304)(1-\xi)(1+\eta)[-390625(\xi^9-\eta^9)+468750(\xi^7-\eta^7)$ $-170625(\xi^5-\eta^5)+20500(\xi^3-\eta^3)-576(\xi-\eta)-72576]$
5	$(-4/5,-1)$	$N_5^{(10)}$	$(25/145152)(1-\xi^2)(1-\eta)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)\xi(5\xi-4)$
6	$(-3/5,-1)$	$N_6^{(10)}$	$(-25/32256)(1-\xi^2)(1-\eta)(25\xi^2-16)(25\xi^2-4)(25\xi^2-1)\xi(5\xi-3)$
7	$(-2/5,-1)$	$N_7^{(10)}$	$(25/12096)(1-\xi^2)(1-\eta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-1)\xi(5\xi-2)$
8	$(-4/5,-1)$	$N_8^{(10)}$	$(-25/6912)(1-\xi^2)(1-\eta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)\xi(5\xi-1)$
9	$(-4/5,-1)$	$N_9^{(10)}$	$(1/1152)(1-\xi^2)(1-\eta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)$
10	$(1/5,-1)$	$N_{10}^{(10)}$	$(-25/6912)(1-\xi^2)(1-\eta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)\xi(5\xi+1)$
11	$(2/5,-1)$	$N_{11}^{(10)}$	$(25/12096)(1-\xi^2)(1-\eta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-1)\xi(5\xi+2)$
12	$(3/5,-1)$	$N_{12}^{(10)}$	$(-25/32256)(1-\xi^2)(1-\eta)(25\xi^2-16)(25\xi^2-4)(25\xi^2-1)\xi(5\xi+3)$
13	$(4/5,-1)$	$N_{13}^{(10)}$	$(25/145152)(1-\xi^2)(1-\eta)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)\xi(5\xi+4)$
14	$(1,-4/5)$	$N_{14}^{(10)}$	$(25/145152)(1+\xi)(1-\eta^2)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)\eta(5\eta-4)$
15	$(1,-3/5)$	$N_{15}^{(10)}$	$(-25/32256)(1+\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-4)(25\eta^2-1)\eta(5\eta-3)$
16	$(1,-2/5)$	$N_{16}^{(10)}$	$(25/12096)(1+\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-9)(25\eta^2-1)\eta(5\eta-2)$
17	$(1,-1/5)$	$N_{17}^{(10)}$	$(-25/6912)(1+\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)\eta(5\eta-1)$
18	$(1,0)$	$N_{18}^{(10)}$	$(1/1152)(1+\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)$
19	$(1,1/5)$	$N_{19}^{(10)}$	$(-25/6912)(1+\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)\eta(5\eta+1)$
20	$(1,2/5)$	$N_{20}^{(10)}$	$(25/12096)(1+\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-9)(25\eta^2-1)\eta(5\eta+2)$
21	$(1,3/5)$	$N_{21}^{(10)}$	$(-25/32256)(1+\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-4)(25\eta^2-1)\eta(5\eta+3)$
22	$(1,4/5)$	$N_{22}^{(10)}$	$(25/145152)(1+\xi)(1-\eta^2)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)\eta(5\eta+4)$
23	$(4/5,1)$	$N_{23}^{(10)}$	$(25/145152)(1-\xi^2)(1+\eta)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)\xi(5\xi+4)$
24	$(3/5,1)$	$N_{24}^{(10)}$	$(-25/32256)(1-\xi^2)(1+\eta)(25\xi^2-16)(25\xi^2-4)(25\xi^2-1)\xi(5\xi+3)$
25	$(2/5,1)$	$N_{25}^{(10)}$	$(25/12096)(1-\xi^2)(1+\eta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-1)\xi(5\xi+2)$
26	$(1/5,1)$	$N_{26}^{(10)}$	$(-25/6912)(1-\xi^2)(1+\eta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)\xi(5\xi+1)$
27	$(0,1)$	$N_{27}^{(10)}$	$(1/1152)(1-\xi^2)(1+\eta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)$
28	$(-1/5,1)$	$N_{28}^{(10)}$	$(-25/6912)(1-\xi^2)(1+\eta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)\xi(5\xi-1)$
29	$(-2/5,1)$	$N_{29}^{(10)}$	$(25/12096)(1-\xi^2)(1+\eta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-1)\xi(5\xi-2)$
30	$(-3/5,1)$	$N_{30}^{(10)}$	$(-25/32256)(1-\xi^2)(1+\eta)(25\xi^2-16)(25\xi^2-4)(25\xi^2-1)\xi(5\xi-3)$
31	$(-4/5,1)$	$N_{31}^{(10)}$	$(25/145152)(1-\xi^2)(1+\eta)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)\xi(5\xi-4)$
32	$(-1,4/5)$	$N_{32}^{(10)}$	$(25/145152)(1-\xi)(1-\eta^2)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)\eta(5\eta+4)$
33	$(-1,3/5)$	$N_{33}^{(10)}$	$(-25/32256)(1-\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-4)(25\eta^2-1)\eta(5\eta+3)$
34	$(-1,2/5)$	$N_{34}^{(10)}$	$(25/12096)(1-\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-9)(25\eta^2-1)\eta(5\eta+2)$

Table 14 (Continued)

Node i	Co-ordinates (ξ_i, η_i)	$\hat{N}_i^{(10)}$	Shape function
35	$(-1, 1/5)$	$N_{35}^{(10)}$	$(-25/6912)(1-\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)\eta(5\eta+1)$
36	$(-1, 0)$	$N_{36}^{(10)}$	$(1/1152)(1-\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)$
37	$(-1, -1/5)$	$N_{37}^{(10)}$	$(-25/6912)(1-\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)\eta(5\eta-1)$
38	$(-1, -2/5)$	$N_{38}^{(10)}$	$(25/12096)(1-\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-9)(25\eta^2-1)\eta(5\eta-2)$
39	$(-1, -3/5)$	$N_{39}^{(10)}$	$(-25/32256)(1-\xi)(1-\eta^2)(25\eta^2-16)(25\eta^2-4)(25\eta^2-1)\eta(5\eta-3)$
40	$(-1, -4/5)$	$N_{40}^{(10)}$	$(25/145152)(1-\xi)(1-\eta^2)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)\eta(5\eta-4)$
41	$(-1/2, -1/2)$	$\hat{N}_{41}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_1^{(6)}(\xi/2, \eta/2)$
42	$(1/2, -1/2)$	$\hat{N}_{42}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_2^{(6)}(\xi/2, \eta/2)$
43	$(1/2, 1/2)$	$\hat{N}_{43}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_3^{(6)}(\xi/2, \eta/2)$
44	$(-1/2, 1/2)$	$\hat{N}_{44}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_4^{(6)}(\xi/2, \eta/2)$
45	$(-2/6, -1/2)$	$\hat{N}_{45}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_5^{(6)}(\xi/2, \eta/2)$
46	$(-1/6, -1/2)$	$\hat{N}_{46}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_6^{(6)}(\xi/2, \eta/2)$
47	$(0, -1/2)$	$\hat{N}_{47}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_7^{(6)}(\xi/2, \eta/2)$
48	$(1/6, -1/2)$	$\hat{N}_{48}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_8^{(6)}(\xi/2, \eta/2)$
49	$(2/6, -1/2)$	$\hat{N}_{49}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_9^{(6)}(\xi/2, \eta/2)$
50	$(1, 2/6)$	$\hat{N}_{50}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{10}^{(6)}(\xi/2, \eta/2)$
51	$(1, 1/6)$	$\hat{N}_{51}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{11}^{(6)}(\xi/2, \eta/2)$
52	$(1, 0)$	$\hat{N}_{52}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{12}^{(6)}(\xi/2, \eta/2)$
53	$(1, -1/6)$	$\hat{N}_{53}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{13}^{(6)}(\xi/2, \eta/2)$
54	$(1, -2/6)$	$\hat{N}_{54}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{14}^{(6)}(\xi/2, \eta/2)$
55	$(2/6, 1)$	$\hat{N}_{55}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{15}^{(6)}(\xi/2, \eta/2)$
56	$(1/6, 1)$	$\hat{N}_{56}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{16}^{(6)}(\xi/2, \eta/2)$
57	$(0, 1)$	$\hat{N}_{57}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{17}^{(6)}(\xi/2, \eta/2)$
58	$(-1/6, 1)$	$\hat{N}_{58}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{18}^{(6)}(\xi/2, \eta/2)$
59	$(-2/6, 1)$	$\hat{N}_{59}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{19}^{(6)}(\xi/2, \eta/2)$
60	$(-1, 2/6)$	$\hat{N}_{60}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{20}^{(6)}(\xi/2, \eta/2)$
61	$(-1, 1/6)$	$\hat{N}_{61}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{21}^{(6)}(\xi/2, \eta/2)$
62	$(-1, 0)$	$\hat{N}_{62}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{22}^{(6)}(\xi/2, \eta/2)$
63	$(-1, 1/6)$	$\hat{N}_{63}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{23}^{(6)}(\xi/2, \eta/2)$
64	$(-1, -2/6)$	$\hat{N}_{64}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{24}^{(6)}(\xi/2, \eta/2)$
65	$(-1/4, -1/4)$	$\hat{N}_{65}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{25}^{(6)}(\xi/2, \eta/2)$
66	$(1/4, -1/4)$	$\hat{N}_{66}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{26}^{(6)}(\xi/2, \eta/2)$
67	$(1/2, 1/2)$	$\hat{N}_{67}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{27}^{(6)}(\xi/2, \eta/2)$
68	$(-1/4, 1/4)$	$\hat{N}_{68}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{28}^{(6)}(\xi/2, \eta/2)$
69	$(0, -1/4)$	$\hat{N}_{69}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{29}^{(6)}(\xi/2, \eta/2)$
70	$(1/4, 0)$	$\hat{N}_{70}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{30}^{(6)}(\xi/2, \eta/2)$
71	$(0, 1/4)$	$\hat{N}_{71}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{31}^{(6)}(\xi/2, \eta/2)$
72	$(-1/4, 0)$	$\hat{N}_{72}^{(10)}$	$(1-\xi^2)(1-\eta^2) \quad N_{32}^{(6)}(\xi/2, \eta/2)$

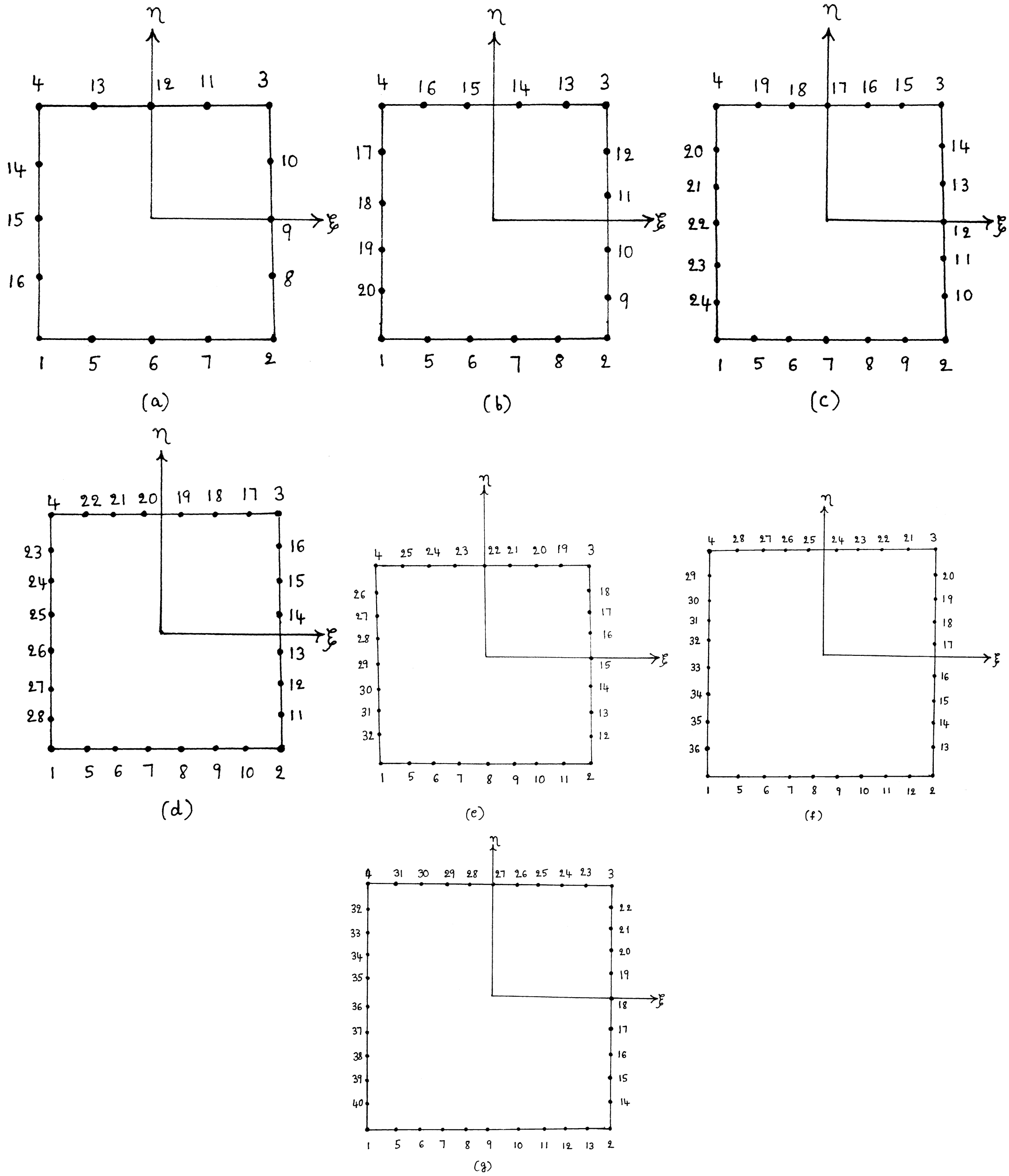

Fig. 5. Serendipity elements over $-1 \leqslant \xi, \eta \leqslant 1$ with equal number of nodes on each Side which allow uniform spacing as defined in Eq. 20): (a) Quartic element, (b) Quintic element, (c) Sextic element, (d) Septic element, (e) Octic element, (f) Nineth-order element, (g) Tenth-order element.

$$\hat{N}_{J_i}^{(10)}(\xi,\eta) = \frac{(1-\xi^2)(1-\eta^2)}{\left[1-\left\{\xi_{J_i}^{(10)}\right\}^2\right]\left[1-\left\{\eta_{J_i}^{(10)}\right\}^2\right]} N_{J_i}^{(6)}(s,t), \tag{36}$$

where

$$J_i = i + 40, \quad i = 1(1)32, \quad s = \frac{\xi}{\theta}, \quad t = \frac{\eta}{\theta}$$

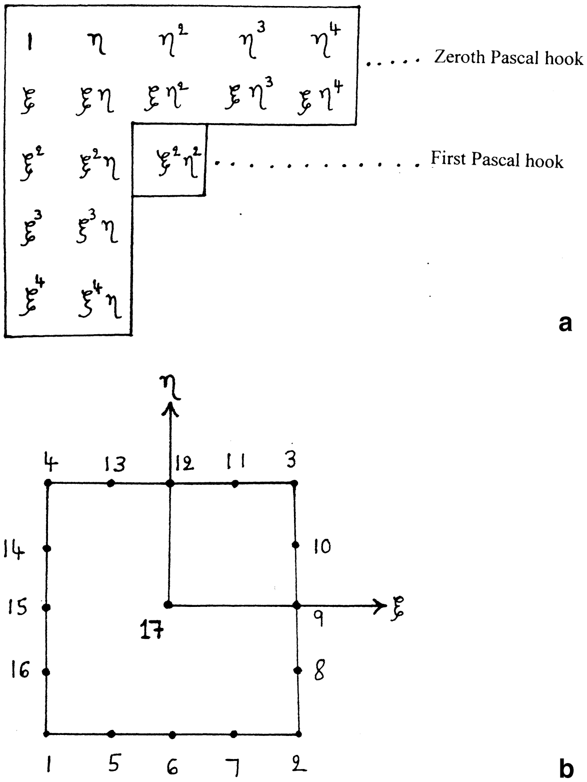

Fig. 6. (a) Monomial Basis for the Quartic General Complete Lagrange element, (b) Quartic General Complete Lagrange element.

and

$$N_{J_i}^{(6)}(s,t)$$

are the Shape functions of Sextic Complete Lagrange element over the 2-square $-1 \leqslant s,\ t \leqslant 1$.
Further

$$N_{J_i}^{(6)}(s,t) = N_{J_i}^{(6)}(s,t) - \sum_{k=25}^{32} N_{J_i}^{(6)}\left(s_{J_k}^{(6)}, t_{J_K}^{(6)}\right)\hat{N}_{J_K}^{(6)}(s,t), \quad i = 1(1)24$$

and

$$\hat{N}_{J_{i+24}}^{(6)}(s,t) = \frac{(1-s^2)(1-t^2)}{\left[1-\left\{s_{J_{i+24}}^{(6)}\right\}^2\right]\left[1-\left\{t_{J_i}^{(6)}\right\}^2\right]}, \tag{37}$$

$$i = 1(1)4, \quad p = \frac{s}{\varepsilon}, \quad q = \frac{t}{\varepsilon}.$$

7. Modified Shape functions for complete Lagrange elements

The Shape functions of Complete Lagrange element can be modified so that they correctly interpolate pth-order displacement states under the same conditions as a pth-order Regular Lagrange element. We assume for the present that these modifications are applicable to *straight-edged Quadrilateral elements* in the global space which can be mapped to a 2-square in the local space by the standard bilinear Shape functions

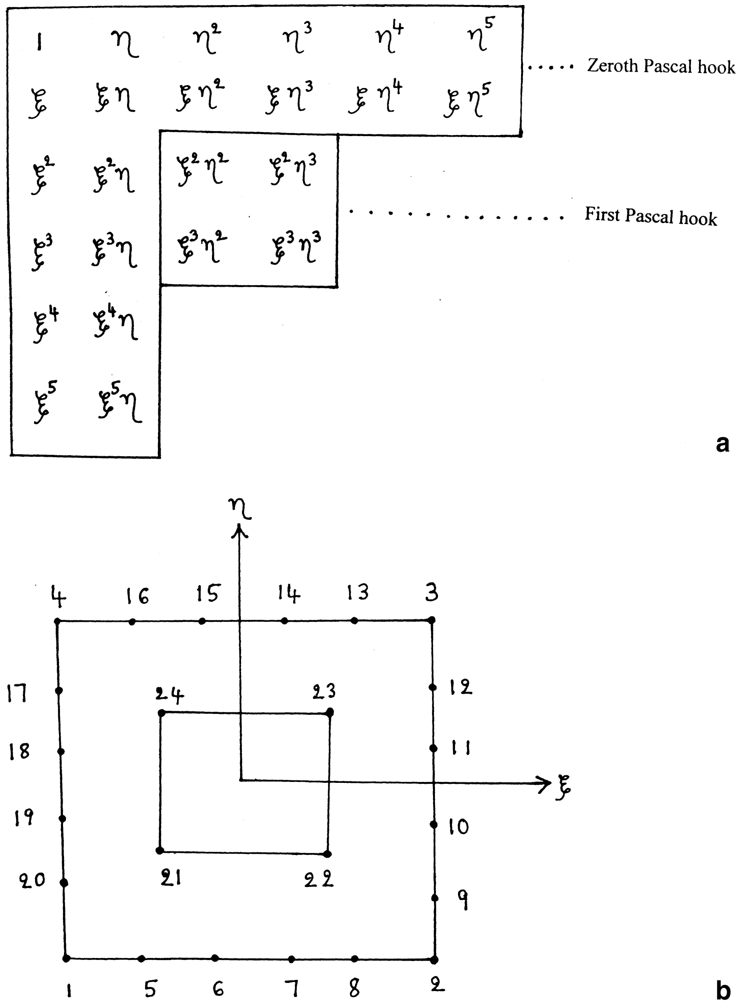

Fig. 7. (a) Monomial Basis for the Quintic General Complete Lagrange element, (b) Quintic General Complete Lagrange element.

in the natural co-ordinates (ξ, η).This is achieved by the application of a suitably designed constraint which constrains the displacement at the nodes of a Complete Lagrange element. Any component of the elements displacement field can then be expressed interms of the Shape functions of a pth-order regular element and the equation of the constraint is

$$u = \sum_{j=1}^{(p+1)^2} M_j^{(p)}(\xi, \eta) u_j, \tag{38}$$

$$u_k = \sum_{i=1}^{c_p} T_i^{k,p} u_i, \tag{39}$$

$$k = c_p + 1, c_p + 2, \ldots, (p+1)^2,$$
$$p = 2, 3, \ldots, 10,$$

where

$$M_j^{(p)}(\xi, \eta)$$

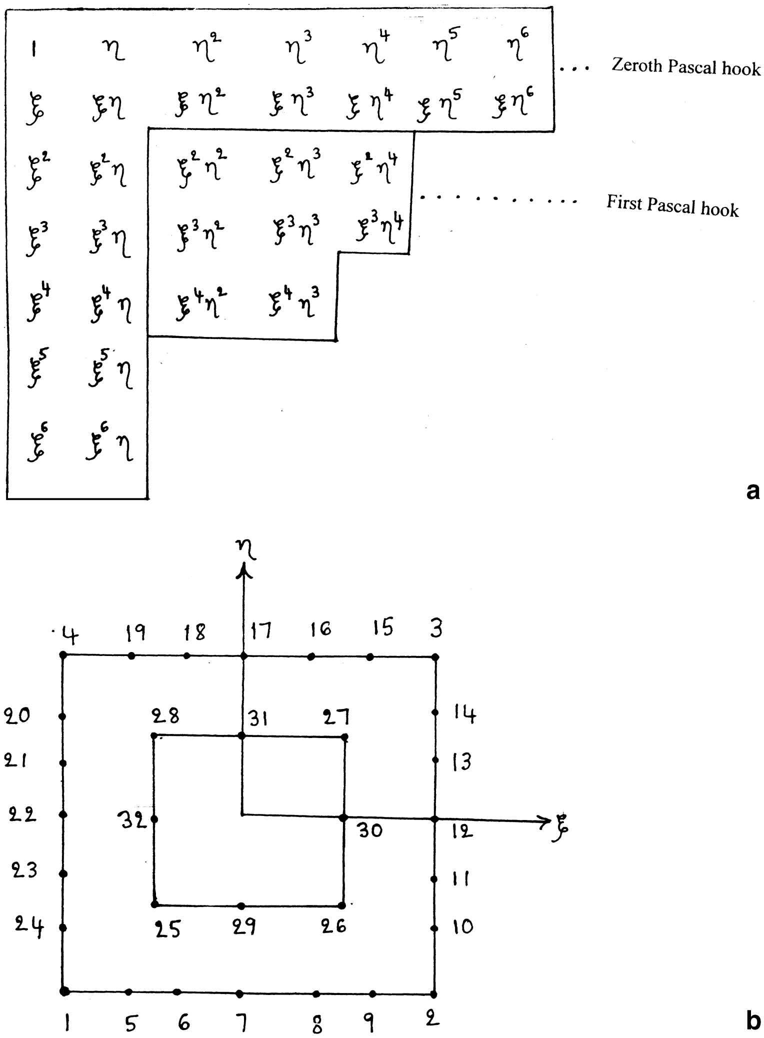

Fig. 8. (a) Monomial Basis for the Sextic General Complete Lagrange element, (b) Sextic General Completed Lagrange element.

is the pth-order Regular Lagrange element Shape function and

$$T_i^{k,p}$$

is a co-efficient of constraint for kth interior node of the Regular Lagrange element of pth-order and c_p is the number of nodes of the pth-order Complete Lagrange element.

The element is, practically speaking, a pth-order Complete Lagrange element with modified Shape functions. Thus

$$u = \sum_{i=1}^{c_p} \overline{M}_i^{(p)}(\xi, \eta) u_i, \tag{40}$$

where

$$\overline{M}_i^{(p)}(\xi, \eta) = M_j^{(p)}(\xi, \eta) + \sum_{k=c_p+1}^{(p+1)^2} M_k^{(p)}(\xi, \eta) T_i^{k,p}. \tag{41}$$

This result can also be expressed in terms of standard Complete Lagrange element Shape functions

$$\hat{N}_i^{(p)}.$$

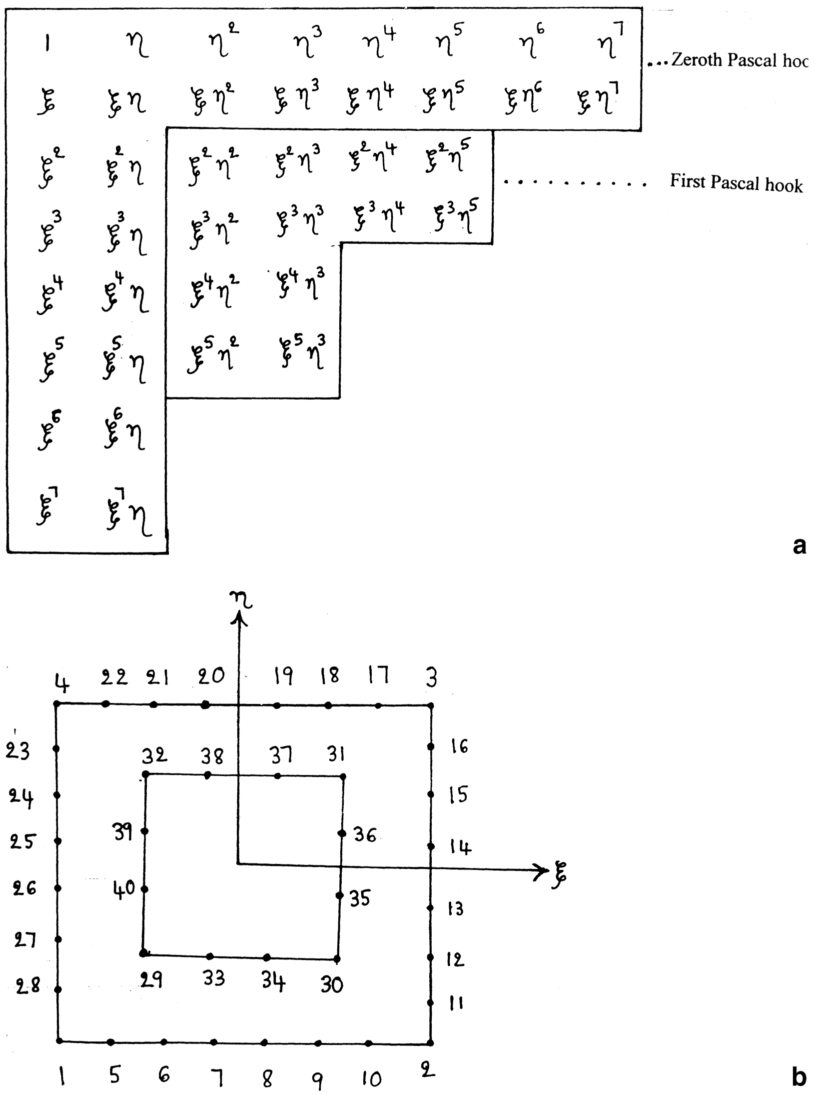

Fig. 9. (a) Monomial Basis for the Septic General Complete Lagrange element, (b) Septic General Complete Lagrange element.

To accomplish this we put Eq. (40) in hierarchical form

$$u = \sum_{i=1}^{c_p} \hat{N}_i^{(p)}(\xi, \eta) u_i + \sum_{k=c_p+1}^{(p+1)^2} M_k^{(p)} \Delta u_k, \tag{42}$$

where assuming that the node k is located at the interior point (ξ_k, η_k) of the 2-square $-1 \leqslant \xi, \eta \leqslant 1$ so that we obtain

$$u_k = u(\xi_k, \eta_k) = \sum_{i=1}^{c_p} \hat{N}_i^{(p)}(\xi_k, \eta_k) u_i + \Delta u_k, \tag{43}$$

$$k = c_p + 1, c_p + 2, \ldots, (p+1)^2.$$

Since the Shape functions

$$M_i^{(p)}$$

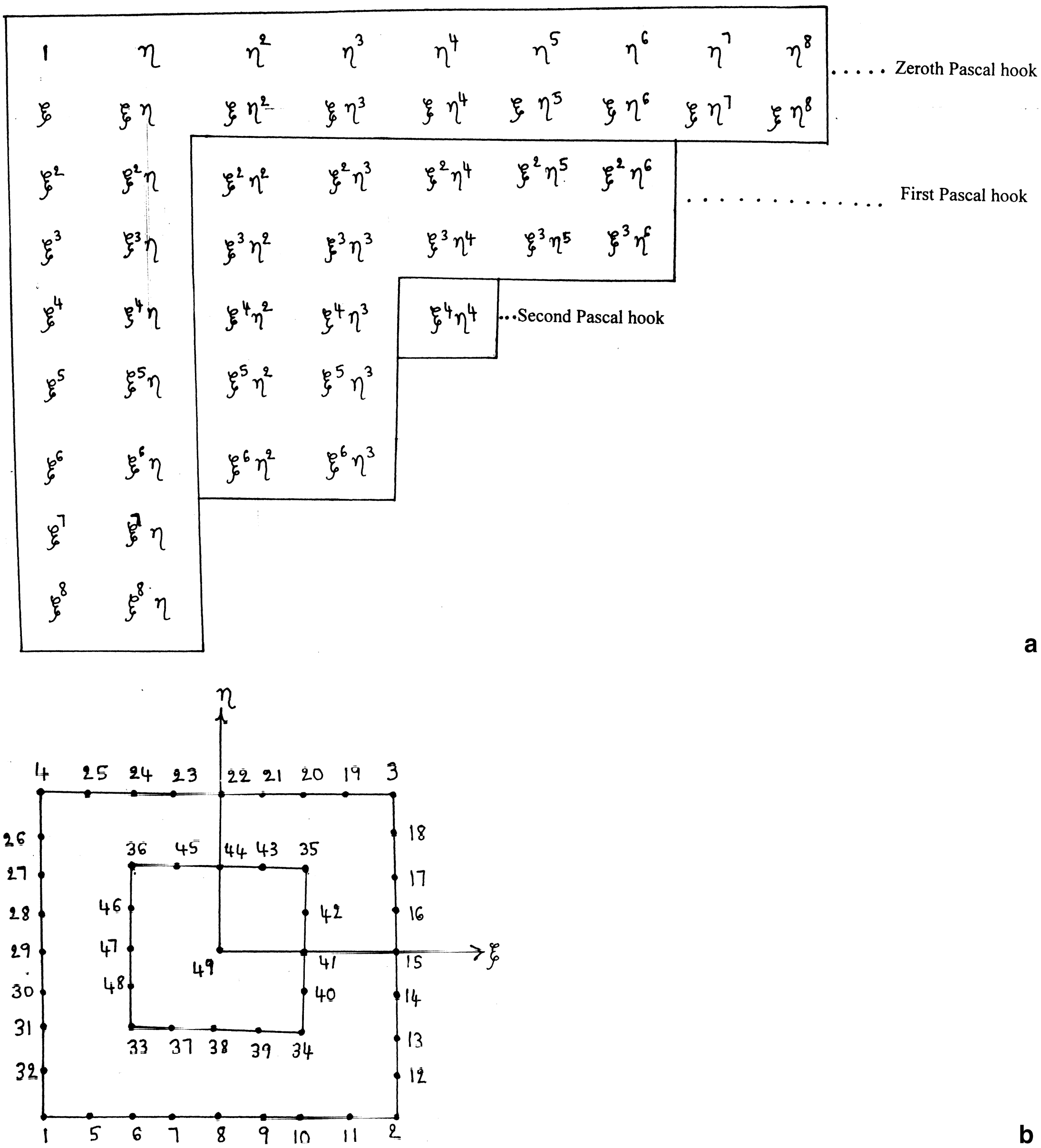

Fig. 10. (a) Monomial Basis for the Octic General Complete Lagrange element, (b) Octic General Complete Lagrange element.

satisfy the property

$$M_i^{(p)}(\xi_k, \eta_k) = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}$$

and thus we have from Eq. (42)

$$\Delta u_k = u_k - \sum_{i=1}^{c_p} \hat{N}_i^{(p)}(\xi_k, \eta_k) u_i. \tag{44}$$

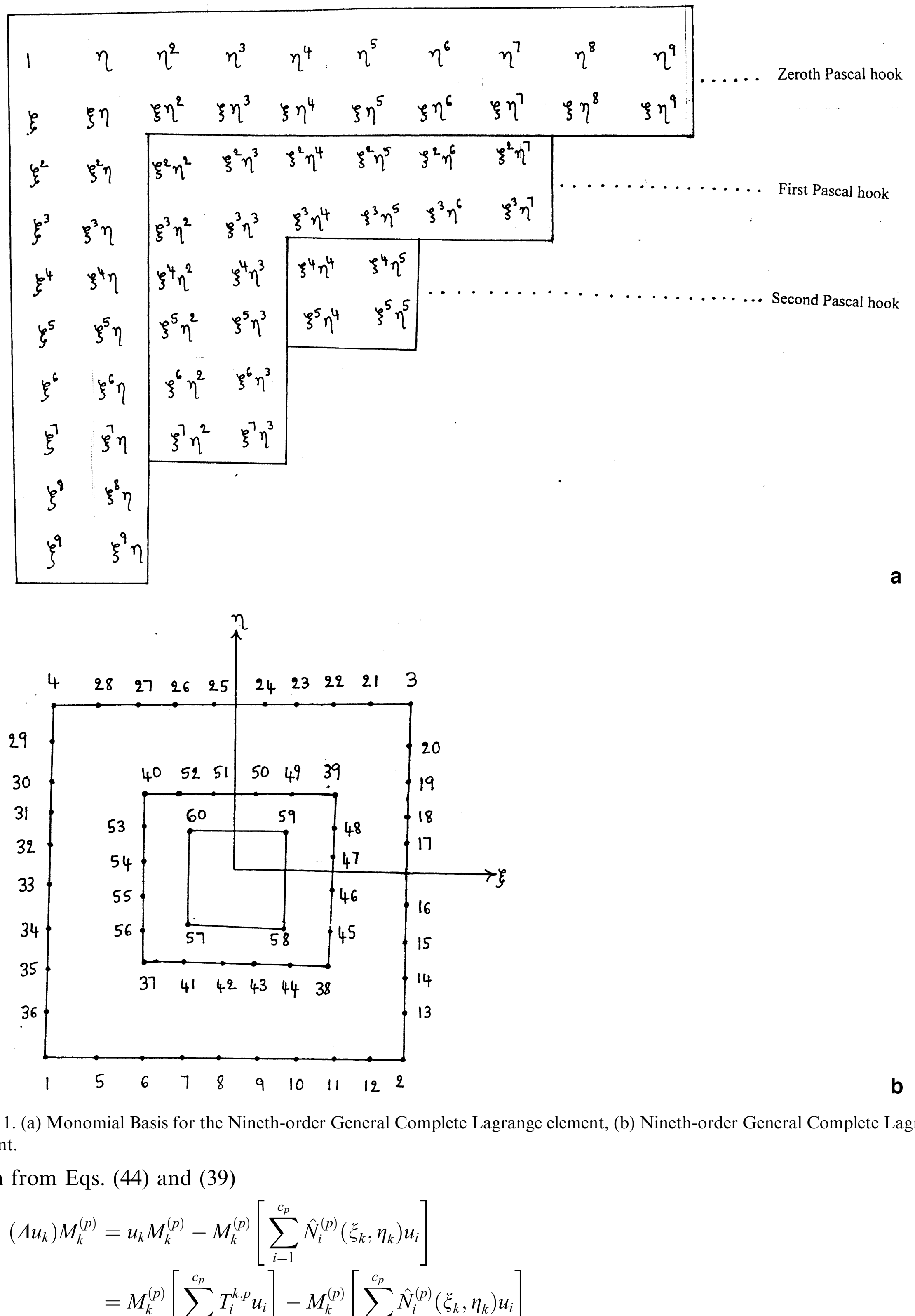

Fig. 11. (a) Monomial Basis for the Nineth-order General Complete Lagrange element, (b) Nineth-order General Complete Lagrange element.

Then from Eqs. (44) and (39)

$$
\begin{aligned}
(\Delta u_k)M_k^{(p)} &= u_k M_k^{(p)} - M_k^{(p)}\left[\sum_{i=1}^{c_p}\hat{N}_i^{(p)}(\xi_k,\eta_k)u_i\right] \\
&= M_k^{(p)}\left[\sum_{i=1}^{c_p}T_i^{k,p}u_i\right] - M_k^{(p)}\left[\sum_{i=1}^{c_p}\hat{N}_i^{(p)}(\xi_k,\eta_k)u_i\right] \\
&= M_k^{(p)}\sum_{i=1}^{c_p}\left[T_i^{k,p} - \hat{N}_i^{(p)}(\xi_k,\eta_k)\right]u_i.
\end{aligned}
\tag{45}
$$

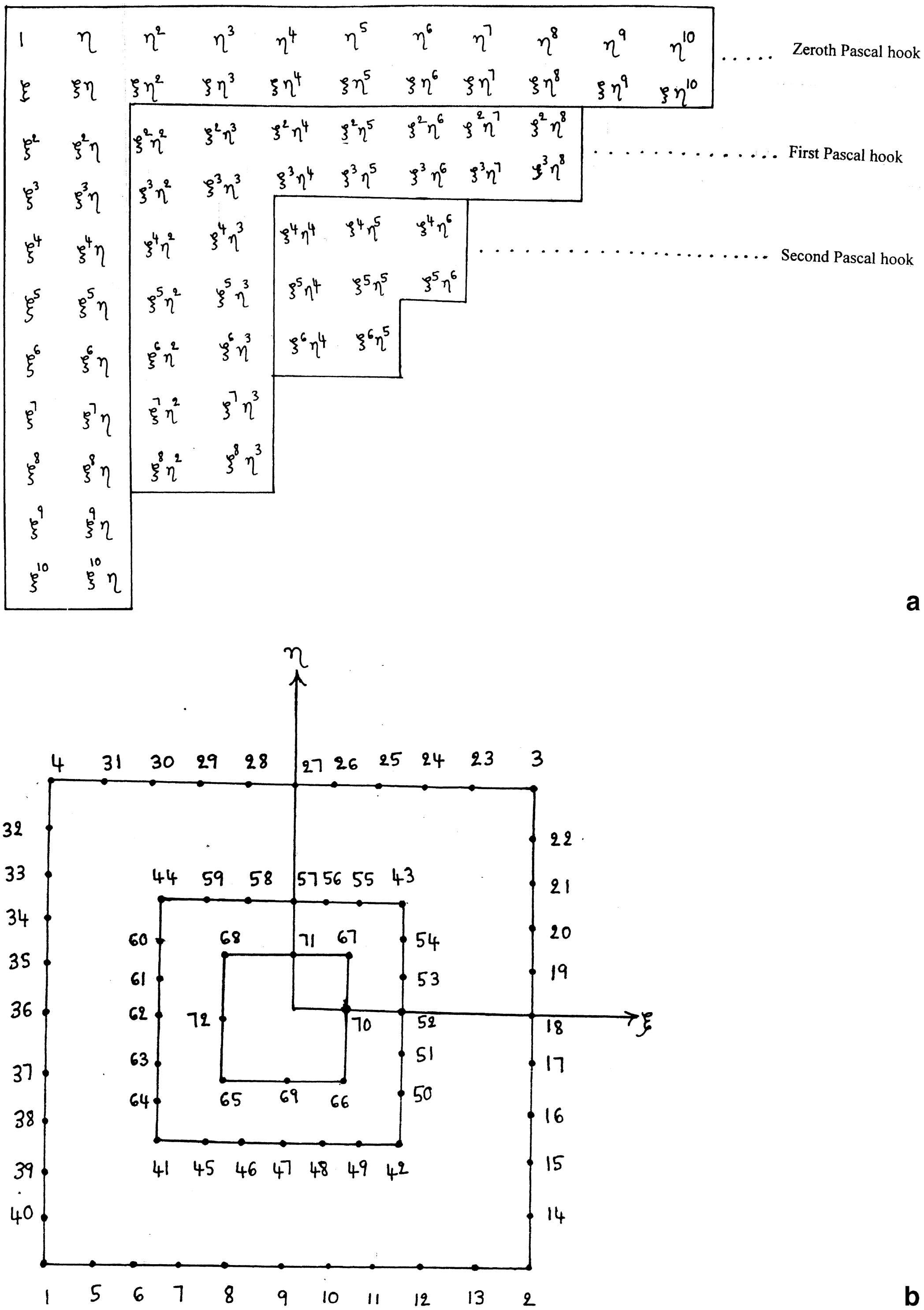

Fig. 12. (a) Monomial Basis for the Tenth-order General Complete Lagrange element, (b) Tenth-order General Complete Lagrange element.

Substituting Eq. (45) in Eq. (42), we obtain

$$u = \sum_{i=1}^{c_p} \left[\hat{N}_i^{(p)}(\xi,\eta) + \sum_{k=c_p+1}^{(p+1)^2} M_k^{(p)} \left\{ T_i^{k,p} - \hat{N}_i^{(p)}(\xi_k,\eta_k) \right\} \right] u_i. \tag{46}$$

Thus from Eqs. (41) and (46)

$$\overline{M}_i^{(p)}(\xi,\eta) = \hat{N}_i^{(p)}(\xi,\eta) + \sum_{k=c_p+1}^{(p+1)^2} M_k^{(p)}(\xi,\eta) \left[T_i^{k,p} - \hat{N}_i^{(p)}(\xi_k,\eta_k) \right]. \tag{47}$$

We see that the Shape functions are revised from those of a Complete Lagrange element of pth-order to the extent that $T_i^{k,p}$ differ from $\hat{N}_i^{(p)}(\xi_k,\eta_k)$. The trick is to design $T_i^{k,p}$ so that it gives the correct value to $u_k,\ k = c_p+1, c_p+2, \ldots, (p+1)^2$ at the intererior nodes for the pth-order displacement field, If that is done, the the pth-order ($p \geqslant 2$). Regular Lagrange Shape functions in Eq. (38) and consequently the Shape functions in Eq. (40) or (46) will correctly interpolate a pth-order displacement field for bilinear element shapes.

The constraint co-efficients $T_i^{k,p}$ are constructed with the aid of a special interpolation formula

$$\begin{aligned} u &= \sum_{i=1}^{c_p} N_i^{*(p)} u_i \\ &= \left[N_i^{*(p)} \right] \{u_i\}, \end{aligned} \tag{48}$$

where

$$\left[N_i^{*(p)} \right] = \left[X_m^{(p)} \right] \left[A_{mi}^{(p)} \right] \tag{49a}$$

and the elements of $\left[X_m^{(p)} \right]$ are selected as follows:

$$\begin{aligned}
\left[X_m^{(2)}\right] &= \left[1, x, y, x^2, xy, y^2, \xi^2\eta, \xi\eta^2\right], \\
\left[X_m^{(3)}\right] &= \left[1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, \xi^3\eta, \xi\eta^3\right], \\
\left[X_m^{(4)}\right] &= \left[1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, x^3y, x^2y^2, xy^3, y^4, \xi^4\eta, \xi\eta^4\right], \\
\left[X_m^{(5)}\right] &= \left[1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, x^3y, x^2y^2, xy^3, y^4, x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5, \xi^5\eta, \xi^3\eta^3, \xi\eta^5\right], \\
\left[X_m^{(6)}\right] &= \left[1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, x^3y, x^2y^2, xy^3, y^4, x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5, x^6, x^5y, x^4y^2,\right. \\
&\qquad \left. x^3y^3, x^2y^4, xy^5, y^6, \xi^6\eta, \xi^4\eta^3, \xi^3\eta^4, \xi\eta^6\right], \\
\left[X_m^{(7)}\right] &= \left[1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, x^3y, x^2y^2, xy^3, y^4, x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5,\right. \\
&\qquad \left. x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6, x^7, x^6y, x^5y^2, x^4y^3, x^3y^4, x^2y^5, xy^6, y^7, \xi^7\eta, \xi^5\eta^3, \xi^3\eta^5, \xi\eta^7\right], \\
\left[X_m^{(8)}\right] &= \left[1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, x^3y, x^2y^2, xy^3, y^4, x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5, x^6, x^5y, x^4y^2, x^3y^3,\right. \\
&\qquad x^2y^4, xy^5, y^6, x^7, x^6y, x^5y^2, x^4y^3, x^3y^4, x^2y^5, xy^6, y^7, x^8, x^7y, x^6y^2, x^5y^3, x^4y^4, x^3y^5, x^2y^6, \\
&\qquad \left. xy^7, y^8, \xi^8\eta, \xi^6\eta^3, \xi^3\eta^6, \xi\eta^8\right], \\
\left[X_m^{(9)}\right] &= \left[1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, x^3y, x^2y^2, xy^3, y^4, x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5, x^6, x^5y,\right. \\
&\qquad x^4y^2, x^3y^3, x^2y^4, xy^5, y^6, x^7, x^6y, x^5y^2, x^4y^3, x^3y^4, x^2y^5, xy^6, y^7, x^8, x^7y, x^6y^2, x^5y^3, x^4y^4, x^3y^5, \\
&\qquad \left. x^2y^6, xy^7, y^8, x^9, x^8y, x^7y^2, x^6y^3, x^5y^4, x^4y^5, x^3y^6, x^2y^7, xy^8, y^9, \xi^9\eta, \xi^7\eta^3, \xi^5\eta^5, \xi^3\eta^7, \xi\eta^9\right],
\end{aligned}$$

$$\left[X_m^{(10)}\right] = \left[1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, x^3y, x^2y^2, xy^3, y^4, x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5, x^6, x^5y, x^4y^2,\right.$$
$$x^3y^3, x^2y^4, xy^5, y^6, x^7, x^6y, x^5y^2, x^4y^3, x^3y^4, x^2y^5, xy^6, y^7, x^8, x^7y, x^6y^2, x^5y^3, x^4y^4, x^3y^5,$$
$$x^2y^6, xy^7, y^8, x^9, x^8y, x^7y^2, x^6y^3, x^5y^4, x^4y^5, x^3y^6, x^2y^7, xy^8, y^9, x^{10}, x^9y, x^8y^2, x^7y^3, x^6y^4, x^5y^5,$$
$$\left. x^4y^6, x^3y^7, x^2y^8, xy^9, y^{10}, \xi^{10}\eta, \xi^8\eta^3, \xi^6\eta^5, \xi^5\eta^6, \xi^3\eta^8, \xi\eta^{10}\right]. \tag{49b}$$

The elements of

$$\left[A_{mi}^{(p)}\right]$$

are just constant co-efficients to be determined. The bivariate monomial terms of degree p in x, y specified at the exterior and interior nodes of Complete Lagrange element correctly interpolate any bivariate complete polynomial function of degree p in x, y specified at the interior and exterior nodes. This would not be possible if these global (metric) terms were replaced by corresponding powers of ξ and η some monomial terms like

$$\xi^\alpha \eta^\beta, \quad \alpha + \beta = p + 1$$

are somewhat arbitrary and are required to ensure the existence of $\left[A_{mi}^{(p)}\right]$. To find $\left[A_{mi}^{(p)}\right]$ note that Eqs. (48) and (49a) give the same values of u at nodes

$$u_i = \left[X_{im}^{(p)}\right]\left[A_{mi}^{(p)}\right]\{u_i\}, \tag{50}$$

where $\left[X_{im}^{(p)}\right]$ is the value of $\left[X_m^{(p)}\right]$ at node i.

Consequently

$$\left[A_{mi}^{(p)}\right] = \left[X_{im}^{(p)}\right]^{-1}, \tag{51}$$

we require the value of u from Eq. (48) at the interior points

$$k = c_p + 1(1)(p+1)^2$$

and we thus have

$$\begin{aligned} u_k &= \sum_{i=1}^{c_p} N_i^{*(p)}(x_k, y_k) u_i \\ &= \left[N_i^{*(p)}(x_k, y_k)\right]\{u_i\} \\ &= \left[X_{km}^{(p)}\right]\left[A_{mi}^{(p)}\right]\{u_i\} \\ &= \left[X_{k1}^{(p)}, X_{k2}^{(p)}, \ldots, X_{km}^{(p)}\right] \begin{bmatrix} A_{11}^{(p)} & A_{12}^{(p)} & \cdot & A_{1i}^{(p)} & \cdots & A_{1m}^{(p)} \\ A_{21}^{(p)} & A_{22}^{(p)} & \cdot & A_{2i}^{(p)} & \cdots & A_{2m}^{(p)} \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ A_{m1}^{(p)} & A_{m2}^{(p)} & \cdot & A_{mi}^{(p)} & \cdots & A_{mm}^{(p)} \end{bmatrix} \{u_i\} \\ &= \left[\sum_{\alpha=1}^{m} X_{k\alpha}^{(p)} A_{\alpha 1}^{(p)}, \ldots, \sum_{\alpha=1}^{m} X_{k\alpha}^{(p)} A_{\alpha i}^{(p)}, \ldots, \sum_{\alpha=1}^{m} X_{k\alpha}^{(p)} A_{\alpha m}^{(p)}\right]\{u_i\} \\ &= \sum_{i=1}^{c_p}\left(\sum_{\alpha=1}^{m} X_{k\alpha}^{(p)} A_{\alpha i}^{(p)}\right) u_i. \end{aligned} \tag{52}$$

Now comparing Eqs. (52) with (39), we obtain

$$T_i^{k,p} = \sum_{\alpha=1}^{m} X_{k\alpha}^{(p)} A_{\alpha i}^{(p)} \tag{53}$$

or

$$T_i^{k,p} = N_i^{*(p)}(x_k, y_k). \tag{54}$$

8. Constraint co-efficients $T_i^{9,2}$ $i = 1(1)8$ for the Quadratic (Serendipity) Complete Lagrange element

Algorithm 1. Consider the eight-noded Quadratic (Serendipity) Complete Lagrange element. We can map the arbitrary straight-edged eight-noded Quadrilateral in global (X, Y) space to new local space (x, y) as shown in Fig. 13, where the origin is now located at the centroid of the element.

Let

$$\begin{aligned} u(X,Y) &= \sum_{i=1}^{8} N^{*(2)}(X,Y)u_i \\ &= \sum_{i=1}^{8} N_i^{*(2)}(x,y)u_i \\ &= \sum_{i=1}^{8} N_i^{*(2)}(\xi,\eta)u_i, \end{aligned} \tag{55}$$

where

$$\begin{aligned} X &= X_9 z + X_6 x + X_7 y, \\ Y &= Y_9 z + Y_6 x + Y_7 y, \\ z &= 1 - x - y, \end{aligned} \tag{56}$$

$$\begin{aligned} X_9 &= \frac{1}{4}(X_1 + X_2 + X_3 + X_4), \\ Y_9 &= \frac{1}{4}(Y_1 + Y_2 + Y_3 + Y_4), \end{aligned}$$

$$\begin{aligned} X_5 &= \frac{X_1 + X_2}{2}, \quad Y_5 = \frac{Y_1 + Y_2}{2}, \\ X_6 &= \frac{X_2 + X_3}{2}, \quad Y_6 = \frac{Y_2 + Y_3}{2}, \\ X_7 &= \frac{X_3 + X_4}{2}, \quad Y_7 = \frac{Y_3 + Y_4}{2}, \\ X_8 &= \frac{X_1 + X_4}{2}, \quad Y_8 = \frac{Y_1 + Y_4}{2}. \end{aligned} \tag{57}$$

We can rewrite Eq. (56) as

$$X = \frac{1}{4}(X_1 + X_2 + X_3 + X_4) + \frac{1}{4}(-X_1 + X_2 + X_3 - X_4)x + \frac{1}{4}(-X_1 - X_2 + X_3 + X_4)y, \tag{58}$$

$$Y = \frac{1}{4}(Y_1 + Y_2 + Y_3 + Y_4) + \frac{1}{4}(-Y_1 + Y_2 + Y_3 - Y_4)x + \frac{1}{4}(-Y_1 - Y_2 + Y_3 + Y_4)y. \tag{59}$$

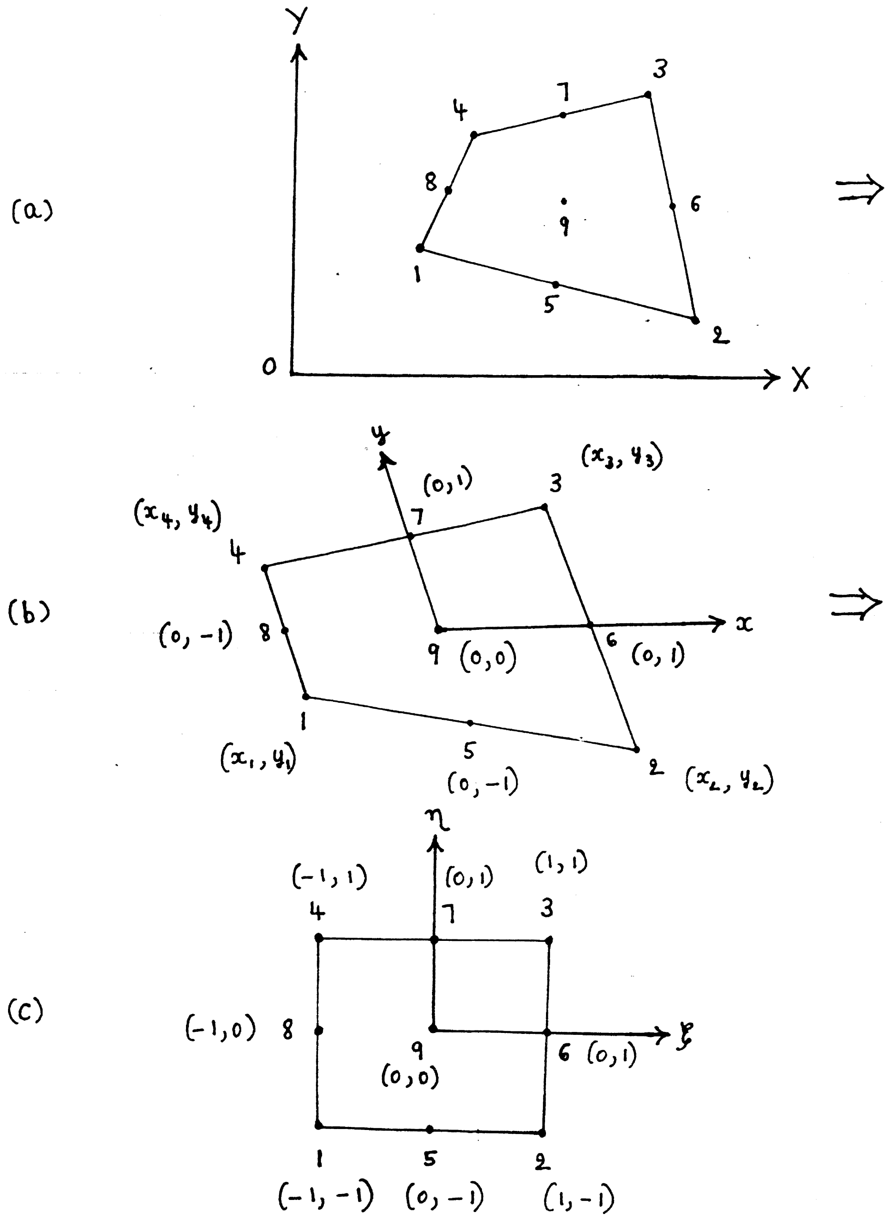

Fig. 13. Transformation of a Quadrilateral element: (a) Quadrilateral in global (X, Y) space, (b) Quadrilateral in local (x, y) space, (c) Mapped 2-square $-1 \leqslant \xi, \eta \leqslant 1$ in local ξ, η, space.

The mapping which transforms the Quadrilateral in (x, y) co-ordinate system to a 2-square $-1 \leqslant \xi, \eta \leqslant 1$ in the local co-ordinate system (ξ, η) is

$$
\begin{aligned}
x &= \xi + \frac{\xi\eta}{4}(x_1 - x_2 + x_3 - x_4), \\
y &= \eta + \frac{\xi\eta}{4}(y_1 - y_2 + y_3 - y_4),
\end{aligned}
\tag{60}
$$

where the correspondence between

(x, y) and (ξ, η)

space is

$$\begin{aligned}
&(x=0,\ y=0) \quad \text{corresponds to} \quad (\xi=0,\ \eta=0),\\
&(x=0,\ y=\pm1) \quad \text{corresponds to} \quad (\xi=0,\ \eta=\pm1),\\
&(x=\pm1,\ y=0) \quad \text{corresponds to} \quad (\xi=\pm1,\ \eta=0).
\end{aligned} \tag{61}$$

We note that mapping from global (X,Y) to (x,y) is linear whereas local (x,y) space to local (ξ,η) space is non-linear. We shall therefore consider the Quadrilateral element in (x,y) space to be in a global space, as any global space (X,Y) can be related to local (x,y) space in a linear manner.

We can therefore write

$$\begin{aligned}
\left[N_i^{*(2)}(x,y)\right] &= \left[1,x,y,x^2,xy,y^2,\xi^2\eta,\xi\eta^2\right] \times \begin{bmatrix} A_{11}^{(2)} & A_{12}^{(2)} & \cdots & \cdots & A_{18}^{(2)} \\ A_{21}^{(2)} & A_{21}^{(2)} & \cdots & \cdots & A_{211}^{(2)} \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ A_{81}^{(2)} & A_{82}^{(2)} & & & A_{88}^{(2)} \end{bmatrix} \\
&= \left[X_m^{(2)}\right]\left[A_{mi}^{(2)}\right].
\end{aligned} \tag{62}$$

From Eq. (50), we have

$$\{u_i\} = \left[X_{im}^{(2)}\right]\left[A_{mi}^{(2)}\right]\{u_i\},$$

where

$$\left[X_{im}^{(2)}\right] = \begin{bmatrix}
1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 & -1 & -1 \\
1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 & -1 & 1 \\
1 & x_3 & y_3 & x_3^2 & x_3y_3 & y_3^2 & 1 & 1 \\
1 & x_4 & y_4 & x_4^2 & x_4y_4 & y_4^2 & 1 & -1 \\
1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{63}$$

and

$$A_{mi}^{(2)} = \left[X_{im}^{(2)}\right]^{-1}. \tag{64}$$

We see that the above procedure simplifies the determination of Shape functions

$$N_i^{*(2)}(x,y)$$

considerably.

We require the value of u from Eq. (48) at only one point – the nineth interior node.

From Eqs. (55) and (62), we have

$$\begin{aligned}
u_9 &= u(x_9,y_9) \\
&= [1,0,0,0,0,0,0,0,0]\left[A_{mi}^{(2)}\right]\{u_i\} \\
&= \left[A_{1i}^{(2)}\right]\{u_i\} \\
&= \sum_{i=1}^{8} A_{1i}u_i,
\end{aligned} \tag{65}$$

where

$$
\begin{aligned}
\Delta^{(2)}A_{11}^{(2)} &= x_2y_2 + x_4y_4,\\
\Delta^{(2)}A_{12}^{(2)} &= -(x_1y_1 + x_3y_3),\\
\Delta^{(2)}A_{13}^{(2)} &= x_2y_2 + x_4y_4,\\
\Delta^{(2)}A_{14}^{(2)} &= -(x_1y_1 + x_3y_3),\\
\Delta^{(2)}A_{15}^{(2)} &= \frac{1}{2}\left[\{(y_1+y_3)-(y_1^2+y_3^2)\}(x_2y_2+x_4y_4) - \{(y_2^2+y_4^2)-(y_2+y_4)\}(x_2y_1+x_3y_3)\right],\\
\Delta^{(2)}A_{16}^{(2)} &= \frac{1}{2}\{(x_2+x_2^2+x_4+x_4^2)(x_1y_1+x_3y_3)-(x_1+x_1^2+x_3+x_3^2)(x_2y_2+x_4y_4)\},\\
\Delta^{(2)}A_{17}^{(2)} &= \frac{1}{2}\{(y_2+y_2^2+y_4+y_4^2)(x_1y_1+x_3y_3)-(y_1+y_1^2+y_3+y_3^2)(x_2y_2+x_4y_4)\},\\
\Delta^{(2)}A_{18}^{(2)} &= \frac{1}{2}\left[\{(x_1+x_3)-(x_1^2+x_3^2)\}(x_2y_2+x_4y_4) - \{(x_2+x_4)-(x_2^2+x_4^2)\}(x_1y_1+x_3y_3)\right],\\
\Delta^{(2)} &= \{2-(x_1^2+y_1^2+x_3^2+y_3^2)\}(x_2y_2+x_4y_4) - \{2-(x_2^2+y_2^2+x_4^2+y_4^2)\}(x_1y_1+x_3y_3).
\end{aligned}
\tag{66}
$$

Finally on comparison with Eq. (39), we obtain

$$T_i = A_{1i} \tag{67}$$

and the revised Shape functions can then be obtained from Eq. (47).

The above algorithm is similar to what has been proposed in Ref. [15] and in addition, we have also given explicit expressions for A_{1i} values. However, we observe that it may not be possible in general to find transformations which coincide with the interior nodal points of the higher-order Regular Lagrange Quadrilateral elements as it was fortunately done in the case of Quadratic element. Hence we propose an alternative algorithm which can work well for all elements of Regular Lagrange family. This is explained in Algorithm 2 for Quadratic to Quintic elements.

Algorithm 2. Let

$$
\begin{aligned}
u(X,Y) &= \sum_{i=1}^{8} N^{*2}(X,Y)u_i\\
&= \sum_{i=1}^{8} N_i^{*2}(x,y)u_i\\
&= \sum_{i=1}^{8} N_i^{*2}(\xi,\eta)u_i,
\end{aligned}
\tag{68}
$$

where

$$
\begin{aligned}
X &= X_1z + X_2x + X_4y,\\
Y &= Y_1z + Y_2x + Y_4y,\\
z &= 1 - x - y.
\end{aligned}
\tag{69}
$$

We can rewrite Eq. (69) as (see Fig. 14a,b)

$$
\begin{aligned}
X &= X_1 + (X_2 - X_1)x + (X_4 - X_1)y,\\
Y &= Y_1 + (Y_2 - Y_1)x + (Y_4 - Y_1)y.
\end{aligned}
\tag{70}
$$

The above transformation of Eqs. (69) and (70) maps an arbitrary Quadrilateral in global (X,Y) space to a new Quadrilateral in the local (x,y) space. We can now map the Quadrilateral in (x,y) space into a unit square in the local (ξ,η) space by the following equation (see Fig. 14c)

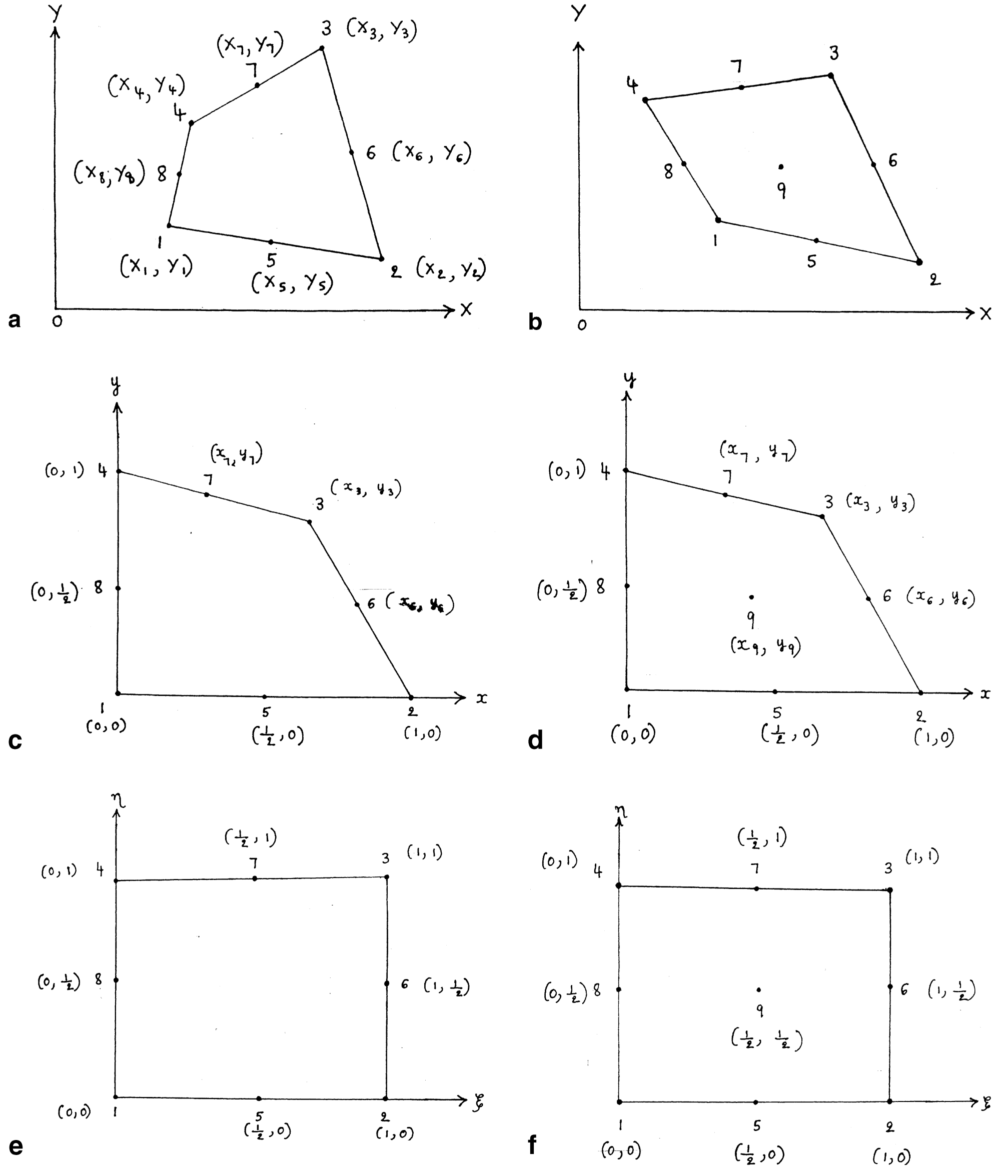

Fig. 14. Transformation of a Quadratic Quadrilateral element: (a–b) Quadrilateral in global (X, Y) space as Serendipity and Lagrange elements, (c–d) Quadrilateral in local (x, y) space as Serendipity and Lagrange elements, (e–f) Mapped 1-square $0 \leqslant \xi, \eta \leqslant 1$, in local ξ, η, space as Serendipity and Lagrange elements.

$$
\begin{aligned}
x &= \xi + (x_3 - 1)\xi\eta, \\
y &= \eta + (y_3 - 1)\xi\eta.
\end{aligned} \tag{71}
$$

We also note that the correspondence between (x, y) and (ξ, η) spaces is such that

$$
\begin{aligned}
&(x = 0,\ y = 0) \quad \text{corresponds to} \quad (\xi = 0,\ \eta = 0), \\
&\left(x = \frac{1}{2},\ y = 0\right) \quad \text{corresponds to} \quad \left(\xi = \frac{1}{2},\ \eta = 0\right), \\
&\left(x = 0,\ y = \frac{1}{2}\right) \quad \text{corresponds to} \quad \left(\xi = 0,\ \eta = \frac{1}{2}\right), \\
&(x = 0,\ y = 1) \quad \text{corresponds to} \quad (\xi = 0,\ \eta = 1),
\end{aligned}
$$

$$
\begin{aligned}
&(x = x_6,\ y = y_6) \quad \text{corresponds to} \quad \left(\xi = 1,\ \eta = \frac{1}{2}\right),\\
&(x = x_3,\ y = y_3) \quad \text{corresponds to} \quad (\xi = 1,\ \eta = 1),\\
&(x = x_7,\ y = y_7) \quad \text{corresponds to} \quad \left(\xi = \frac{1}{2},\ \eta = 1\right).
\end{aligned}
$$

That is, the (x, y) co-ordinates and (ξ, η) co-ordinates are same for nodal points along $x = 0$ corresponding to $\xi = 0$ and $y = 0$ corresponding to $\eta = 0$ and this has advantages over other transformations. Moreover the mapping from (X, Y) space to (x, y) space is linear and the mapping from (x, y) space to (ξ, η) space is non-linear.

We can therefore write

$$
\begin{aligned}
N_i^{*2}(x,y) &= \left[1, x, y, x^2, xy, y^2, \xi^2\eta, \xi\eta^2\right] \times \begin{bmatrix} A_{11}^{(2)} & A_{12}^{(2)} & \cdots & \cdots & A_{18}^{(2)} \\ A_{21}^{(2)} & A_{22}^{(2)} & \cdots & \cdots & A_{28}^{(2)} \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ A_{81}^{(2)} & A_{82}^{(2)} & \cdots & \cdots & A_{88}^{(2)} \end{bmatrix} \\
&= \left[X_m^{(2)}\right]\left[A_{mi}^{(2)}\right]
\end{aligned} \tag{72}
$$

and from Eq. (50), we have

$$
\{u_i\} = \left[X_{im}^{(2)}\right]\left[A_{mi}^{(2)}\right]\{u_i\}, \tag{73a}
$$

where

$$
\left[X_{im}^{(2)}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & x_3 & y_3 & x_3^2 & x_3y_3 & y_3^2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 & \frac{1}{2} & \frac{1}{4} \\ 1 & x_7 & y_7 & x_7^2 & x_7y_7 & y_7^2 & \frac{1}{4} & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 & 0 \end{bmatrix}. \tag{73b}
$$

From the property of Shape functions

$$
N_i^{*(2)}(x_k, y_k) = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases} \tag{74}
$$

and from Eqs. (72), (73a) and (73b), we obtain

$$
\begin{aligned}
&N_i^{*(2)}(x,y) = -H_i^{(2)}(x,y) + A_{5i}^{(2)}xy + A_{7i}^{(2)}\xi^2\eta + A_{8i}^{(2)}\xi\eta^2,\\
&i = 1(1)8,
\end{aligned} \tag{75a}
$$

where

$$
\begin{aligned}
&H_i^{(2)}(x,y) = 0, \qquad i = 3, 6, 7,\\
&H_1^{(2)}(x,y) = -\left\{1 - 3(x+y) + 2(x^2+y^2)\right\},\\
&H_2^{(2)}(x,y) = -2x^2 + x,\\
&H_4^{(2)}(x,y) = -2y^2 + y,\\
&H_5^{(2)}(x,y) = 4x^2 - 4x,\\
&H_8^{(2)}(x,y) = 4y^2 - 4y.
\end{aligned} \tag{75b}
$$

From Eqs. (75a) and (75b), we find

$$\begin{bmatrix} A^{(2)}_{5,3} & A^{(2)}_{5,6} & A^{(2)}_{5,7} \\ A^{(2)}_{7,3} & A^{(2)}_{7,6} & A^{(2)}_{7,7} \\ A^{(2)}_{8,3} & A^{(2)}_{8,6} & A^{(2)}_{8,7} \end{bmatrix} = \begin{bmatrix} x_3y_3 & 1 & 1 \\ x_6y_6 & \frac{1}{2} & \frac{1}{4} \\ x_7y_7 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}^{-1} \tag{76}$$

and since

$$N^{*(2)}_i(x_k, y_k) = 0 \quad \text{for} \quad i \neq k$$

we can write

$$\begin{aligned} H^{(2)}_i(x_3, y_3) &= A^{(2)}_{5i} x_3y_3 + A^{(2)}_{7i} \xi_3^2\eta_3 + A^{(2)}_{8i} \xi_3\eta_3^2, \\ H^{(2)}_i(x_6, y_6) &= A^{(2)}_{5i} x_6y_6 + A^{(2)}_{7i} \xi_6^2\eta_6 + A^{(2)}_{8i} \xi_6\eta_6^2, \\ H^{(2)}_i(x_7, y_7) &= A^{(2)}_{5i} x_7y_7 + A^{(2)}_{7i} \xi_7^2\eta_7 + A^{(2)}_{8i} \xi_7\eta_7^2 \end{aligned} \tag{77}$$

for

$$i = 1, 2, 4, 5, 8.$$

From Eq. (77), we can immediately write

$$\begin{bmatrix} A^{(2)}_{5i} \\ A^{(2)}_{7i} \\ A^{(2)}_{8i} \end{bmatrix} = \begin{bmatrix} x_3y_3 & 1 & 1 \\ x_6y_6 & \frac{1}{2} & \frac{1}{4} \\ x_7y_7 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}^{-1} \times \begin{bmatrix} H^{(2)}_i(x_3,\ y_3) \\ H^{(2)}_i(x_6,\ y_6) \\ H^{(2)}_i(x_7,\ y_7) \end{bmatrix}. \tag{78}$$

Thus from Eqs. (76) and (78),

$$\begin{bmatrix} A^{(2)}_{5,3} & A^{(2)}_{5,6} & A^{(2)}_{5,7} \\ A^{(2)}_{7,3} & A^{(2)}_{7,6} & A^{(2)}_{7,7} \\ A^{(2)}_{8,3} & A^{(2)}_{8,6} & A^{(2)}_{8,7} \end{bmatrix} = \frac{1}{\Delta^{(2)}} \times \begin{bmatrix} \frac{3}{16} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4}(x_7y_7 - 2x_6y_6) & \frac{1}{2}(x_3y_3 - 2x_7y_7) & \frac{1}{4}(4x_6y_6 - x_3y_3) \\ \frac{1}{4}(x_6y_6 - 2x_7y_7) & \frac{1}{4}(4x_7y_7 - x_3y_3) & \frac{1}{2}(x_3y_3 - 2x_6y_6) \end{bmatrix}, \tag{79a}$$

where

$$\begin{aligned} A^{(2)}_{5i} &= \frac{1}{\Delta^{(2)}} \left\{ \frac{3}{16} H^{(2)}_i(x_3, y_3) - \frac{1}{4} H^{(2)}_i(x_6, y_6) - \frac{1}{4} H^{(2)}_i(x_7, y_7) \right\}, \\ A^{(2)}_{7i} &= \frac{1}{\Delta^{(2)}} \left\{ \frac{1}{4}(x_7y_7 - 2x_6y_6) H^{(2)}_i(x_3, y_3) + \frac{1}{2}(x_3y_3 - 2x_7y_7) H^{(2)}_i(x_6, y_6) + \frac{1}{4}(4x_6y_6 - x_3y_3) H^{(2)}_i(x_7, y_7) \right\}, \\ A^{(2)}_{8i} &= \frac{1}{\Delta^{(2)}} \left\{ \frac{1}{4}(x_6y_6 - 2x_7y_7) H^{(2)}_i(x_3, y_3) + \frac{1}{4}(4x_7y_7 - x_3y_3) H^{(2)}_i(x_6, y_6) + \frac{1}{2}(x_3y_3 - 2x_6y_6) H^{(2)}_i(x_7, y_7) \right\}, \\ &\quad i = 1, 2, 4, 5, 8, \end{aligned} \tag{79b}$$

$$\Delta^{(2)} = \frac{3}{16} x_3y_3 - \frac{1}{4}\ x_6y_6 - \frac{1}{4} x_7y_7. \tag{79c}$$

Finally from Eqs. (54) and (75a) (75b),

$$T_i^{9,2} = N_i^{*(2)}(x_9, y_9) \tag{80}$$

and the revised Shape functions can then be obtained on using Eq. (47).

9. Constraint co-efficients $T_i^{k,3}$ ($i = 1(1)12,\ k = 13, 14, 15, 16$) for the Cubic Complete Lagrange element

Following the procedure outlined in Eqs. (68)–(80), it can be shown that (see Fig. 15)

$$N_i^{*(3)}(x,y) = -H_i^{(3)}(x,y) + xyA^{(3)}_{5,i} + x^2y\,A^{(3)}_{8,i} + xy^2A^{(3)}_{9,i} + \xi^3\eta\,A^{(3)}_{11,i} + \xi\eta^3A^{(3)}_{12,i}, \quad i = 1(1)12, \tag{81a}$$

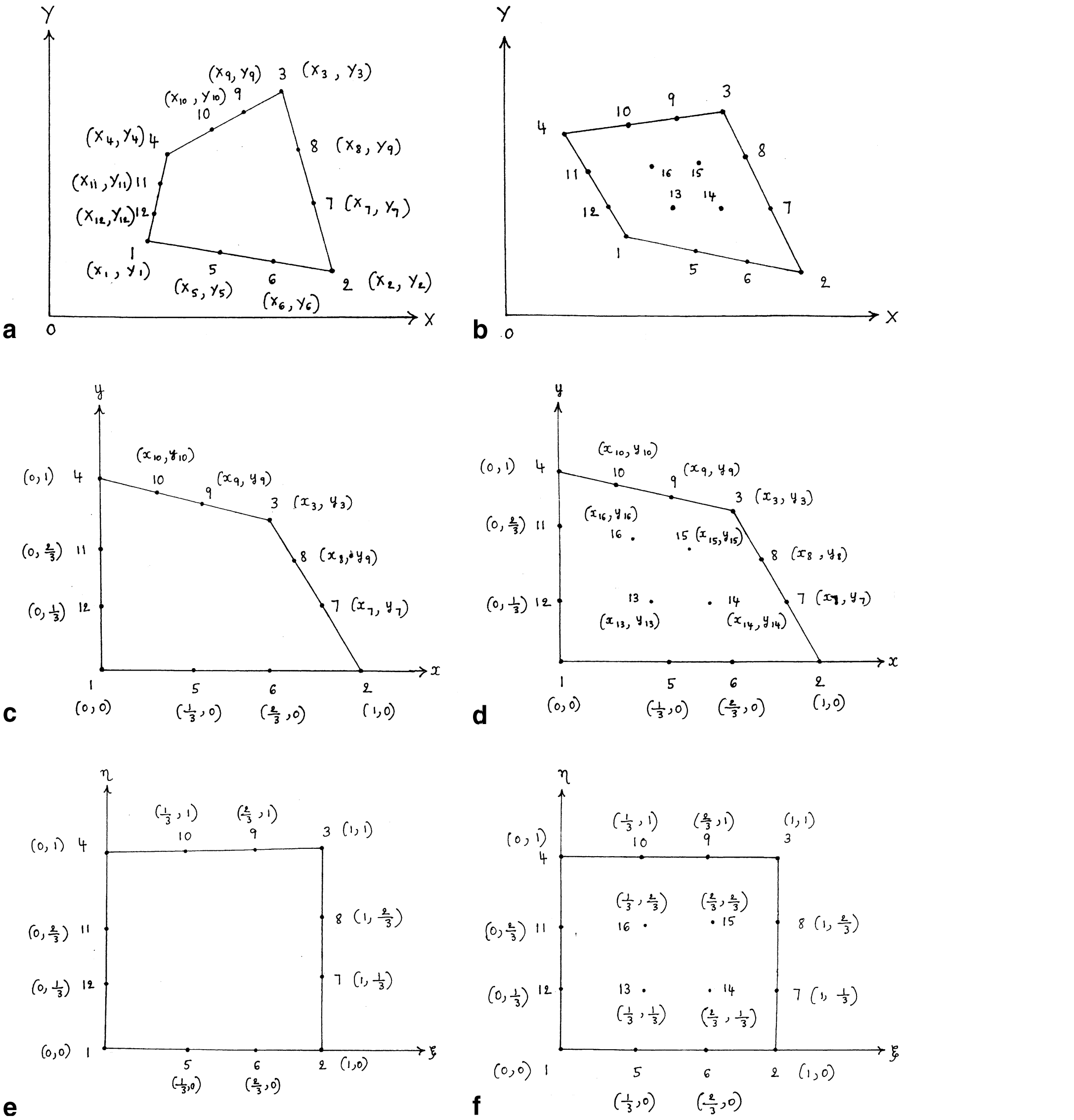

Fig. 15. Transformation of a Cubic Quadrilateral element: (a–b) Quadrilateral in global (X, Y) space as Serendipity and Lagrange elements, (c–d) Quadrilateral in local (x, y) space as Serendipity and Lagrange elements, (e–f) Mapped 1-square $0 \leqslant \xi, \eta \leqslant 1$, in local ξ, η, space as Serendipity and Lagrange elements.

where

$$
\begin{aligned}
H_1^{(3)}(x,y) &= 1+\frac{9}{2}\left(x-\frac{1}{3}\right)\left(x-\frac{2}{3}\right)(x-1)+\frac{9}{2}\left(y-\frac{1}{3}\right)\left(y-\frac{2}{3}\right)(y-1),\\
H_2^{(3)}(x,y) &= -\frac{9}{2}x\left(x-\frac{1}{3}\right)\left(x-\frac{2}{3}\right),\\
H_4^{(3)}(x,y) &= -\frac{9}{2}y\left(y-\frac{1}{3}\right)\left(y-\frac{2}{3}\right),\\
H_5^{(3)}(x,y) &= -\frac{27}{2}x\left(x-\frac{2}{3}\right)(x-1),\\
H_6^{(3)}(x,y) &= \frac{27}{2}x\left(x-\frac{1}{3}\right)(x-1),\\
H_{11}^{(3)}(x,y) &= \frac{27}{2}y\left(y-\frac{1}{3}\right)(y-1),\\
H_{12}^{(3)}(x,y) &= -\frac{27}{2}y\left(y-\frac{2}{3}\right)(y-1),\\
H_i^{(3)}(x,y) &= 0 \quad \text{for} \quad i=3,7,8,9,10.
\end{aligned}
\tag{81b}
$$

Then, we obtain

$$
\begin{bmatrix}
A_{5,3}^{(3)} & A_{5,7}^{(3)} & A_{5,8}^{(3)} & A_{5,9}^{(3)} & A_{5,10}^{(3)}\\
A_{8,3}^{(3)} & A_{8,7}^{(3)} & A_{8,8}^{(3)} & A_{8,9}^{(3)} & A_{8,10}^{(3)}\\
A_{9,3}^{(3)} & A_{9,7}^{(3)} & A_{9,8}^{(3)} & A_{9,9}^{(3)} & A_{9,10}^{(3)}\\
A_{11,3}^{(3)} & A_{11,7}^{(3)} & A_{11,8}^{(3)} & A_{11,9}^{(3)} & A_{11,10}^{(3)}\\
A_{12,3}^{(3)} & A_{12,7}^{(3)} & A_{12,8}^{(3)} & A_{12,9}^{(3)} & A_{12,10}^{(3)}
\end{bmatrix}
=
\begin{bmatrix}
x_3y_3 & x_3^2y_3 & x_3y_3^2 & 1 & 1\\
x_7y_7 & x_7^2y_7 & x_7y_7^2 & \frac{1}{3} & \frac{1}{27}\\
x_8y_8 & x_8^2y_8 & x_8y_8^2 & \frac{2}{3} & \frac{8}{27}\\
x_9y_9 & x_9^2y_9 & x_9y_9^2 & \frac{8}{27} & \frac{2}{3}\\
x_{10}y_{10} & x_{10}^2y_{10} & x_{10}y_{10}^2 & \frac{1}{27} & \frac{1}{3}
\end{bmatrix}^{-1}
\tag{82}
$$

and

$$
\begin{bmatrix}
A_{5,i}^{(3)}\\ A_{8,i}^{(3)}\\ A_{9,i}^{(3)}\\ A_{11,i}^{(3)}\\ A_{12,i}^{(3)}
\end{bmatrix}
=
\begin{bmatrix}
x_3y_3 & x_3^2y_3 & x_3y_3^2 & 1 & 1\\
x_7y_7 & x_7^2y_7 & x_7y_7^2 & \frac{1}{3} & \frac{1}{27}\\
x_8y_8 & x_8^2y_8 & x_8y_8^2 & \frac{2}{3} & \frac{8}{27}\\
x_9y_9 & x_9^2y_9 & x_9y_9^2 & \frac{8}{27} & \frac{2}{3}\\
x_{10}y_{10} & x_{10}^2y_{10} & x_{10}y_{10}^2 & \frac{1}{27} & \frac{1}{3}
\end{bmatrix}^{-1}
\times
\begin{bmatrix}
H_i^{(3)}(x_3,y_3)\\ H_i^{(3)}(x_7,y_7)\\ H_i^{(3)}(x_8,y_8)\\ H_i^{(3)}(x_9,y_9)\\ H_i^{(3)}(x_{10},y_{10})
\end{bmatrix},
\quad i=1,2,4,5,6,11,12.
\tag{83}
$$

Finally from Eqs. (54), (81a) and (81b)

$$
T_i^{k,3} = N_i^{*(3)}(x_k,y_k), \quad k=13,14,15,16
\tag{84}
$$

and the revised Shape functions can then be obtained from Eq. (47).

10. Constraint co-efficients $T_i^{k,4}$ $(i = 1(1)17,\ k = 17(1)25)$ for Quartic Complete Lagrange element

Following the procedure outlined in Eqs. (68)–(80), it can be shown that (see Fig. 16)

$$N_i^{*(4)}(x,y) = -H_i^{(4)}(x,y) + xyA_{5,i}^{(4)} + x^2yA_{8,i}^{(4)} + xy^2A_{9,i}^{(4)} + x^3yA_{12,i}^{(4)} + x^2y^2A_{13,i}^{(4)} + xy^3A_{14,i}^{(4)}$$

$$+ \xi^4\eta A_{16,i}^{(4)} + \xi\eta^4A_{17,i}^{(4)} \quad i = 1(1)17, \tag{85a}$$

where

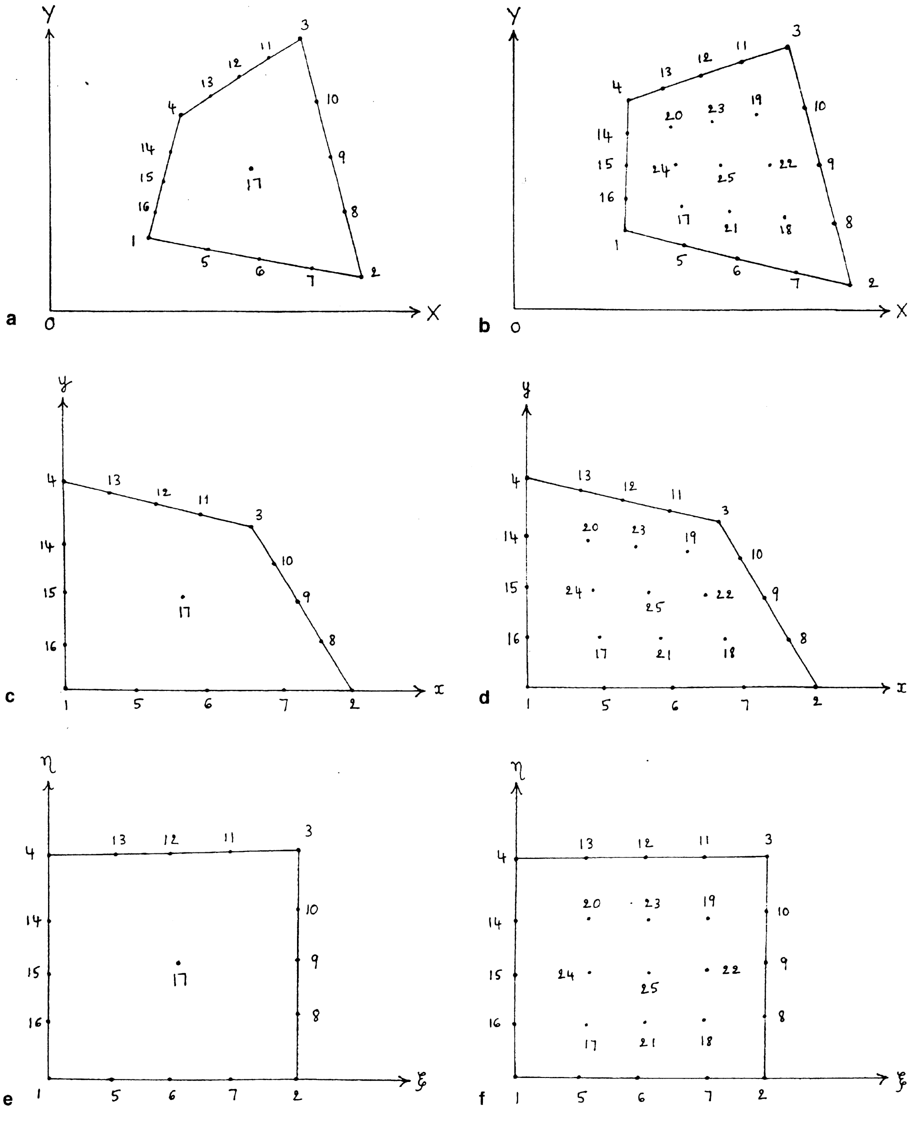

Fig. 16. Transformation of a Quartic Quadrilateral element: (a–b) Quadrilateral in global (X, Y) space as Serendipity and Lagrange elements, (c–d) Quadrilateral in local (x, y) space as Serendipity and Lagrange elements, (e–f) Mapped 1-square $0 \leqslant \xi, \eta \leqslant 1$, in local ξ, η, space as Serendipity and Lagrange elements.

$$
\begin{aligned}
H_1^{(4)}(x,y) &= 1 - \frac{1}{3}\{(4x-1)(2x-1)(4x-3)(x-1) \\
&\qquad + (4y-1)(2y-1)(4y-3)(y-1)\}, \\
H_2^{(4)}(x,y) &= -\frac{1}{3}x(4x-1)(2x-1)(4x-3), \\
H_4^{(4)}(x,y) &= -\frac{1}{3}y(4y-1)(2y-1)(4y-3), \\
H_5^{(4)}(x,y) &= \frac{16}{3}x(2x-1)(4x-3)(x-1), \\
H_6^{(4)}(x,y) &= -4x(4x-1)(4x-3)(x-1), \\
H_7^{(4)}(x,y) &= \frac{16}{3}x(4x-1)(2x-1)(x-1), \\
H_{14}^{(4)}(x,y) &= \frac{16}{3}y(4y-1)(2y-1)(y-1), \\
H_{15}^{(4)}(x,y) &= -4y(4y-1)(4y-3)(y-1), \\
H_{16}^{(4)}(x,y) &= \frac{16}{3}y(2y-1)(4y-3)(y-1), \\
H_i^{(4)}(x,y) &= 0, \quad i = 3,8,9,10,11,12,13.
\end{aligned} \tag{85b}
$$

From Eq. (85a), we have on using the property of Shape functions

$$
\begin{aligned}
\delta_{ij} + H_i^{(4)}(x_j,y_j) &= x_jy_jA_{5,i}^{(4)} + x_j^2y_jA_{8,i}^{(4)} + x_jy_j^2A_{9,i}^{(4)} + x_j^3y_jA_{12,i}^{(4)} + x_j^2y_j^2A_{13,i}^{(4)} + x_jy_j^3A_{14,i}^{(4)} \\
&\quad + \xi_j^4\eta_jA_{16,i}^{(4)} + \xi_j\eta_j^4A_{17,i}^{(4)}, \quad i,j = 1(1)17.
\end{aligned} \tag{86}
$$

The solution to the above set of linear simultaneous equations (86) is given by

$$
\begin{aligned}
&\left\{A_{mi}^{*(4)}\right\} = \left[X_{jm}^{*(4)}\right]^{-1}\left\{H_i^{(4)}\left(x_j,y_j\right)\right\} \quad \text{and} \\
&\left\{A_{mj}^{*(4)}\right\} = \left[X_{jm}^{*(4)}\right]^{-1},
\end{aligned} \tag{87}
$$

where

$$
\begin{aligned}
&\left\{A_{mk}^{*(4)}\right\} = \left[A_{5,k}^{(4)}\, A_{8,k}^{(4)}\, A_{9,k}^{(4)}\, A_{12,k}^{(4)}\, A_{13,k}^{(4)}\, A_{14,k}^{(4)}\, A_{16,k}^{(4)}\, A_{17,k}^{(4)}\right]^{\mathrm{T}}, \\
&\left[X_m^{*(4)}\right] = \left[xy\; x^2y\; xy^2; x^3y; x^2y^2; xy^3; \xi^4\eta; \xi\eta^4\right], \\
&k = i,j, \\
&i = 1,2,4,5,6,7,14,15,16, \\
&j = 3,8,9,10,11,12,13,17.
\end{aligned} \tag{88}
$$

Clearly for the Quartic element, we have

$$
[X_{jm}^{*(4)}] = \begin{bmatrix}
x_3y_3 & x_3^2y_3 & x_3y_3^2 & x_3^3y_3x_3^2y_3^2 & x_3y_3^3 & \xi_3^4\eta_3 & \xi_3\eta_3^4 \\
x_8y_8 & x_8^2y_8 & x_8y_8^2 & x_8^3y_8x_8^2y_8^2 & x_8y_8^3 & \xi_8^4\eta_8 & \xi_8\eta_8^4 \\
x_9y_9 & x_9^2y_9 & x_9y_9^2 & x_9^3y_9x_9^2y_9^2 & x_9y_9^3 & \xi_9^4\eta_9 & \xi_9\eta_9^4 \\
x_{10}y_{10} & x_{10}^2y_{10} & x_{10}y_{10}^2 & x_{10}^3y_{10}x_{10}^2y_{10}^2 & x_{10}y_{10}^3 & \xi_{10}^4\eta_{10} & \xi_{10}\eta_{10}^4 \\
x_{11}y_{11} & x_{11}^2y_{11} & x_{11}y_{11}^2 & x_{11}^3y_{11}x_{11}^2y_{11}^2 & x_{11}y_{11}^3 & \xi_{11}^4\eta_{11} & \xi_{11}\eta_{11}^4 \\
x_{12}y_{12} & x_{12}^2y_{12} & x_{12}y_{12}^2 & x_{12}^3y_{12}x_{12}^2y_{12}^2 & x_{12}y_{12}^3 & \xi_{12}^4\eta_{12} & \xi_{12}\eta_{12}^4 \\
x_{13}y_{13} & x_{13}^2y_{13} & x_{13}y_{13}^2 & x_{13}^3y_{13}x_{13}^2y_{13}^2 & x_{13}y_{13}^3 & \xi_{13}^4\eta_{13} & \xi_{13}\eta_{13}^4 \\
x_{17}y_{17} & x_{17}^2y_{17} & x_{17}y_{17}^2 & x_{17}^3y_{17}x_{17}^2y_{17}^2 & x_{17}y_{17}^3 & \xi_{17}^4\eta_{17} & \xi_{17}\eta_{17}^4
\end{bmatrix}. \tag{89}
$$

Finally from Eqs. (54), (85a) and (85b) and Fig. 16f

$$T_i^{k,4} = N_i^{*(4)}(x_k, y_k),$$
$$k = 17, 18, 19, 20, 21, 22, 23, 24, 25, \tag{90}$$
$$i = 1(1)17$$

and the revised Shape functions can then be obtained from Eq. (47).

11. Constraint co-efficients $T_i^{k,5}$ $(i = 1(1)24,\ k = 21(1)36)$ for Quintic Complete Lagrange element

Following the procedure outlined in Eqs. (68)–(80), it can be shown that

$$\begin{aligned} N_i^{*(5)}(x,y) = & -H_i^{(5)}(x,y) + xyA_{5,i}^{(5)} + x^2yA_{8,i}^{(5)} + xy^2A_{9,i}^{(5)} + x^3yA_{12,i}^{(5)} + x^2y^2A_{13,i}^{(5)} + xy^3A_{14,i}^{(5)} \\ & + x^4yA_{17,i}^{(5)} + x^3y^2A_{18,i}^{(5)} + x^2y^3A_{19,i}^{(5)} + xy^4A_{20,i}^{(5)} + \xi^5\eta A_{22,i}^{(5)} + \xi^3\eta^3A_{23,i}^{(5)} \\ & + \xi\eta^5A_{24,i}^{(5)}, \quad i = 1(1)24, \end{aligned} \tag{91a}$$

where

$$\begin{aligned} & H_i^{(5)}(x,y) = 0, \quad i = 3, 9(1)16, 21(1)24, \\ & H_1^{(5)}(x,y) = 1 - F_1^{(5)}(x) - G_1^{(5)}(y), \\ & H_i^{(5)}(x,y) = -F_i^{(5)}(x), \quad i = 2, 5, 6, 7, 8, \\ & H_i^{(5)}(x,y) = -G_i^{(5)}(y), \quad i = 4, 17, 18, 19, 20, \\ & F_i^{(5)}(x) = \frac{\Pi^{(5)}(x)}{(x-x_i)\Pi^{(5)^1}(x_i)}, \\ & G_i^{(5)}(x) = \frac{\phi^{(5)}(x)}{(y-y_i)\phi^{(5)^1}(y_i)}, \\ & \Pi^{(5)}(x) = (x-x_1)(x-x_5)(x-x_6)(x-x_7)(x-x_8)(x-x_2), \\ & \Phi^{(5)}(y) = (y-y_1)(y-y_{20})(y-y_{19})(y-y_{18})(y-y_{17})(y-y_4), \\ & x_1 = y_1 = 0, \quad x_5 = y_{20} = \frac{1}{5}, \quad x_6 = y_{19} = \frac{2}{5}, \\ & x_7 = y_{18} = \frac{3}{5}, \quad x_8 = y_{17} = \frac{4}{5}, \quad x_2 = y_4 = 1. \end{aligned} \tag{91b}$$

From Eq. (91a), we have onusing the property of Shape functions

$$\begin{aligned} \delta_{ij} + H_i^{(5)}(x_j,y_j) = & \ x_jy_jA_{5,i}^{(5)} + x_j^2y_jA_{8,i}^{(5)} + x_jy_j^2A_{9,i}^{(5)} + x_j^3y_jA_{12,i}^{(5)} + x_j^2y_j^2A_{13,i}^{(5)} + x_jy_j^3A_{14,i}^{(5)} \\ & + x_j^4y_jA_{17,i}^{(5)} + x_j^3y_j^2A_{18,i}^{(5)} + x_j^2y_j^3A_{19,i}^{(5)} + x_jy_j^4A_{20,i}^{(5)} + \xi_j^5\eta_jA_{22,i}^{(5)} + \xi_j^3\eta_j^3A_{23,i}^{(5)} \\ & + \xi_jn_j^5A_{24,i}^{(5)} \quad i,j = 1(1)24. \end{aligned} \tag{92}$$

The solution to the above set of linear simultaneous equations (92) is given by

$$\begin{aligned} & \{A_{mi}^{*(5)}\} = [X_{im}^{*(5)}]^{-1}\{H_i^{(5)}(x_j,y_j)\} \quad \text{and} \\ & \{A_{mj}^{*(5)}\} = [X_{jm}^{*(5)}]^{-1}, \end{aligned} \tag{93}$$

where

$$\{A_{mk}^{*(5)}\} = [A_{5,k}^{(5)} \ A_{8,k}^{(5)} \ A_{9,k}^{(5)} \ A_{12,k}^{(5)} \ A_{13,k}^{(5)} \ A_{14,k}^{(5)} \ A_{17,k}^{(5)} \ A_{18,k}^{(5)} \ A_{19,k}^{(5)} \ A_{20,k}^{(5)} \ A_{22,k}^{(5)} \ A_{23,k}^{(5)} \ A_{24,k}^{(5)}]^{\mathrm{T}},$$
$$\left[X_m^{*(5)}\right] = [xy \ \ x^2y \ \ xy^2 \ \ x^3y \ \ x^2y^2 \ \ xy^3 \ \ x^4y \ \ x^3y^2 \ \ x^2y^3 \ \ xy^4 \ \ \xi^5\eta \ \ \xi^3\eta^3 \ \ \xi\eta^5],$$

$$k = i, j, \qquad i = 1, 2, 4, 5, 6, 7, 8, 17, 18, 19, 20, \qquad j = 3, 9, 10, 11, 12, 13, 14, 15, 16, 21, 22, 23, 24. \tag{94}$$

Finally from Eqs. (54), (91a) and (91b) and Fig. 17f

$$T_i^{k,5} = N_i^{*(5)}(x_k, y_k), \qquad k = 21(1)36, \qquad i = 1(1)24 \tag{95}$$

and the revised Shape functions can then be obtained from Eq. (47).

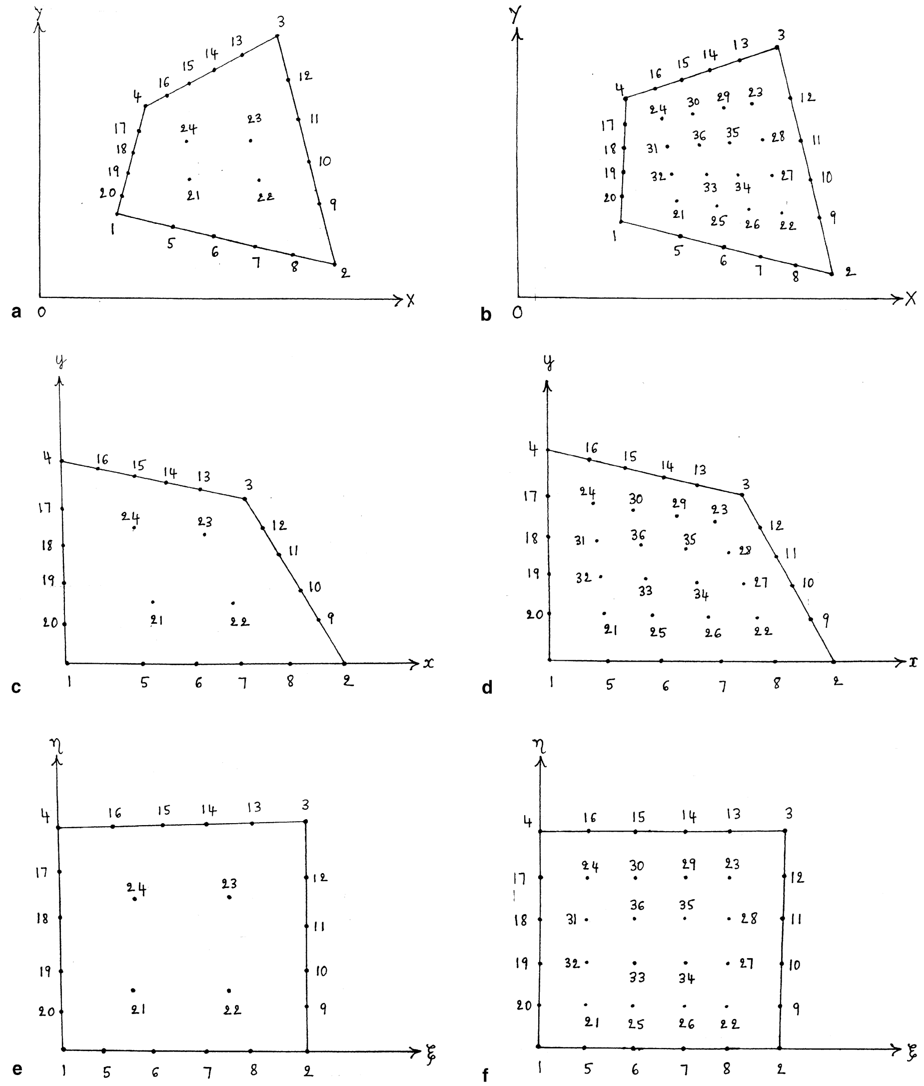

Fig. 17. Transformation of a Quintic Quadrilateral element: (a–b) Quadrilateral in global (X, Y) space as Serendipity and Lagrange elements, (c–d) Quadrilateral in local (x, y) space as Serendipity and Lagrange elements, (e–f) Mapped 1-square $0 \leqslant \xi, \eta \leqslant 1$, in local ξ, η, space as Serendipity and Lagrange elements.

We observe that the constraint co-efficients for higher-order (Sextic- to Tenth-order) Complete Lagrange elements can be similarly determined. We also note that the procedure outlined above requires the inversion of a matrix of smaller size than that required in the ordinary procedure.

12. Conclusions

A general formula is derived for the Shape functions of General Serendipity family for the 2-cube with arbitrarily placed nodes along each side. It is then shown that the interpolation functions for the General Complete Lagrange family are derivable through the Bases transformation without matrix inversion.

Explicit expressions for interpolation functions of the Serendipity family and the Complete Lagrange family elements which allow uniform spacing of nodes over the domain of the element $-1 \leqslant \xi, \eta \leqslant 1$ from Fourth- to Tenth-orders are obtained for the first time.

Recent successes of adaptive Finite Element procedures [10–13] justify the use of higher-order approximations with harmonious combination of lower order elements.

The native Lagrange family [8] may produce a variety of higher-order and any trasitive elements combining the different order Finite Elements conformably. However from the viewpoint of polynomial completeness, Serendipity elements are undoubtedly preferable.

The Complete Lagrange family presented here realises polynomial completeness with necessary minimum nodes without destroying the nodal symmetry. We have not pursued the derivation of explicit interpolation functions for the mixed Lagrange family which neglects nodal symmetry of interior nodes as regularity of nodal placement may be important in practical Finite Element applications. Needless to say these reconstructed interpolation functions produce a wide range of possibilities with reference to improved isoparametric transformation for Finite Element Analysis [14].

We have also derived modified Shape functions of pth-order Complete Lagrange elements ($p = 2(1)10$) in such a way that pth-order displacement states are correctly interpolated for bilinear Element Geometry which refers to angular distortions.

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