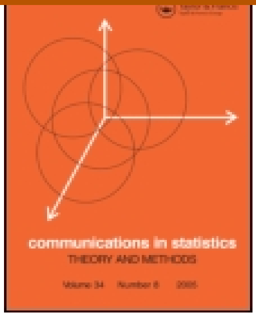

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Estimation of Measures in M/M/1 Queue

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Maximum likelihood and uniform minimum variance unbiased estimators of steady-state probability distribution of system size, probability of at least ℓ customers in the system in steady state, and certain steady-state measures of effectiveness in the M/M/1 queue are obtained/derived based on observations on X, the number of customer arrivals during a service time. The estimators are compared using Asympotic Expected Deficiency (AED) criterion leading to recommendation of uniform minimum variance unbiased estimators over maximum likelihood estimators for some measures.

Keywords Asymptotic expected deficiency; Maximum likelihood estimator; Nonlinear program; Queues; Uniform minimum variance unbiased estimator.

1. Introduction

Queueing theory has attracted researchers from various disciplines, due to many interesting probabilistic, operational, and statistical problems that naturally arise in the study of queueing systems. A problem that gains importance due to practical application of queues is that of statistical inference for queueing parameters and related parametric functions. A state-of-the-art review of this and associated areas is due to Bhat et al. (1997). The literature on statistical inference in the post review period includes Conti (1998), Armero and Conesa (1998), Armero and Bayarri (1999), Sharma and Kumar (1999), Zheng and Seila (2000), Armero and Conesa (2000), Butler and Huzurbazaar (2000), Huang and Brill (2001), Ausin et al. (2005), Ramirez et al. (2008a,b), Choudhury and Borthakur (2008), and Kiessler and Lund (2009). However, the problem of Maximum Likelihood (ML) and Uniform Minimum Variance Unbiased (UMVU) estimation of traffic intensity, system size probabilities and measures of effectiveness in steady state, based on a random sample of fixed size n on number of customer arrivals during the service time of a customer, in $M/M/1$ queue has not been considered. A necessary and sufficient

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condition for a $M/M/1$ queue to be in steady state is that the traffic intensity, denoted by ρ, must be restricted to the interval [0, 1). This article uses this fact to solve the problem with the objective of classical Maximum Likelihood and Uniform Minimum Variance Unbiased estimation of: (i) traffic intensity in steady state; (ii) Steady state probabilities of system size; and (iii) measures of effectiveness in steady state, in the $M/M/1$ queue.

The outline of this article is as follows. The problem of ML estimation (MLE) of the traffic intensity in the steady state is solved as a constrained optimization problem in Sec. 2. In Sec. 3, ML estimators of the steady state measures mentioned in (ii) and (iii) above are obtained. Furthermore, best unbiased estimators of all the above-mentioned measures are obtained by the application of Lehmann–Scheffe theorem in Sec. 4. In particular, UMVU estimator of steady state probability of system size greater than l is derived in Sec. 4.1 and this is used to obtain UMVU estimators of steady state system size probabilities in Sec. 4.2. Also, UMVU estimators of steady state measures of effectiveness are derived in Sec. 4.3. A comparison of ML and UMVU estimators of various measures is discussed in Sec. 5 leading to a recommendation to the queueing analyst on choice of one of the classical estimators. Since we are interested in only steady-state measures we will suppress the usage of "steady state" in the sequel.

2. Constrained Optimization for MLE of Traffic Intensity

In the imbedded Markov chain analysis of the M/G/1 queue, $X_1, X_2, \ldots, X_n, \ldots$ are i.i.d. random variables, where X_n denotes the number of customer arrivals during the service time of n-th customer. For the $\mathrm{M/E}_r/1$ queue, the probability mass function (pmf) of X is given by

$$p(x; \rho, r) = \binom{x+r-1}{x}\left(\frac{\rho}{\rho+r}\right)^x\left(\frac{r}{\rho+r}\right)^r, x = 0, 1, 2, \ldots, \tag{2.1}$$

the negative binomial distribution with parameters r and ρ. Notice that when $r = 1$, X is a geometric random variable with the following geometric pmf for X in M/M/1 queue:

$$p(x; \rho) = \left(\frac{\rho}{\rho+1}\right)^x\left(\frac{1}{\rho+1}\right), \quad x = 0, 1, 2, \ldots. \tag{2.2}$$

The pmf in (2.1) and its reparametrized versions have received importance in the biological sciences due to its many applications as evidenced in Saha and Paul (2005) and related articles. In the queueing context, Harishchandra and Subba Rao (1988) studied statistical inference aspects related to ρ in the $\mathrm{M/E}_k/1$ queue based on a random sample on X with pmf (2.1). They showed that the ML estimator of ρ, $\hat{\rho}$, in the $\mathrm{M/E}_k/1$ queue, assuming k to be known, is the sample mean estimator and thus is the ML estimator of ρ in the M/M/1 queue as well. However, they did not derive the ML estimator of the traffic intensity in the $M/M/1$ queue under steady state conditions. This is required to obtain ML estimators of measures in the next section and thus ML estimator of ρ is derived by solving a constrained optimization problem. This is based on a random sample $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ on x with pmf (2.2) and this is true of all estimation procedures in this article.

The problem of ML estimation of ρ in the $M|M|1$ queue in steady state is the problem of maximizing the likelihood function (or equivalently log-likelihood function), subject to the constraint that $0 \leq \rho < 1$, with respect to ρ. This optimization problem is:

$$\max_{\rho} L(\rho; \tilde{\mathbf{x}}) = t \log \rho - (n+t)\log(1+\rho)$$
$$\text{s.t} \quad \rho \leq 1^-$$
$$\rho \geq 0,$$

where 1^- is a value of ρ close to 1 but less than one and is based on known outside information about the queueing system and $t = \sum_{i=1}^{n} x_i$. Clearly, it is a Nonlinear Program (NLP) in a single variable ρ. Also,

$$\frac{\partial L}{\partial \rho} = \frac{t}{\rho} - \frac{n+t}{1+\rho} \quad \text{and} \quad \frac{\partial^2 L}{\partial \rho^2} = -\frac{t}{\rho^2} + \frac{n+t}{(1+\rho)^2} < 0 \quad \forall \rho.$$

Thus, $L(\rho; \mathbf{x})$ is a strictly concave function of ρ. As the left-hand side of linear constraint is such that $\frac{\partial^2 \rho}{\partial \rho^2} = 0$ this function is both convex and concave. Hence the optimal solution $\hat{\rho}^*$ to this NLP is provided by the following Kuhn–Tucker (see Hillier and Lieberman, 1974) conditions:

1. $\dfrac{t}{\hat{\rho}^*} - \dfrac{n+t}{1+\hat{\rho}^*} - u \leq 0$
2. $\hat{\rho}^* \left[\dfrac{t}{\hat{\rho}^*} - \dfrac{n+t}{1+\hat{\rho}^*} - u \right] = 0$
3. $\hat{\rho}^* - 1^- \leq 0$
4. $u[\hat{\rho}^* - 1^-] = 0$
5. $u \geq 0$
6. $\hat{\rho}^* \geq 0$

Here, u is a real number. The optimal solution is given by

$$\hat{\rho}^* = \begin{cases} \overline{x}_n, & \overline{x}_n < 1^- \\ 1^-, & \overline{x}_n \geq 1^- \end{cases}. \tag{2.3}$$

That is the MLE of ρ is the classical sample mean estimate if its value is less than or equal to 1^-, otherwise, meaning that the value is greater than 1^-, the ML estimate is 1^-. We shall use $\hat{\rho}$ for $\hat{\rho}^*$ in the next section for notational convenience.

3. ML Estimation of Measures

The ML estimator of the steady state probability distribution and certain measures of effectiveness in the M/M/1 queue can be obtained using the fact that $\hat{\rho}$ is given by (2.3) in such a queue and by the application of the invariance property of ML estimators (see Kale, 1999).

The steady-state probability distribution of system size in M/M/1 queue is given by

$$p_k = (1-\rho)\rho^k, \quad k = 0, 1, 2, \ldots \tag{3.1}$$

and its ML estimator, $\hat{p}_k$, is obtained by plugging in the ML estimator of ρ for ρ. Thus,

$$\hat{p}_k = (1 - \hat{\rho})(\hat{\rho})^k, \quad k = 0, 1, 2, \ldots. \tag{3.2}$$

The steady-state measures of effectiveness in M/M/1 queue, namely: (i) expected number in the non empty system (L_q'); (ii) expected number in the system (L); (iii) expected number in the queue (L_q) are, following Gross and Harris (1985), given by

$$L_q', L, L_q = \frac{\rho^{\vartheta}}{1-\rho}, \quad \vartheta = 0, 1, 2 \tag{3.3}$$

and their ML estimators are obtained by substituting $\hat{\rho}$ for ρ in (3.3), which is the application of invariance property of ML estimators.

4. Best Unbiased Estimation

We first obtain the UMVU estimator of $\rho^l (l > 0)$ in the M/M/1 queue, which is the steady-state probability of having l or more customers in the system, to facilitate UMVU estimation of steady state probability distribution of system size in Sec. 4.2. In Sec. 4.3, UMVU estimators of certain measures of effectiveness in the M/M/1 queue are derived. For UMVU estimation we work with a reparametrized geometric distribution.

4.1. *Estimation of ρ^l*

The UMVU estimator of ρ^l, $l \in \{1, 2, \ldots\}$, denoted by T_l, is derived using Lehmann–Scheffe theorem. For this, we reparametrize the geometric pmf in (2.2) with $\theta = \frac{\rho}{1+\rho}$ implying $\rho = \frac{\theta}{1-\theta}$. This leads from the parameter space $\Theta = \{\rho : 0 < \rho < 1\}$ to $\Theta = \{\theta : 0 < \theta < \frac{1}{2}\}$. The geometric pmf is

$$p(x; \theta) = (1 - \theta)\theta^x, \quad x = 0, 1, 2, \ldots, \tag{4.1}$$

which belongs to the power series family of distributions and hence to Koopman-Darmois family. The complete sufficient statistic for this family is $T = \sum_{i=1}^{n} X_i$. The UMVU estimator of ρ^l is, by the application of Lehmann–Scheffe theorem, any function $\phi(T)$ that is an unbiased estimator of ρ^l. Mathematically,

$$E[\phi(T)] = \rho^l$$

which, in terms of θ, is

$$E[\phi(T)] = \left(\frac{\theta}{1-\theta}\right)^l$$

and is equivalent to

$$\sum_{t=0}^{\infty} \phi(t) \binom{t+n-1}{t} \theta^t = \theta^l + (n+l)\theta^{l+1} + (n+l)(n+l+1)\frac{\theta^{l+2}}{2!} + \cdots,$$

where we have used the constraint on θ and thus of ρ. The power series on left-hand side of above equation and right-hand side of above equation are equal if the coefficients of θ^t, $t = 0, 1, 2, \ldots$ are equal. This implies

$$\phi(t)\binom{t+n-1}{t} = \begin{cases} 0, & t < l \\ 1, & t = l \\ \dfrac{(n+t-1)!}{(n+l-1)!(t-l)!}, & t > l \end{cases}$$

which leads to

$$T_l = \phi(t) = \begin{cases} 0, & t < l \\ \binom{t+n-1}{t}^{-1}, & t = l \\ \dfrac{t!\,(n-1)!}{(n+l-1)!\,(t-l)!}, & t > l, \end{cases} \tag{4.2}$$

the UMVU estimator of ρ^l. Clearly, the UMVU estimator of ρ is obtained when $l = 1$ and is the sample mean estimator, as it should be, for ρ is the mean of the pmf in (2.2).

4.2. *Estimation of System Size Distribution*

The UMVU estimator of the steady-state probability distribution of system size in the M/M/1 queue is obtained by a direct application of the well known theorem (see Patel et al., 1976) which states that if $T_1, T_2, \ldots, T_k$ are UMVU estimators of $g_1(\theta), g_2(\theta), \ldots, g_k(\theta)$, respectively, then $\sum_{i=1}^k c_i T_i$ is the UMVU estimator of $\sum_{i=1}^k c_i g_i(\theta)$ for any constants $c_i, i = 1, 2, \ldots, k$. Thus, the UMVU estimator of p_k, $\tilde{p}_k$, is given by

$$\tilde{p}_k = T_k - T_{k+1},$$

where p_k is given by (3.1) and T_ℓ is UMVU estimator of ρ^ℓ given in (4.2).

4.3. *Estimation of Measures of Effectiveness*

We now derive the UMVU estimators of measures of effectiveness given by (3.3). For this we again work with the reparametrized geometric distribution in (4.1) with constrained parameter space $\Theta = \left\{\theta : 0 < \theta < \frac{1}{2}\right\}$. Recalling that $\sum_{i=1}^n X_i$ is a complete sufficient statistic, the UMVU estimator of L_q' is derived by an application of the Lehmann–Scheffe theorem. The measure L_q' in (3.3) in terms of θ is $\frac{1-\theta}{1-2\theta}$, $0 < \theta < \frac{1}{2}$. Thus, by Lehmann–Scheffe theorem, $\phi(T)$ is UMVU estimator of L_q' if

$$E_\theta\{\phi(T)\} = \frac{1-\theta}{1-2\theta}, \quad 0 < \theta < \frac{1}{2}$$

which is equivalent to

$$\sum_{t=0}^{\infty} \phi(t) \binom{n+t-1}{t} \theta^t (1-\theta)^n = \frac{1-\theta}{1-2\theta}, \quad 0 < \theta < \frac{1}{2}$$

and this can be rewritten as

$$\sum_{t=0}^{\infty} \left\{ \phi(t) \binom{n+t-1}{t} \right\} \theta^t = \frac{1-\theta}{1-2\theta} \cdot \frac{1}{(1-\theta)^n}. \tag{4.3}$$

Expanding the right-hand side of (4.3) and collecting the coefficients of θ^j, $j = 0, 1, 2, \ldots$ on right-hand side of (4.3) gives

$$\text{coefficient of } \theta^t = \sum_{r=0}^{t} 2^{t-r} \binom{n+r-2}{r},$$

where r is just a running subscript and not the parameter. Thus (4.3) is

$$\sum_{t=0}^{\infty} \left\{ \binom{n+t-1}{t} \phi(t) \right\} \theta^t = \sum_{t=0}^{\infty} \left\{ \sum_{r=0}^{t} \binom{n+r-2}{r} 2^{t-r} \right\} \theta^t.$$

The power series on left-hand side and right-hand side are equal if coefficient of θ^t for every value of t are equal. Hence,

$$\phi(t) = \frac{\sum_{r=0}^{t} \binom{n+r-2}{r} 2^{t-r}}{\binom{n+t-1}{t}}, \quad n \geq 2$$

is the UMVU estimator of L'_q.

We now turn to UMVU estimation of L, which in terms of parameter θ is $\theta(1-2\theta)^{-1}, 0 < \theta < \frac{1}{2}$. By Lehmann–Scheffé theorem, $\phi_1(T)$ is UMVU estimator of L if

$$E\{\phi_1(T)\} = \theta(1-2\theta)^{-1}, \quad 0 < \theta < \frac{1}{2}$$

That is,

$$\sum_{t=0}^{\infty} \left\{ \phi_1(t) \binom{n+t-1}{t} \right\} \theta^t = \theta(1-2\theta)^{-1}(1-\theta)^{-n}$$

and thus

$$\sum_{t=0}^{\infty} \left\{ \phi_1(t) \binom{n+t-1}{t} \right\} \theta^t = \sum_{t=1}^{\infty} \sum_{r=1}^{t} 2^{t-r} \binom{n+r-2}{r-1} \theta^t.$$

Equating the coefficients of θ^t, $t = 0, 1, 2, \ldots$ on both sides we get

$$\phi_1(t) = \begin{cases} 0, & t = 0 \\ \dfrac{\sum_{r=1}^{n} 2^{t-r} \binom{n+r-2}{r-1}}{\binom{n+t-1}{t}}, & t = 1, 2, \ldots. \end{cases}$$

the UMVU estimator of L.

The UMVU estimator of $L_q = \theta^2(1-\theta)^{-1}(1-2\theta)^{-1}$ is $\phi_2(T)$ if it satisfies

$$E_\theta\{\phi_2(T)\} = \theta^2(1-\theta)^{-1}(1-2\theta), 0 < \theta < \frac{1}{2},$$

which is equivalent to

$$\sum_{t=0}^{\infty}\left\{\phi_2(t)\binom{n+t-1}{t}\right\}\theta^t = \sum_{t=2}^{\infty}\sum_{r=2}^{t}2^{t-r}\binom{n+r-2}{r-2}$$

and thus leading to

$$\phi_2(t) = \begin{cases} 0, & t = 0, 1 \\ \dfrac{\sum_{r=2}^{t} 2^{t-r}\binom{n+r-2}{r-2}}{\binom{n+t-1}{t}}, & t = 2, 3, \ldots \end{cases}$$

the UMVU estimator of L_q. The UMVU estimator of L_q can also be obtained using the UMVU estimators of L and ρ and the application of the theorem in Sec. 4.2. Thus, $\widetilde{L}_q = \widetilde{L} + \tilde{\rho}$, where $\sim$ indicates the UMVU estimator of the corresponding measure.

The results of this section can also be derived working in the framework of modified power series distribution following Gupta (1974, 1977) and Gupta (1982).

5. Comparison of ML and UMVU Estimators

Whenever two or more estimators of a parametric function are proposed, it is natural to compare and contrast them. Many different criteria for such a comparison have been suggested. They are Mean Square Error (MSE), Pitman nearness, Asymptotic Relative Efficency (ARE), second-order efficiency, and Asymptotic Expected Deficiency (AED), to name a few. We use the AED citerion for comparison of ML and UMVU estimators proposed in Secs. 2 and 4 in the case of ML estimates being naturally less than one. A reason for such a choice is the "ease of computation".

5.1. *AED*

Let $T_1(X_1, X_2, \ldots, X_n)$ and $T_2(X_1, X_2, \ldots, X_n)$ be two estimators of $\varphi(\theta)$. Let the measure of performance of T_i be taken as the expected squared error, denoted by $V_n(T_i), i = 1, 2$. Hodges and Lehmann (1970) showed that there exists a unique k_n such that $V_{k_n}(T_2) = V_n(T_1)$ and $\frac{K_n}{n} \to 1$ as $n \to \infty$. This led Hodges and Lehmann to define the AED of T_2 relative to T_1 as

$$\text{AED}(T_2, T_1) = \lim_{n\to\infty}(k_n - n),$$

provided the limit exists. Hwang and Hu (1990) derived the expression for AED without any constraint on the variance function of ML estimator of $\varphi(\theta)$ relative to the UMVU estimator of $\varphi(\theta)$, under certain regularity conditions, in the one parameter exponential family with probability distribution given by

$$f(x;\theta) = \exp\{\Phi_1(\theta)T(x) + \Phi_2(\theta) + d(x)\}, \quad x \in S, \ \theta \in \Theta \tag{5.1}$$

Table 1
AED computation for comparison

Estimable function	Range of ρ	Relatively better estimator
$p_k, k = 1$	$\rho < 0.41$	ML
	$\rho > 0.41$	UMVU
$p_k, k = 2$	$\rho < 0.19$ or $\rho > 0.62$	UMVU
	$0.19 \le \rho \le 0.62$	ML
$p_k, k = 3$	$\rho < 0.36$ or $\rho > 0.72$	UMVU
	$0.36 \le \rho \le 0.72$	ML
$p_k, k = 4$	$\rho < 0.48$ or $\rho > 0.77$	UMVU
	$0.48 \le \rho \le 0.77$	ML
$p_k, k = 5$	$\rho < 0.56$ or $\rho > 0.81$	UMVU
	$0.56 \le \rho \le 0.81$	ML
$p_k, k = 6$	$\rho < 0.62$ or $\rho > 0.84$	UMVU
	$0.62 \le \rho \le 0.84$	ML
L, L'_q	$\rho < 1$	UMVU
L_q	$\rho < 0.235$	UMVU
	$\rho \ge 0.235$	ML

Note: Range of ρ is approximate.

with respect to a fixed σ-finite measure μ, where S is a set of real numbers and Θ is the parameter space. Let $\mathbf{Z} = \sum_{i=1}^{n} \frac{T(X_i)}{n}$. The result of Hwang and Hu (1990) is reproduced below as Theorem 5.1.

Theorem 5.1. *Under certain regularity conditions, the AED of ML estimator $\varphi(\mathbf{Z})$ of $\varphi(\theta)$ relative to the UMVU estimator $U(\mathbf{Z})$ for the exponential family* (5.1) *is given by*

$$AED\ g(\mathbf{Z}), U(\mathbf{Z})) = V(\theta)\left\{\frac{\varphi'''(\theta)}{\varphi'(\theta)} + \frac{1}{4}\left(\frac{\varphi''(\theta)}{\varphi'(\theta)}\right)^2\right\} + V'(\theta)\frac{\varphi''(\theta)}{\varphi'(\theta)},$$

where $V'(\theta) = \{\Phi_1'(\theta)\}^{-1}$.

5.2. *Comparison of Estimators*

We use Theorem 5.1 to compute the AED of ML estimator relative to UMVU estimator, of p_k for $k = 1, 2, \ldots, 6$ for different values of ρ. Also, the AED of ML estimators, of measures, relative to their corresponding UMVU estimators are computed. The results are summarized in Table 1.

6. Recommendation and Related Work

The nature of observations required involves less book keeping in relation to recording of actual interarrival and service times. The use of UMVU estimator in relation to ML estimator is recommended for estimation of L and L'_q. This should aid the queueing analyst in choosing one of the classical estimators. However such a categorical recommendation of an estimator of system size distribution is not possible as is evident from Table 1. But the UMVU estimator can be recommended for small and large values of traffic intensity.

Classical Estimation in $M/D/1$ and $M/E_r/1$ queues and Bayesian estimation in $M/M/1$, $M/E_r/1$, and $M/D/1$ queues are studied and will be part of different communications.

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