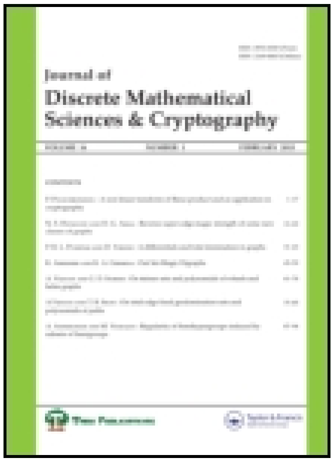

Journal of Discrete Mathematical Sciences and Cryptography
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tdmc20

Discrete quaternion Fourier transform in signal processing systems

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Published online: 03 Jun 2013.

To cite this article: H. Chandrashekhar & M. Nagaraj (2000) Discrete quaternion Fourier transform in signal processing systems, Journal of Discrete Mathematical Sciences and Cryptography, 3:1-3, 95-112, DOI: 10.1080/09720529.2000.10697900

To link to this article: http://dx.doi.org/10.1080/09720529.2000.10697900

Discrete quaternion Fourier transform in signal processing systems

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ABSTRACT

We define n^{th} root of unity in quaternion space and then we define discrete quaternion Fourier transform. We use first order quaternion filter for implementing fourth order real co-efficient filter.

0. INTRODUCTION

The concept of signals and systems arise in an extremely wide variety of fields, and the ideas and techniques associated with these concepts play an important role in such diverse areas of science and technology as communications, aeronautics and astronautics, circuit design, acoustics, seismology, biomedical engineering, energy generation and distribution systems, chemical process control and speech processing. Although the physical nature of the signals and systems that arise in these various disciplines may be drastically different, they all have two very basic features in common. The signals are functions of one or more independent variables and typically contain information about the behaviour or nature of some phenomenon, where as the systems respond to particular signals by producing other signals. Voltages and currents as a function of time in an electrical circuit are examples of signals, and a circuit is itself an example of a system which in this case responds to applied voltages and currents. A computer programme for the automated diagnosis of electrocardiograms can be viewed as a system which has as its input a digitized electrocardiogram and which produces estimates of parameters such as

Journal of Discrete Mathematical Sciences & Cryptography
Vol. 3 (2000), Nos. 1-3, pp. 95-112

heart rate as outputs. A camera is a system that receives light from different sources and reflected from objects and produces a photograph.

Fourier transforms are alternative representations for signals. These representations can be used to construct broad and useful classes of signals. The response of an LTI system to each basic signal should be simple enough in structure to provide us with a convenient representation for the response of the system to any signal constructed as a linear combination of these basic signals.

We shall extend discrete Fourier transforms using quaternion valued functions which may find application in analyzing the signals particularly received from space. Also they seem to be useful in implementing the fourth order digital filters with real coefficients.

1. THE n^{th} ROOTS OF UNITY LYING ON THE UNIT 3-SPHERE

1.1 *Introduction*

The discrete Fourier transform over the complex plane maps a complex function f defined on the integers mod n to another such function Tf defined by

$$(Tf)(k) = \sum_{j=0}^{n-1} \omega^{kj} f(j) \tag{1.1.1}$$

where ω is the n^{th} root of unity having the property:

$$\sum_{k=0}^{n-1} \omega^{k} = 0 \qquad \text{when} \quad \omega \neq 1$$

$$= n \qquad \text{when} \quad \omega = 1.$$

These things lead us to develop discrete quaternion Fourier transform. We first define the n^{th} roots of unity over the quaternion domain.

It is well-known that usual powers of a quaternion $\underset{\sim}{q}$ are not regular (see Deavours [1] and Sudbery [7]). M. Nagaraj and B. S. Suresh [5] have investigated these aspects by means of polar co-ordinates.

They have defined the formal powers $\underset{\sim}{q}^{n}$ of a quaternion $\underset{\sim}{q}$ and discovered an elegant expression for $\underset{\sim}{q}^{n}$ analogous to z^{n} in polar coordinates, where z is a complex number, and have obtained an analogue of the De Moivre's theorem of the complex domain. We shall state here the definition and the theorem regarding the formal powers of a quaternion.

Let

$$\underset{\sim}{q} = q^4 + \hat{u}r$$

$$= q^4 + q^1 i + q^2 j + q^3 k$$

be a quaternion variable where $\hat{u}$ is the pure unit quaternion part of $\underset{\sim}{q}$ given by

$$\hat{u} = i \sin\theta \cos\phi + j \sin\theta \sin\phi + k \cos\theta.$$

Definition 1.1.1. The n^{th} **formal power** of $\underset{\sim}{q}$ is defined by

$$\underset{\sim}{q}^n = \underset{\sim}{q} \cdot \underset{\sim}{q} \ldots \underset{\sim}{q} \to n \textit{ times}$$

that is, $\underset{\sim}{q}^n$ is the quaternion product of $\underset{\sim}{q}$ with itself n times, where n is a positive integer, and define $\underset{\sim}{q}^0 = 1$.

THEOREM (1.1.2). *If the polar representation of a quaternion*

$$\underset{\sim}{q} = q^4 + \hat{u}r$$

is $\quad q^4 = \rho\cos\chi$ and $r = \rho\sin\chi$

then the n^{th} *formal power of* $\underset{\sim}{q}$ *is*

$$\underset{\sim}{q}^n = \rho^n(\cos n\chi + \hat{u}\sin n\chi), \quad n \in Z. \tag{1.1.2}$$

1.2 *The Pure Unit Quaternion* $\hat{u}_{ab}$

We shall require certain pure unit quaternions in order to define discrete quaternion Fourier transform (DQFT).

Let $\underset{\sim}{q} = q^4 + \hat{u}r$

with $\quad q^4 = \rho\cos\chi$ and $r = \rho\sin\chi$

be a quaternion variable. Also we have

$$\hat{u} = i \sin\theta \cos\phi + j \sin\theta \sin\phi + k \cos\theta.$$

Since the pure unit quaternion $\hat{u}$ depends on θ and ϕ, it is not constant. Therefore, for fixed values of θ and ϕ we define the pure unit quaternion.

Definition 1.2.1. Let θ_a and ϕ_b be fixed values of θ varying from 0 to π and ϕ varying from 0 to 2π respectively. Then the pure unit quaternion denoted by $\hat{u}_{ab}$ is defined by

$$\hat{u}_{ab} = i \sin\theta_a \cos\phi_b + j \sin\theta_a \sin\phi_b + k\cos\theta_a .$$

We shall now see the condition for the product of two pure unit quaternious $\hat{u}_{ab}$ and $\hat{u}_{cd}$ to be another pure unit quaternion.

1.2.2 *The condition for the product $\hat{u}_{ab}$ and $\hat{u}_{cd}$ to be another pure unit quaternion*

Consider

$$\hat{u}_{ab} = i \sin\theta_a \cos\phi_b + j \sin\theta_a \sin\phi_b + k\cos\theta_a \tag{1.2.1}$$

and

$$\hat{u}_{cd} = i \sin\theta_c \cos\phi_d + j \sin\theta_c \sin\phi_d + k\cos\theta_c . \tag{1.2.2}$$

To obtain, the product $\hat{u}_{ab}\cdot\hat{u}_{cd}$ as another pure unit quaternion $\hat{u}_{ef}$, we equate the real part of the product to zero which gives the desired condition.

That is, $\mathrm{Re}(\hat{u}_{ab}\,\hat{u}_{cd}) = 0$ implies

$$\sin\theta_a \sin\theta_c \cos(\phi_b - \phi_d) + \cos\theta_a \cos\theta_c = 0. \tag{1.2.3}$$

Case i): Suppose $\phi_b = \phi_d$. (1.2.4)

The equation (1.2.3) becomes

$$\cos(\theta_a - \theta_c) = 0. \tag{1.2.5}$$

Therefore,

$$\theta_a - \theta_c = \frac{\pi}{2} \cdot \tag{1.2.6}$$

Case ii): Suppose $\phi_b - \phi_d = \frac{\pi}{2}\cdot$ (1.2.7)

The equation (1.2.3) becomes

$$\cos\theta_a \cos\theta_c = 0 \tag{1.2.8}$$

which implies

$$\theta_a = \frac{\pi}{2} \quad \text{or} \quad \theta_c = \frac{\pi}{2}\cdot \tag{1.2.9}$$

Therefore, the conditions are either

$$\phi_b = \phi_d \qquad \text{and} \qquad \theta_a - \theta_c = \frac{\pi}{2}$$

or

$$\phi_b - \phi_d = \frac{\pi}{2} \qquad \text{and} \qquad \theta_a = \frac{\pi}{2} \text{ or } \theta_c = \frac{\pi}{2}\cdot$$

Example 1.2.3. Let

$$\theta_a = \pi, \quad \theta_c = \frac{\pi}{2} \text{ and } \phi_b = \phi_d = \frac{\pi}{4}\cdot$$

Then

$$\hat{u}_{ab} = i \sin \pi \cos \frac{\pi}{4} + j \sin \pi \sin \frac{\pi}{4} + k \cos \pi$$

$$= -k$$

$$\hat{u}_{cd} = i \sin \frac{\pi}{2} \cos \frac{\pi}{4} + j \sin \frac{\pi}{2} \sin \frac{\pi}{4} + k \cos \frac{\pi}{2}$$

$$= \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$$

$$\hat{u}_{ab}\, \hat{u}_{cd} = -k\left(\frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j\right)$$

$$= -\frac{1}{\sqrt{2}} j + \frac{1}{\sqrt{2}} i$$

$$= \frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j$$

$$= \hat{u}_{ef}$$

which is the pure unit quaternion.

1.3. *The Primitive n^{th} Root of Unity*

We know that, any complex number z with $z^n = 1$ for some positive integer n is the n^{th} root of unity in complex plane. The n^{th} roots of unity are the numbers

$$\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$$

for $k = 0, 1, 2, \ldots, n-1$. There are n, n^{th} roots of unity, and they are equally spaced around the unit circle in the complex plane. A primitive n^{th} root of unity is an n^{th} root of unity which is not a root of unity of a lower order than n. In other words, the primitive n^{th} root of unity is a power z^k with $z^k \neq 1$ where k and n are coprime and $0 < k < n$. All complex numbers of absolute value 1, that is, those on the unit circle, constitute a commutative group under multiplication. When $n > 0$ is an integer, all z such that $z^n = 1$, the n^{th} roots of unity, constitute a subgroup of order n. Generators of this group are the primitive n^{th} roots of unity.

Now we study the existence of n^{th} roots of unity on the unit 3-sphere.

We define the n^{th} root of unity on the unit 3-sphere as follows.

Definition 1.3.1. Let $\underset{\sim}{q}$ be any quaternion number with $\underset{\sim}{q}^n = 1$ for some positive integer n. Then $\underset{\sim}{q}$ is called the n^{th} root of unity and is expressed in the form

$$\underset{\sim}{q} = \cos\frac{2k\pi}{n} + \hat{u}_{ab}\sin\frac{2k\pi}{n} \tag{1.3.1}$$

for $k = 0, 1, 2, \ldots, n-1$, where $\hat{u}_{ab}$ is the pure unit quaternion for fixed values of θ_a and ϕ_b.

For each pair of θ and ϕ we have n, n^{th} roots of unity. We suppose, there are l values of θ and m values of ϕ:

$$\left.\begin{array}{l} 0 \leq \theta_1, \theta_2, \ldots, \theta_l < \pi \\ 0 \leq \phi_1, \phi_2, \ldots, \phi_m < 2\pi \end{array}\right\}. \tag{1.3.2}$$

From the equations (1.3.1) and (1.3.2) there are lmn n^{th} roots of unity. The **primitive** n^{th} root of unity over the **quaternion domain** is a power $\underset{\sim}{q}^k$ with $\underset{\sim}{q}^k \neq 1$ and $0 < k < n$, where k and n are coprime.

We denote the n^{th} root of unity on unit 3-sphere by ω_{ab}.

That is,

$$\omega_{ab} = \cos\frac{2k\pi}{n} + \hat{u}_{ab}\sin\frac{2k\pi}{n} \tag{1.3.3}$$

for $k = 0, 1, 2, \ldots, n-1$.

We shall extend the following property:

$$\sum_{k=0}^{n-1} \omega^k = 0 \qquad \text{when} \quad \omega \neq 1$$

$$= n \qquad \text{when} \quad \omega = 1$$

(where ω is the n^{th} root of unity in complex plane) to n^{th} root of unity in quaternion space.

THEOREM 1.3.2. *Let* ω_{ab} *be the* n^{th} *root of unity lying on unit* 3-*sphere. Then*

$$\sum_{k=0}^{n-1} (\omega_{ab})^k = \begin{cases} 0 & \textit{when} \quad \omega_{ab} \neq 1 \\ n & \textit{when} \quad \omega_{ab} = 1 \end{cases} \tag{1.3.4}$$

for each pair of fixed θ_a *and* ϕ_b.

PROOF. The proof of this theorem is trivial. □

We shall define an analogue of the Euler's relation of the complex domain.

Definition 1.3.3. The Euler's relation for the quaternion domain is defined by

$$e^{\hat{u}_{ab}\chi} = \cos\chi + \hat{u}_{ab}\sin\chi \tag{1.3.5}$$

where $\hat{u}_{ab}$ is pure unit quaternion for fixed values of θ and ϕ.

Remark 1.3.4. The multiplication of $e^{\hat{u}_{ab}\chi}$ and $e^{\hat{u}_{ab}\psi}$ is commutative since

$$e^{\hat{u}_{ab}\chi} \cdot e^{\hat{u}_{ab}\psi} = e^{\hat{u}_{ab}(\chi+\psi)}$$

$$= e^{\hat{u}_{ab}(\psi+\chi)}$$

$$= e^{\hat{u}_{ab}\psi} \cdot e^{\hat{u}_{ab}\chi}.$$

Remark 1.3.5. Using equation (1.3.5), we can express the n^{th} roots of unity over quaternion domain as exponentials in the form

$$\omega_{ab} = e^{\hat{u}_{ab}\frac{2k\pi}{n}} \tag{1.3.6}$$

for $\quad k = 0, 1, 2, \ldots, n-1.$

2. DISCRETE QUATERNION FOURIER TRANSFORM

Let $\underset{\sim}{F}$ be a quaternion valued function. Now we shall define the discrete quaternion Fourier transform (DQFT).

Definition 2.1. The discrete Fourier transform maps a quaternion function $\underset{\sim}{F}$ defined on the integers *mod* n to another such function $T\underset{\sim}{F}$ defined by

$$(T\underset{\sim}{F})(k) = \sum_{l=0}^{n-1} (\omega_{ab})^{kl}\, \underset{\sim}{F}(l) \tag{2.1}$$

where ω_{ab} is the primitive n^{th} root of unity defined as in the equation (1.3.3) and having the property expressed by the equation (1.3.4)

We shall extend the following property to discrete quaternion Fourier transform:

Let $(T'f)(k) = (Tf)(-k)$, the maps TT' and $T'T$ are both n times the identity. In other words, $T^{-1} = T'/n$. (see L. Garding, T. Tambour [14])

THEOREM 2.2. *Let* $\underset{\sim}{F}$ *be quaternion function. Let* $(T'\underset{\sim}{F})(k) = (T\underset{\sim}{F})(-k)$. *Then*

$$\begin{aligned}(T'T\underset{\sim}{F})(k) &= n\,\underset{\sim}{F}(k)\\ &= (TT'\underset{\sim}{F})(k).\end{aligned} \tag{2.2}$$

PROOF. We have

$$\begin{aligned}(T'T\underset{\sim}{F})(k) &= \sum_{l=0}^{n-1} \omega_{ab}^{-kl}\,(T\underset{\sim}{F})(l)\\
&= \sum_{l=0}^{n-1} \omega_{ab}^{-kl} \sum_{p=0}^{n-1} \omega_{ab}^{lp}\,\underset{\sim}{F}(p)\\
&= \sum_{l=0}^{n-1}\sum_{p=0}^{n-1} \omega_{ab}^{l(p-k)}\,\underset{\sim}{F}(p)\\
&= \sum_{p=0}^{n-1}\sum_{l=0}^{n-1} (\omega_{ab}^{p-k})^{l}\,\underset{\sim}{F}(p)\\
&= \sum_{\substack{l=0\\ p=k}}^{n-1} (1)^{l}\,\underset{\sim}{F}(k) + \sum_{p\neq k}\sum_{l=0}^{n-1} (\omega_{ab}^{p-k})^{l}\,\underset{\sim}{F}(p)\end{aligned} \tag{2.3}$$

$= n \underset{\sim}{F}(k)$ (from equation (1.3.4)).

Similarly $(TT'\underset{\sim}{F})(k) = n \underset{\sim}{F}(k)$. □

Denoting $T\underset{\sim}{F}$ by $\underset{\sim}{G}$, the Fourier transform and its inverse are given by the following formulae

$$\left.\begin{aligned} \underset{\sim}{G}(l) &= \sum_{k=0}^{n-1} (\omega_{ab})^{lk} \underset{\sim}{F}(k) \\ \text{and} \quad & \\ \underset{\sim}{F}(k) &= \sum_{l=0}^{n-1} (\omega_{ab})^{-lk} \underset{\sim}{G}(l)/n \end{aligned}\right\}. \tag{2.4}$$

We shall prove some properties of the discrete quaternion Fourier transforms in Section 3 below.

3. PROPERTIES OF DQFT

The Fourier transform converts a convolution into multiplication of the transforms.

Let $\underset{\sim}{F}$ and $\underset{\sim}{H}$ be two quaternion functions. The convolution sum is defined as

$$(\underset{\sim}{F} * \underset{\sim}{H})(k) = \sum_{l=0}^{n-1} \underset{\sim}{F}(k-l)\, \underset{\sim}{H}(l). \tag{3.1}$$

We shall prove that the Fourier transform of $\underset{\sim}{F} * \underset{\sim}{H}$ is the product of Fourier transforms.

THEOREM 3.1. *The Fourier transform of the convolution $\underset{\sim}{F} * \underset{\sim}{H}$ is the product of Fourier transforms of $\underset{\sim}{F}$ and $\underset{\sim}{H}$, that is*

$$T(\underset{\sim}{F} * \underset{\sim}{H}) = (T\underset{\sim}{F})\,(T\underset{\sim}{H}).$$

PROOF. From equation (3.1), we have

$$(\underset{\sim}{F} * \underset{\sim}{H})(k) = \sum_{l=0}^{n-1} \underset{\sim}{F}(k-l)\, \underset{\sim}{H}(l).$$

Using the definition (2.1) the Fourier transform of $\underset{\sim}{F} * \underset{\sim}{H}$ is

$$T(\underset{\sim}{F} * \underset{\sim}{H})(p) = \sum_{k=0}^{n-1} \omega_{ab}^{pk}\, (\underset{\sim}{F} * \underset{\sim}{H})(k)$$

$$= \sum_{k=0}^{n-1} \omega_{ab}^{pk} \sum_{l=0}^{n-1} \underset{\sim}{F}(k-l)\, \underset{\sim}{H}(l).$$

$$= \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \omega_{ab}^{pk}\, \omega_{ab}^{pl-pl}\, \underset{\sim}{F}(k-l)\, \underset{\sim}{H}(l)$$

$$= \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \omega_{ab}^{p(k-l)}\, \omega_{ab}^{pl}\, \underset{\sim}{F}(k-l)\, \underset{\sim}{H}(l) \tag{3.2}$$

Summing independently over $k-l$ and l, the right hand side of equation (3.2) becomes

$$(T\underset{\sim}{F})T(\underset{\sim}{H}).$$

Therefore

$$T(\underset{\sim}{F} * \underset{\sim}{H}) = (T\underset{\sim}{F})(T\underset{\sim}{H}). \quad \square \tag{3.3}$$

The next theorem proves the linear property of Fourier transform.

THEOREM 3.2. *If* $\underset{\sim}{F}$ *and* $\underset{\sim}{H}$ *are two quaternion valued functions and* c *and* d *are real numbers then*

(i) $T[c, \underset{\sim}{F}] = cT\underset{\sim}{F}$

(ii) $T[c\underset{\sim}{F} + d\underset{\sim}{H}] = cT\underset{\sim}{F} + dT\underset{\sim}{H}$.

PROOF. (i) From the equation (2.1)

$$\{T(c\underset{\sim}{F})\}(k) = \sum_{l=0}^{n-1} (\omega_{ab})^{kl}\, c\, \underset{\sim}{F}(l)$$

$$= c \sum_{l=0}^{n-1} (\omega_{ab})^{kl}\, \underset{\sim}{F}(l)$$

$$= c(T\underset{\sim}{F})(k).$$

Therefore, $T(c\underset{\sim}{F}) = c(T\underset{\sim}{F})$. (3.4)

(ii) Again from the equation (2.1)

$$\{T(c\underset{\sim}{F} + d\underset{\sim}{H})\}(k) = \sum_{l=0}^{n-1} (\omega_{ab})^{kl} (c\underset{\sim}{F} + d\underset{\sim}{H})(l)$$

$$= \sum_{l=0}^{n-1} (\omega_{ab})^{kl} \, c \, \underset{\sim}{F}(l) + \sum_{l=0}^{n-1} (\omega_{ab})^{kl} \, d \, \underset{\sim}{H}(l)$$

$$= c \sum_{l=0}^{n-1} (\omega_{ab})^{kl} \, \underset{\sim}{F}(l) + d \sum_{l=0}^{n-1} (\omega_{ab})^{kl} \, \underset{\sim}{H}(l)$$

(from the equation (3.4))

$$= c(T\underset{\sim}{F})(k) + d(T\underset{\sim}{H})(k). \tag{3.5}$$

Therefore, $T[c\underset{\sim}{F} + d\underset{\sim}{H}] = cT\underset{\sim}{F} + dT\underset{\sim}{H}$. □

We shall prove another property of Fourier transform, that is the time-invariance of Fourier transform. Here we consider the quaternion valued function $\underset{\sim}{F}(n)$ as input and its Fourier transform $(T\underset{\sim}{F})(n)$ as output, at discrete time n.

THEOREM 3.3. *The Fourier transform $T\underset{\sim}{F}$ of a quaternion valued function $\underset{\sim}{F}$ is time invariant.*

PROOF. Let $\underset{\sim}{F}_1(n)$ be any input and let $(T\underset{\sim}{F}_1)(n)$ be its output.

That is, $$(T\underset{\sim}{F}_1)(n) = \sum_{p=0}^{n-1} (\omega_{ab})^{np} \, \underset{\sim}{F}_1(p). \tag{3.6}$$

Now consider a second input obtained by shifting $\underset{\sim}{F}_1(n)$:

$$\underset{\sim}{F}_2(n) = \underset{\sim}{F}_1(n - n_0). \tag{3.7}$$

The output corresponding to this input is

$$(T\underset{\sim}{F}_2)(n) = (T\underset{\sim}{F}_1)(n - n_0)$$

$$= \sum_{p=0}^{n-1} (\omega_{ab})^{n-n_0)p} \, \underset{\sim}{F}_1(p). \tag{3.8}$$

Similarly from equation (3.6)

$$(T\underset{\sim}{F}_1)(n - n_0) = \sum_{p=0}^{n-1} (\omega_{ab})^{(n-n_0)p} \, \underset{\sim}{F}_1(p). \tag{3.9}$$

Comparing equations (3.8) and (3.9), we see that

$$(T\underset{\sim}{F}_2)(n) = (T\underset{\sim}{F}_1)(n - n_0).$$

Therefore, the Fourier transform of a quaternion valued function $\underset{\sim}{F}$ is time invariant. □

4. APPLICATION TO SIGNAL PROCESSING SYSTEMS

A system can be viewed as any process that results in the transformation of signals. Thus, a system has an input signal and an output signal which is related to the input through the system transformation. For example, an automobile can also be viewed as a system in which the input is the depression of the accelerator pedal and the output is the motion of the vechicle.

A **discrete-time system,** that is, one that transforms discrete-time inputs into discrete-time outputs, will be represented symbolically as

$$x[n] \rightarrow y[n]. \tag{4.1}$$

A system is **time-invariant** if a time shift in the input signal causes the same time shift in the output signal. Specifically,

if $\quad x[n] \rightarrow y[n]$

then $\quad x[n - n_0] \rightarrow y[n - n_0]$.

A system is **linear** if

$$ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n] \tag{4.2}$$

where a and b are any complex constants.

The systems possessing these two properties are known as **linear, time-invariant (LTI) systems.** These systems play a particularly important role in system analysis and design, in part due to the fact that many systems encountered in nature can be successfully modeled as linear and time-invariant. (Ref. Oppenheim, Willsky with Young [6], De Fatta, Lucas and Hodgkiss [2])

4.1 *Filtering*

A digital filter is a discrete-time system that is designed to pass the spectral content of the input signal in a specified band of frequencies, that is, the filter transfer function forms a spectral window through which only the desired portion of the input spectrum is allowed to pass. For linear time-invariant systems, the spectrum of the output is that of the input multiplied by the frequency response of the system. An example in which linear time-invariant filtering is encountered is in audio systems. In such systems, a filter is typically included to permit the listener to modify the relative amounts of low-frequency energy and high-frequency energy. The filter corresponds to a linear time-invariant system whose frequency response is changed by manipulating the tone controls.

The behaviour of the discrete-time (linear time-invariant) system is described by the N^{th} order linear difference equation with constant coefficients:

$$\sum_{k=0}^{N} b_k y(n-k) = \sum_{k=0}^{N} a_k x(n-k). \tag{4.3}$$

Rewriting equation (4.3) to express the present output in terms of present and past inputs with $b_0 = 1$ yields:

$$y(n) = \sum_{k=0}^{N} a_k x(n-k) - \sum_{k=1}^{N} b_k y(n-k). \tag{4.4}$$

The topic of filtering encompasses many issues, such as those involving design and implementation. The basic concept stem directly from the notions and properties of the Fourier transform.

4.2 *Implementation of the Fourth order real coefficient digital filters using First order quaternion filters*

We claim that four fourth order real coefficient digital filters can be implemented by the first order quaternion filters with quaternion coefficients.

From equation (4.4) the fourth order digital filter is

$$y(n) = \sum_{k=0}^{4} a_k\, x(n-k) - \sum_{k=1}^{4} b_k\, y(n-k). \tag{4.5}$$

We assume the integers k for the input $x(n)$ and output $y(n)$ as

$$\left.\begin{array}{l} k \bmod (N+1) \\ \text{and} \\ k \bmod N \end{array}\right\} \tag{4.6}$$

respectively.

We shall consider the first order digital filter discussed in [3]. It is described by the following equation :

$$Y(n) = aY(n-1) + c[X(n) - bX(n-1)] \tag{4.7}$$

where $X(n)$ and $Y(n)$ are quaternion input and output respectively and a, b, c are quaternion constants.

We define

$$\left.\begin{aligned} X(n) &= x(n) + ix(n-1) + jx(n-2) + kx(n-3) \\ Y(n) &= y(n) + iy(n-1) + jy(n-2) + ky(n-3) \end{aligned}\right\}. \tag{4.8}$$

The constants a, b, c are given by

$$\left.\begin{aligned} a &= a_0 + ia_1 + ja_2 + ka_3 \\ b &= b_0 + ib_1 + jb_2 + kb_3 \\ c &= c_0 + ic_1 + jc_2 + kc_3 \end{aligned}\right\}. \tag{4.9}$$

From equations (4.8) and (4.9) the equation (4.7) can be simplified as follows:

$$\begin{aligned} Y(n) = {} & (a_0 + ia_1 + ja_2 + ka_3)\{y(n-1) + iy(n-2) + jy(n-3) \\ & + k(n-4)\} \\ & + (c_0 + ic_1 + jc_2 + kc_3)[x(n) + ix(n-1) + jx(n-2) + kx(n-3) \\ & - (b_0 + ib_1 + jb_2 + kb_3)\{x(n-1) + ix(n-2) + jx(n-3) \\ & + kx(n-4)\}]. \end{aligned} \tag{4.10}$$

The product

$$\begin{aligned} -cb &= -(c_0 + ic_1 + jc_2 + kc_3)(b_0 + kb_1 + jb_2 + kb_3) \\ &= d_0 + id_1 + jd_2 + kd_3 \end{aligned}$$

where

$$\left.\begin{aligned} -d_0 &= c_0b_0 - c_1b_1 - c_2b_2 - c_3b_3 \\ -d_1 &= c_0b_1 + c_1b_0 + c_2b_3 - c_3b_2 \\ -d_2 &= c_0b_2 + c_2b_0 + c_3b_1 - c_1b_3 \\ -d_3 &= c_0b_3 + c_3b_0 + c_1b_2 - c_2b_1 \end{aligned}\right\}. \tag{4.11}$$

Now the equation (4.10) is

$$\begin{aligned} Y(n) = {} & (a_0\, y(n-1) - a_1\, y(n-2) - a_2\, y(n-3) - a_3\, y(n-4) \\ & + i\{a_0\, y(n-2) + a_1\, y(n-1) + a_2\, y(n-4) - a_3\, y(n-3)\} \\ & + j\{a_0\, y(n-3) + a_2\, y(n-1) + a_3\, y(n-2) - a_1\, y(n-4)\} + \end{aligned}$$

$$
\begin{aligned}
&+ k\{a_0\, y(n-4) + a_3\, y(n-1) + a_1\, y(n-3) - a_2\, y(n-2)\}\\
&+ c_0\, x(n) - c_1\, x(n-1) - c_2\, x(n-2) - c_3\, x(n-3)\\
&+ i\{c_0\, x(n-1) + c_1\, x(n) + c_2\, x(n-3) - c_3\, x(n-2)\}\\
&+ j\{c_0\, x(n-2) + c_2\, x(n) + c_3\, x(n-1) - c_1\, x(n-3)\}\\
&+ k\{c_0\, x(n-3) + c_3\, x(n) + c_1\, x(n-2) - c_2\, x(n-1)\}\\
&+ d_0\, x(n-1) - d_1\, x(n-2) - d_2\, x(n-3) - d_3\, x(n-4)\\
&+ i\{d_0\, x(n-2) + d_1\, x(n-1) + d_2\, x(n-4) - d_3\, x(n-3)\}\\
&+ j\{d_0\, x(n-3) + d_2\, x(n-1) + d_3\, x(n-2) - d_1\, x(n-4)\}\\
&+ k\{d_0\, x(n-4) + d_3\, x(n-1) + d_1\, x(n-3) - d_2\, x(n-2)\}\\
= {} & a_0\, y(n-1) - a_1\, y(n-2) - a_2\, y(n-3) - a_3\, y(n-4)\\
&+ c_0\, x(n) + (d_0 - c_1)\, x(n-1) - (d_1 + c_2)\, x(n-2)\\
&\qquad - (d_2 + c_3)\, x(n-3) - d_3\, x(n-4)\\
&+ i\{a_1\, y(n-1) + a_0\, y(n-2) - a_3\, y(n-3) + a_2\, y(n-4)\\
&+ c_1\, x(n) + (c_0 + d_1)\, x(n-1) + (d_0 - c_3)\, x(n-2)\\
&\qquad + (c_2 - d_3)\, x(n-3) + d_2\, x(n-4)\}\\
&+ j\{a_2\, y(n-1) + a_3\, y(n-2) + a_0\, y(n-3) - a_1\, y(n-4)\\
&+ c_2\, x(n) + (c_3 + d_2)\, x(n-1) + (c_0 + d_3)\, x(n-2)\\
&+ (d_0 - c_1)\, x(n-3) - d_1\, x(n-4)\}\\
&+ k\{a_3\, y(n-1) - a_2\, y(n-2) + a_1\, y(n-3) + a_0\, y(n-4)\\
&+ c_3\, x(n) + (d_3 - c_2)\, x(n-1) + (c_1 - d_2)\, x(n-2)\\
&+ (c_0 + d_1)\, x(n-3) + d_0 x(n-4)\}.
\end{aligned}
\tag{4.12}
$$

Equating the corresponding components of both sides of equation (4.12),

$$
\begin{aligned}
y(n) &= a_0\, y(n-1) - a_1\, y(n-2) - a_2\, y(n-3) - a_3\, y(n-4) + c_0\, x(n)\\
&\quad + (d_0 - c_1)\, x(n-1) - (d_1 + c_2)\, x(n-2) - (d_2 + c_3)\, x(n-3)\\
&\quad - d_3\, x(n-4)\\
&= \sum_{k=0}^{4} a'_k\, x(n-k) - \sum_{k=1}^{4} b'_k\, y(n-k).
\end{aligned}
\tag{4.13}
$$

The relations for the coefficients a'_k and b'_k are given by

$$\left.\begin{array}{ll} a'_0 = c_0 & b'_1 = -a_0 \\ a'_1 = d_0 - c_1 & b'_2 = a_1 \\ a'_2 = -d_1 - c_2 & b'_3 = a_2 \\ a'_3 = -d_2 - c_3 & b'_4 = a_3 \\ a'_4 = -d_3 & \end{array}\right\} \tag{4.14}$$

$$\begin{aligned} y(n-1) = {} & a_1\, y(n-1) + a_0\, y(n-2) - a_3\, y(n-3) + a_2\, y(n-4) \\ & + c_1\, x(n) + (c_0 + d_1)\, x(n-1) + (d_0 - c_3)\, x(n-2) \\ & + (c_2 - d_3)\, x(n-3) + d_2\, x(n-4). \end{aligned} \tag{4.15}$$

Using the expression (4.6) we can write the equation (4.15) as

$$\begin{aligned} y(n-1) &= a_0\, y(n-2) - a_3\, y(n-3) + a_2\, y(n-4) + a_1\, y(n-5) \\ &\quad + (c_0 + d_1)\, x(n-1) + (d_0 - c_3)\, x(n-2) \\ &\quad + (c_2 - d_3)\, x(n-3) + d_2\, x(n-4) + c_1\, x(n-5) \\ &= \sum_{k=0}^{4} a''_k\, x(n-1-k) - \sum_{k=1}^{4} b''_k\, y(n-1-k). \end{aligned} \tag{4.16}$$

The relations for the coefficients a''_k and b''_k are as follows:

$$\left.\begin{array}{ll} a''_0 = c_0 + d_1 & b''_1 = -a_0 \\ a''_1 = d_0 - c_3 & b''_2 = a_3 \\ a''_2 = c_2 - d_3 & b''_3 = -a_2 \\ a''_3 = d_2 & b''_4 = -a_1 \\ a''_4 = c_1 & \end{array}\right\} \tag{4.17}$$

$$\begin{aligned} y(n-2) = {} & a_2\, y(n-1) + a_3\, y(n-2) + a_0\, y(n-3) - a_1\, y(n-4) \\ & + c_2\, x(n) + (c_3 + d_2)\, x(n-1) + (c_0 - d_3)\, x(n-2) \\ & + (d_0 - c_1)\, x(n-3) - d_1\, x(n-4). \end{aligned} \tag{4.18}$$

Again using the expression (4.6), the equation (4.18) becomes

$$y(n-2) = a_0\, y(n-3) - a_1\, y(n-4) + a_2\, y(n-5) + a_3\, y(n-6)$$
$$+ (c_0 + d_3)\, x(n-2) + (d_0 - c_1)\, x(n-3) - d_1\, x(n-4)$$
$$+ c_2\, x(n-5) + (c_3 + d_2)\, x(n-6)$$
$$= \sum_{k=0}^{4} u_k\, x(n-2-k) - \sum_{k=1}^{4} v_k\, y(n-2-k). \tag{4.19}$$

The relations for the coefficients u_k and v_k are given by

$$\left.\begin{array}{ll} u_0 = c_0 + d_3 & v_1 = -a_0 \\ u_1 = d_0 - c_1 & v_2 = a_1 \\ u_2 = -d_1 & v_3 = -a_2 \\ u_3 = c_2 & v_4 = -a_3 \\ u_4 = c_3 + d_2 & \end{array}\right\} \tag{4.20}$$

$$y(n-3) = a_3\, y(n-1) - a_2\, y(n-2) + a_1\, y(n-3) + a_0\, y(n-4)$$
$$+ c_3\, x(n) - (d_3 - c_2)\, x(n-1) + (c_1 - d_2)\, x(n-2)$$
$$+ (c_0 + d_1)\, x(n-3) + d_0\, x(n-4). \tag{4.21}$$

From the expression (4.6), the equation (4.21) is given by

$$y(n-3) = a_0\, y(n-4) + a_3\, y(n-5) - a_2\, y(n-6) + a_1\, y(n-7)$$
$$+ (c_0 + d_1)\, x(n-3) + d_0\, x(n-4) + c_3\, x(n-5)$$
$$+ (d_3 - c_2)\, x(n-6) + (c_1 - d_2)\, x(n-7)$$
$$= \sum_{k=0}^{4} u'_k\, x(n-3-k) - \sum_{k=1}^{4} v'_k\, y(n-3-k). \tag{4.22}$$

The relations for the coefficients u'_k and v'_k are given by

$$\left.\begin{array}{ll} u'_0 = c_0 + d_1 & v'_1 = -a_0 \\ u'_1 = d_0 & v'_2 = -a_3 \\ u'_2 = c_3 & v'_3 = a_2 \\ u'_3 = d_3 - c_2 & v'_4 = -a_1 \\ u'_4 = c_1 - d_2 & \end{array}\right\}. \tag{4.23}$$

The equations (4.14), (4.17), (4.20) and (4.23) give the coefficients for the filters of the outputs $y(n)$, $y(n-1)$, $y(n-2)$ and $y(n-3)$.

We can implement the fourth order filters with real coefficients for the above outputs using the first order quaternion filter with quaternion coefficients.

4.3 *Remark*

With the usual convention due to non-commutativity of the quaternion multiplication, all the multiplication in this paper are from left to right and not from right to left.

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