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Onset of Thermogravitational Convection in a Ferrofluid Layer With Temperature Dependent Viscosity

The onset of thermogravitational convection in a horizontal ferrofluid layer is investigated with viscosity depending exponentially on temperature. The bounding surfaces of the ferrofluid layer are considered to be either stress free or rigid-ferromagnetic and insulated to temperature perturbations. The resulting eigenvalue problem is solved numerically using the Galerkin technique and also by a regular perturbation technique for different types of velocity boundary conditions, namely free-free, rigid-rigid, and lower rigid- upper free. It is observed that increasing the viscosity parameter, Λ, and the magnetic number, M_1, is to hasten the onset of ferroconvection, while the nonlinearity of fluid magnetization, M_3, is found to have no influence on the stability of the system. The critical stability parameters are found to be the same in the limiting cases of either no magnetic forces or no buoyancy forces. [DOI: 10.1115/1.4004758]

Keywords: ferroconvection, ferrofluid, temperature dependent viscosity, viscosity parameter

1 Introduction

Ferrofluids are stable colloidal suspensions of magnetic nanoparticles in a carrier fluid such as water, hydrocarbon (mineral oil or kerosene), or fluorocarbon. The unusual behavior of these fluids is a combination of normal liquid behavior with magnetic control of their flow and properties. Presently, these fluids are in wide use in seals, bearings, magnetostatic supports, jet printers, dampers, actuators, sensors, transducers, and medical applications and for separation of nonmagnetic particles, flow control and drag reduction. An authoritative introduction to this fascinating subject along with the applications is provided in Refs. [1–4]. The magnetization of ferromagnetic fluids depends on the magnetic field and the temperature and density of the fluid. Any variation of these quantities can induce a change in body force distribution in the fluid. This leads to convection in ferrofluids in the presence of a magnetic field gradient, known as ferroconvection, which is similar to buoyancy driven convection and has been studied extensively [5–8].

One of the well known phenomena generated by the influence of a magnetic field on ferrofluids is the change of their viscous behavior, an active field in ferrofluid research. In his review article, Odenbach [9] discussed elaborately the development and importance of this topic. Several previous studies have studied thermal convective instability in ferrofluids by considering the variation in viscosity of ferrofluids. Stiles and Kagan [10] were the first to investigate thermal convective instability in a ferrofluid layer heated from below. They considered a linear variation in viscosity with temperature. The effects of magnetic field dependent (MFD) viscosity on the onset of ferroconvection in a rotating ferrofluid layer are discussed by Vaidyanathan et al. [11]. Sunil et al. [12] added consideration of dust particles. Nanjundappa et al. [13] investigated the effect of MFD viscosity on the onset of convection in a ferromagnetic fluid layer in the presence of a vertical magnetic field by considering bounding surfaces that are either rigid-ferromagnetic or are stress-free with constant heat flux conditions. Recently, Nanjundappa et al., [14] analyzed the effects of MFD viscosity on the onset of coupled Benard-Marangoni ferroconvection in a horizontal layer of ferrofluid.

Most ferrofluids are either water based or oil based. The viscosity of water is far more sensitive to temperature variations and oils are known to have viscosity decreasing exponentially with temperature rather than linearly. Realizing the importance, several investigators considered exponential variation in viscosity with temperature in analyzing thermal convective instability in horizontal fluid layers but the studies were limited to ordinary viscous fluids [15–18]. To our knowledge, no attention has been given to convective instability problems involving ferrofluids, despite the importance of ferrofluids in many heat transfer applications. For example, in a rotating shaft seal involving ferrofluids the temperature may rise above 100 °C at high shaft surface speeds. A similar situation may arise in the use of ferrofluids in loud speakers [19]. The aim of the present study is, to investigate thermogravitational convection in a layer of ferrofluid by considering its exponential variation of viscosity with temperature. In investigating the problem, the boundaries of the ferrofluid layer are considered rigid, or free and ferromagnetic. Moreover, actual physical situations suggest that the appropriate thermal boundary conditions are uniform heat flux rather than uniform temperature. The resulting eigenvalue problem is solved both numerically using the Galerkin technique and analytically by a regular perturbation technique for different types of velocity boundary conditions, namely free-free, rigid-rigid, and lower rigid-upper free. The results obtained from the two techniques complement each other. Present results in the literature are obtained as particular cases of the present study.

2 Mathematical Formulation

The physical configuration considered is an initially quiescent horizontal layer of an incompressible ferrofluid of characteristic thickness, d, in the presence of an imposed spatially uniform magnetic field, H_0, oriented in the vertical direction. The lower and upper boundaries are maintained at constant but different temperatures, T_l and $T_u\ (< T_l)$, respectively. A Cartesian co-ordinate system (x, y, z) is used with the origin at the bottom and with the z-axis directed vertically upward. Gravity acts in the negative

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z-direction, $\vec{g} = -g\,\hat{k}$, where $\hat{k}$ is the unit vector in the z-direction. The variation of viscosity, η, of the ferrofluid with temperature is assumed to be exponential, given by $\eta = \eta_0 \exp[-\gamma(T - T_r)]$, where T is the temperature, η_0 is the reference value at the reference temperature, T_r, and γ is a positive constant.

The governing equations under the Oberbeck–Boussinesq approximation are given by the following:

Mass balance

$$\nabla \cdot \vec{q} = 0 \tag{1}$$

Linear momentum balance

$$\rho_0\left[\frac{\partial \vec{q}}{\partial t} + (\vec{q}.\nabla)\,\vec{q}\right] = -\nabla \mathrm{p} + \rho\,\vec{g} + \nabla\cdot\left[\eta\left(\nabla\vec{q} + \nabla\vec{q}^T\right)\right] + \mu_0(\vec{M}\cdot\nabla)\vec{H} \tag{2}$$

Energy balance

$$\left[\rho_0 C_{V,H} - \mu_0 \vec{H}\cdot\left(\frac{\partial \vec{M}}{\partial T}\right)_{V,H}\right]\frac{DT}{Dt} + \mu_0 T\left(\frac{\partial \vec{M}}{\partial T}\right)_{V,H}\cdot\frac{D\vec{H}}{Dt} = k_t \nabla^2 T \tag{3}$$

Equation of state

$$\rho = \rho_0\left[1 - \alpha_t(T - T_a)\right] \tag{4}$$

Maxwell's equations in the magnetostatic limit

$$\nabla\cdot\vec{B} = 0 \tag{5a}$$

$$\nabla\times\vec{H} = 0 \tag{5b}$$

$$\vec{B} = \mu_0\left(\vec{M} + \vec{H}\right) \tag{5c}$$

Here, $\vec{q}$ is the velocity, p is the pressure, ρ is the fluid density, $\vec{M}$ is the magnetization, $\vec{H}$ is the magnetic field intensity, $\vec{B}$ is the magnetic flux density, μ_0 is the magnetic permeability of a vacuum, $\vec{g}$ is the gravitational acceleration, T is the temperature, k_t is the thermal conductivity, $C_{V,H}$ is the specific heat at a constant volume and magnetic field, ρ_0 is the reference density, α_t is the thermal expansion coefficient and $T_a = (T_l + T_u)/2$ is the average temperature. In view of Eq. (5b), $\vec{H}$ can be expressed as

$$\vec{H} = \nabla\varphi \tag{6}$$

where, φ is the magnetic potential.

Since the magnetization depends on the magnitude of magnetic field and temperature, we have

$$\vec{M} = \frac{\vec{H}}{H}M(H,T) \tag{7}$$

The linearized equation of the magnetic state about H_0 and T_a is

$$M = M_0 + \chi(H - H_0) - K(T - T_a) \tag{8}$$

where, $M_0 = M(H_0, T_a)$ is the saturation magnetization, $\chi = (\partial M/\partial H)_{H_0,T_a}$ is the magnetic susceptibility, $K = -(\partial M/\partial T)_{H_0,T_a}$ is the pyromagnetic co-efficient, and $H = |\vec{H}|$ and $M = |\vec{M}|$.

It is clear that there exists the following solution for the quiescent basic state

$$\vec{q}_b = 0$$

$$p_b(z) = p_0 - \rho_0 g\,z - \frac{1}{2}\rho_0\alpha_t g\beta\,z(z-d) - \frac{\mu_0 M_0\kappa\,\beta}{1+\chi}z - \frac{\mu_0\kappa^2\beta^2}{2(1+\chi)^2}z(z-d)$$

$$T_b(z) = T_a - \beta\left(z - \frac{d}{2}\right) \tag{9}$$

$$\vec{H}_b(z) = \left[H_0 - \frac{K\beta}{1+\chi}\left(z - \frac{d}{2}\right)\right]\hat{k}$$

$$\vec{M}_b(z) = \left[M_0 + \frac{K\beta}{1+\chi}\left(z - \frac{d}{2}\right)\right]\hat{k}$$

where $\beta = (T_l - T_u)/d$ is the temperature gradient and the subscript b denotes the basic state. To investigate the conditions under which the quiescent solution is stable against small disturbances, we consider a perturbed state such that

$$\vec{q} = \vec{q}\,', \quad p = p_b(z) + p', \quad T = T_b(z) + T',$$
$$\vec{H} = \vec{H}_b(z) + \vec{H}', \quad \vec{M} = \vec{M}_b(z) + \vec{M}' \tag{10}$$

where, $\vec{q}\,', p', T', \vec{H}'$, and $\vec{M}'$ are perturbed variables, assumed to be small. Then, we note that $\eta = \eta_0\exp[\gamma\beta(z - d/2) + \gamma(T_r - T_a) - \gamma T']$. Substituting Eq. (10) into Eqs. (5c) and (7), and using Eqs. (5a) and (5b), we obtain (after dropping the primes)

$$H_x + M_x = (1 + M_0/H_0)\,H_x,$$
$$H_y + M_y = (1 + M_0/H_0)\,H_y, \tag{11}$$
$$H_z + M_z = (1+\chi)H_z - KT.$$

Again substituting Eq. (10) into the momentum Eq. (2), linearizing, eliminating the pressure term by operating the curl twice and using Eq. (11), the z-component of the resulting equation is obtained as (after dropping the primes)

$$\rho_0\frac{\partial}{\partial t}(\nabla^2 w) = \eta(z)\,\nabla^4 w + 2\frac{\partial\eta(z)}{\partial z}\nabla^2\left(\frac{\partial w}{\partial z}\right) + \frac{\partial^2\eta(z)}{\partial z^2}\left(\nabla^2 w - 2\nabla_h^2 w\right) - \mu_0 K\beta\frac{\partial}{\partial z}\left(\nabla_h^2\varphi\right) + \frac{\mu_0 K^2\beta}{1+\chi}\nabla_h^2 T + \rho_0\alpha_t g\,\nabla_h^2 T \tag{12}$$

where

$$\eta(z) = \eta_0\exp\left[\gamma\beta\left(z - \frac{d}{2}\right) + \gamma(T_r - T_a)\right]$$

and $\nabla_h^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplacian operator.

Equation (3), after using Eq. (10) and linearizing, takes the form (after dropping the primes)

$$(\rho_0 C)\frac{\partial T}{\partial t} - \mu_0 T_0 K\frac{\partial}{\partial t}\left(\frac{\partial\varphi}{\partial z}\right) = k_1\,\nabla^2 T + \left[\rho_0 C - \frac{\mu_0 T_0 K^2}{1+\chi}\right]w\,\beta \tag{13}$$

where, $\rho_0 C = \rho_0 C_{V,H} + \mu_0 H_0 K$. Equations 5(a) and 5(b), after substituting Eq. (10) and using Eq. (11), may be written as (after dropping the primes)

$$\left(1 + \frac{M_0}{H_0}\right)\nabla_h^2\varphi + (1+\chi)\frac{\partial^2\varphi}{\partial z^2} - K\frac{\partial T}{\partial z} = 0 \tag{14}$$

Since the principle of exchange of stability is valid [15], the normal mode expansion of the dependent variables is assumed to be of the form

$$\{w, T, \varphi\} = \{W(z),\, \Theta(z),\, \Phi(z)\}\, \exp[i\,(\ell x + my)] \qquad (15)$$

where ℓ and m are wave numbers in the x and y directions, respectively.

On substituting Eq. (15) into Eqs. (12)–(14) and nondimensionalizing the variables by setting

$$z^* = \frac{z}{d}, W^* = \frac{d}{\kappa} W, \quad \Theta^* = \frac{1}{\beta d}\Theta,$$
$$\Phi^* = \frac{(1+\chi)}{\mathrm{K}\,\beta\, d^2}\Phi, \quad f(z) = \frac{\eta(z)}{\eta_0} \qquad (16)$$

where $\nu = \eta_0/\rho_0$ is the kinematic viscosity and $\kappa = k_t/\rho_0 C$ is the effective thermal diffusivity, we obtain (after dropping the asterisks for simplicity)

$$f\left(D^2 - a^2\right)^2 W + 2Df\left(D^2 - a^2\right)D\,W + D^2 f\left(D^2 + a^2\right) W = -a^2 R_t\,[M_1 D\Phi - (1+M_1)\,\Theta] \qquad (17)$$

$$\left(D^2 - a^2\right)\,\Theta = -(1 - M_2)\,W \qquad (18)$$

$$\left(D^2 - a^2 M_3\right)\Phi - D\Theta = 0 \qquad (19)$$

Here, $D = d/dz$ is the differential operator, $a = \sqrt{\ell^2 + m^2}$ is the overall horizontal wavenumber, W is the amplitude of the vertical component of velocity, Θ is the amplitude of temperature, Φ is the amplitude of magnetic potential, $R_t = \alpha_t g \beta\, d^4/\nu\,\kappa$ is the thermal Rayleigh number (the ratio of buoyant to viscous forces), $M_1 = \mu_0 K^2 \beta/(1+\chi)\alpha_t \rho_0 g$ is the magnetic number (the ratio of magnetic to gravitational buoyancy forces), $M_2 = \mu_0 T_a K^2/(1+\chi)\,\rho_0 C$ is the magnetic parameter, $M_3 = (1 + M_0/H_0)/(1+\chi)$ is the measure of nonlinearity of fluid magnetization parameter and $f(z)$ is given by

$$f(z) = \exp\left[\Lambda\left(z - \frac{1}{2}\right) + \Lambda\frac{(T_r - T_a)}{\beta d}\right] \qquad (20)$$

where $\Lambda = \gamma\beta d$ is the dimensionless viscosity parameter. If the reference temperature, T_r, is same as T_a, then $f(z) = \exp[\Lambda\,(z - 1/2)]$. The typical value of M_2 for magnetic fluids with different carrier liquids turns out to be of the order of 10^{-6} and, hence, its effect is negligible compared to unity.

The boundaries are considered stress free or rigid-ferromagnetic and they are insulated to temperature perturbations.

Thus, on the stress free boundary

$$W = D^2 W = D\Phi = D\Theta = 0 \qquad (21)$$

and on the rigid-ferromagnetic boundary

$$W = DW = \Phi = D\Theta = 0 \qquad (22)$$

3 Method of Solution

Equations (17)–(19) together with the corresponding boundary conditions constitute an eigenvalue problem with R as an eigenvalue. The resulting eigenvalue problem is solved numerically using the Galerkin technique and analytically by the regular perturbation technique for three different types of velocity boundary conditions namely, (i) free-free, (ii) rigid-rigid, and (iii) lower rigid and upper free.

3.1 Solution by the Galerkin Technique. The Galerkin method is used to solve this problem as explained by Finlayson [20]. In this method, the test (weighted) functions are the same as the base (trial) functions. Accordingly, W, Θ and Φ are written as

$$W = \sum_{i=1}^{n} A_i\, W_i(z), \quad \Theta(z) = \sum_{i=1}^{n} C_i \Theta_i(z), \quad \Phi(z) = \sum_{i=1}^{n} D_i \Phi_i(z) \qquad (23)$$

where A_i, C_i, and D_i are unknown constants to be determined. The base functions, $W_i(z)$, $\Theta_i(z)$, and $\Phi_i(z)$, are generally chosen such that they satisfy the boundary conditions. Substituting Eq. (23) into Eqs. (17)–(19), multiplying the momentum equation by $W_j(z)$, the energy equation by $\Theta_j(z)$, and the magnetic potential equation by $\Phi_j(z)$ performing integration by parts with respect to z between $z=0$ and $z=1$ and using the boundary conditions, we obtain the following system of linear homogeneous algebraic equations

$$C_{ji}A_i + D_{ji}C_i + E_{ji}D_i = 0 \qquad (24)$$

$$F_{ji}A_i + G_{ji}C_i = 0 \qquad (25)$$

$$H_{ji}C_i + I_{ji}D_i = 0 \qquad (26)$$

The coefficients $C_{ji} - I_{ji}$ involve the inner products of the base functions and are given by

$$C_{ji} = < D^2 W_j D^2 W_i + (2a^2 - \Lambda^2) DW_j DW_i + a^2(a^2 + \Lambda^2) W_j W_i > - 2\Lambda < DW_j D^2 W_i + a^2 W_j W_i >$$

$$D_{ji} = -a^2 R_t\,(1 + M_1) < \exp[-\Lambda\,(z - 1/2)\, W_j\, \Theta_i] >$$

$$E_{ji} = a^2 R_t M_1 < \exp[-\Lambda\,(z - 1/2)\, W_j D\Phi_i >$$

$$F_{ji} = - < \Theta_j W_i >$$

$$G_{ji} = < D\Theta_j D\Theta_i > + a^2 < \Theta_j \Theta_i >$$

$$H_{ji} = - < D\Phi_j \Theta_i >$$

$$I_{ji} = < D\Phi_j D\Phi_i > + a^2 M_3 < \Phi_j \Phi_i >$$

where, the inner product is defined as $< \cdots\cdots > = \int_0^1 (\cdots)dz$.

The above set of homogeneous algebraic equations can have a nontrivial solution if and only if

$$\begin{vmatrix} C_{ji} & D_{ji} & E_{ji} \\ F_{ji} & G_{ji} & 0 \\ 0 & H_{ji} & I_{ji} \end{vmatrix} = 0 \qquad (27)$$

The eigenvalue has to be extracted from the above characteristic equation. For this, we select the trial functions as follows:

(i) Free-free ferromagnetic boundaries

$$W_i = (z^4 - 2z^3 + z)\,T^*_{i-1}, \quad \Theta_i = z^2(1 - 2z/3)T^*_{i-1},$$
$$\Phi_i = z^2(1 - 2z/3)\,T^*_{i-1} \qquad (28)$$

(ii) Rigid-rigid ferromagnetic boundaries

$$W_i = (z^4 - 2z^3 + z^2)\,T^*_{i-1}, \quad \Theta_i = z^2(1 - 2z/3)T^*_{i-1},$$
$$\Phi_i = (z^2 - z)(z - 2)\,T^*_{i-1} \qquad (29)$$

(iii) Lower rigid- upper free ferromagnetic boundaries

$$W_i = (2z^4 - 5z^3 + 3z^2)\,T^*_{i-1}, \quad \Theta_i = z^2(1 - 2z/3)T^*_{i-1},$$
$$\Phi_i = z^2(1 - 2z/3)\,T^*_{i-1} \qquad (30)$$

Here, $T^*_i s$ are the modified Chebyshev polynomials and note that the above trial functions satisfy all the boundary conditions. Equation (27) is solved numerically to obtain the critical Rayleigh number, R_{tc}, as a function of the wave number, a, for fixed values of Λ, M_1, and M_3 as well as for different velocity boundary conditions.

3.2 Solution by Regular Perturbation Technique. It is known that for insulated boundary conditions the onset of convection corresponds to a vanishingly small wave number (i.e., unicellular convection). The numerical calculations carried out in the previous section also corroborate this fact. Therefore, an attempt is being made to exploit this fact to obtain an analytical formula for the onset of convection using a regular perturbation technique with wave number, a, as a perturbation parameter. Such a study also helps in knowing the accuracy of the numerical method employed in solving the problem. Accordingly, the variables W, Θ and Φ are expanded in powers of a^2 as

$$(W,\Theta,\Phi) = (W_0,\Theta_0,\Phi_0) + a^2(W_1,\Theta_1,\Phi_1) + \cdots \tag{31}$$

Substituting Eq. (31) into Eqs. (17)–(19) and also in the boundary conditions, and collecting the terms of zero-th order, we obtain

$$f\,D^4W_0 + 2\,Df\,D^3W_0 + D^2f\,D^2W_0 = 0 \tag{32a}$$

$$D^2\Theta_0 + W_0 = 0 \tag{32b}$$

$$D^2\Phi_0 + D\Theta_0 = 0 \tag{32c}$$

with boundary conditions $W_0 = D^2W_0 = 0 = D\Theta_0 = D\Phi_0$ on the stress free boundary, and $W_0 = DW_0 = 0 = D\Theta_0 = \Phi_0$ on the rigid-ferromagnetic boundary. The solution to the zero-th order equations for free-free boundaries is: $W_0 = 0$, $\Theta_0 = 1$, and $\Phi_0 = 1$, while the solution for rigid-rigid and lower rigid- upper free boundaries is: $W_0 = 0$, $\Theta_0 = 1$, and $\Phi_0 = 0$.

The first order equations are then

$$f\,D^4W_1 + 2\,Df\,D^3W_1 + D^2f\,D^2W_1 = R_t\,(1+M_1) \tag{33a}$$

$$D^2\Theta_1 + W_1 = 1 \tag{33b}$$

$$D^2\Phi_1 - D\Theta_1 = 0 \tag{33c}$$

with boundary conditions $W_1 = D^2W_1 = D\Phi_1 = D\Theta_1 = 0$ on the free boundary and $W_1 = DW_1 = \Phi_1 = D\Theta_1 = 0$ on the rigid boundary.

The general solution of Eq. (33a) is given by

$$W_1 = c_1 + c_2 z + (c_3 + c_4 z)e^{-\Lambda z} + \frac{R_t(1+M_1)}{2\,\Lambda^2}\, z^2 e^{-\Lambda(z-1/2)} \tag{34}$$

where the arbitrary constants $c_1 - c_4$ are determined using different velocity boundary conditions. They are given below:

(i) Free-free boundaries

$$\begin{aligned} c_1 &= 2\,\Delta_1(-3+\Lambda)\\ c_2 &= -2\,\Delta_1(-3+\Lambda+3\,e^{-\Lambda}+\Lambda\,e^{-\Lambda})\\ c_3 &= -\,c_1\\ c_4 &= -\Delta_1(-4\,\Lambda+\Lambda^2) \end{aligned} \tag{35}$$

where $\Delta_1 = \frac{R\,(1+M_1)\,e^{\Lambda/2}}{2\,\Lambda^4}$.

(ii) Rigid-rigid boundaries

$$\begin{aligned} c_1 &= -\Delta_2(1-e^{\Lambda}+\Lambda\,e^{\Lambda}),\\ c_2 &= \Delta_2(2+\Lambda-2\,e^{\Lambda}+\Lambda\,e^{\Lambda}),\\ c_3 &= -\,c_1,\\ c_4 &= \Delta_2(-2+2\,e^{\Lambda}-2\,\Lambda\,e^{\Lambda}+\Lambda^2\,e^{\Lambda}) \end{aligned} \tag{36}$$

where $\Delta_2 = \frac{R\,(1+M_1)\,e^{\Lambda/2}}{2\,\Lambda^2(1-2\,e^{\Lambda}-\Lambda^2 e^{\Lambda}+e^{2\Lambda})}$.

(iii) Rigid-free boundaries

$$\begin{aligned} c_1 &= \Delta_3(-2+2\,\Lambda+2\,e^{\Lambda}-4\,\Lambda\,e^{\Lambda}+\Lambda^2\,e^{\Lambda}),\\ c_2 &= -\Delta_3(-2+2\,\Lambda+2\,\Lambda^2+2\,e^{\Lambda}-4\,\Lambda\,e^{\Lambda}+\Lambda^2\,e^{\Lambda}),\\ c_3 &= -\,c_1,\\ c_4 &= -\Delta_3(2-4\,\Lambda-2\,e^{\Lambda}+6\,\Lambda\,e^{\Lambda}-5\,\Lambda^2\,e^{\Lambda}+\Lambda^3\,e^{\Lambda}) \end{aligned} \tag{37}$$

where $\Delta_3 = \frac{R\,(1+M_1)\,e^{\Lambda/2}}{2\,\Lambda^3(-2+2\,e^{\Lambda}-2\,\Lambda\,e^{\Lambda}+\Lambda^2 e^{\Lambda})}$.

From Eq. (30b), it follows that

$$1 = \int_0^1 W_1 dz \tag{38}$$

Substituting for W_1 from Eq. (34) into Eq. (38) and carrying out the integration leads to an expression for the critical Rayleigh number, R_{tc}, for free-free, rigid-rigid, and lower rigid-upper free boundaries, respectively, in the form

$$R_{tc} = \frac{\Lambda^6}{(1+M_1)\,\sinh(\Lambda/2)\,[2\,\Lambda+\Lambda\,\cosh\Lambda-3\,\sinh\Lambda]} \tag{39}$$

$$R_{tc} = \frac{2\,\Lambda^5(2+\Lambda^2-2\,\cosh\Lambda)}{(1+M_1)\,[2\,\sinh(\Lambda/2)-\Lambda\,\cosh(\Lambda/2)]\,[4+\Lambda^2-4\,\cosh\Lambda+\Lambda\,\sinh\Lambda)]} \tag{40}$$

$$R_{tc} = \frac{8\,\Lambda^5 e^{\Lambda/2}(-2+2\,e^{\Lambda}-2\,\Lambda\,e^{\Lambda}+\Lambda^2 e^{\Lambda})}{(1+M_1)\,[10+10\,\Lambda+e^{\Lambda}(-18+3\,\Lambda^2-2\,\Lambda^3)+e^{2\Lambda}(6-6\,\Lambda+6\,\Lambda^2)+e^{3\Lambda}(2-4\,\Lambda+\Lambda^2)]} \tag{41}$$

As $\Lambda \to 0$, Eqs. (39)–(41) respectively, reduce to

$$R_{tc} = \frac{120}{1+M_1} \tag{42a}$$

$$R_{tc} = \frac{720}{1+M_1} \tag{42b}$$

$$R_{tc} = \frac{320}{1+M_1} \tag{42c}$$

These are the results for constant viscosity ferrofluids and coincide with Nanjundappa and Shivakumara [8]. When $M_1 = 0$ (i.e., ordinary viscous fluid), Eqs. 42(a–c) reduce to the critical Rayleigh numbers of $R_{tc} = 120,\ 720$, and 320 for free-free, rigid-rigid, and lower rigid-upper free boundaries, respectively, which are the known exact values documented in the literature. Equations (39)–(41) further reveal that the nonlinearity of fluid magnetization (i.e., M_3) has no effect on the onset of ferroconvection; a result which is observed by numerical computations carried out in the previous section. This result is similar to the one noted in the case of constant viscosity ferrofluids [8]. At the onset of convection $a_c = 0$ (very large wave length). Thus, one would expect that, M_3, has no effect on the stability of the system (cf. Eq. 19).

4 Results and Discussion

The linear stability analysis is carried out with viscosity depending exponentially on temperature at the onset of thermogravitational convection in a ferrofluid layer. The bounding surfaces of the ferrofluid layer are either free or rigid-ferromagnetic and insulated to temperature perturbations. The resulting eigenvalue problem is solved numerically by employing the Galerkin technique for free-free, rigid-rigid and lower rigid-upper free boundaries. It is noted that the critical wave number is vanishingly small and this fact is exploited to obtain an analytic expression for the critical Rayleigh number using a regular perturbation technique with wave number a as a perturbation parameter. Such a study also helps in knowing the accuracy of the numerical method employed in solving the problem.

The parameters which are influencing the criterion for the onset of convection are Λ and M_1. The salient characteristics of these parameters are exhibited graphically in Fig. 1 by exhibiting R_{tc} as a function of Λ for different values of M_1 as well as for different types of velocity boundary conditions, namely free-free, rigid-rigid and lower rigid- upper free. The critical Rayleigh numbers calculated from the corresponding analytic expression for the Rayleigh numbers are also exhibited in Fig. 1 by ($\times\ \times\ \times\times$). We note that the results obtained from a simple regular perturbation technique coincide exactly with those computed from time- consuming numerical methods and, thus, provide a justification for the analytically obtained results. From the figure, we note that Λ has a strong influence on thermogravitational ferroconvection. The effect of increase in the value of Λ is to hasten the onset of thermogravitational ferroconvection. In fact, R_{tc} decreases quite rapidly at first, then slowly, with increasing Λ. This is due to the decrease in viscosity of the ferrofluid with temperature. For all the boundary conditions considered, the maximum critical Rayleigh number, R_{tc}, exists at $\Lambda = 0$ (i.e., constant viscosity fluid case) and R_{tc} for rigid-rigid boundaries is greatest, followed by rigid-free boundaries and the least for free-free boundaries. This is because the rigid boundaries suppress perturbations the most and hence higher heating (i.e., higher critical Rayleigh number) is required for the onset of convection. The results for $M_1 = 0$ correspond to the case of ordinary viscous fluids and it is observed that higher heating is required to have instability in that case. However, increasing the value of M_1 is to decrease the value of critical Rayleigh number R_{tc}. Thus, increasing M_1 is to augment thermomagnetic convection. This is due to an increase in the destabilizing magnetic force. In other words, magnetized ferrofluids transport heat more efficiently than ordinary viscous fluids. Equations (39)–(41) can also be expressed as

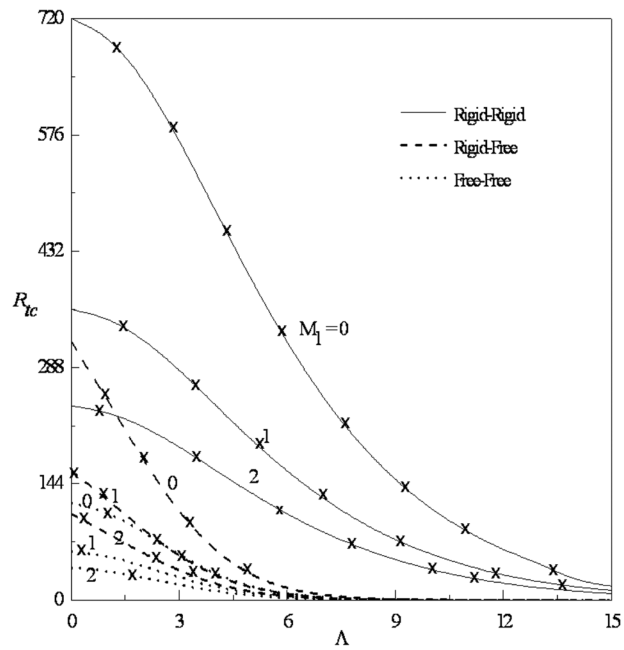

Fig. 1 Variation of R_{tc} as a function of Λ for different values of M_1

$$R_{mc} = \frac{\Lambda^6}{\sinh(\Lambda/2)\,[2\,\Lambda + \Lambda\,\cosh\Lambda - 3\,\sinh\Lambda]} - R_{tc} \tag{43}$$

$$R_{mc} = \frac{2\,\Lambda^5(2 + \Lambda^2 - 2\,\cosh\Lambda)}{[2\,\sinh(\Lambda/2) - \Lambda\,\cosh(\Lambda/2)]\,[4 + \Lambda^2 - 4\,\cosh\Lambda + \Lambda\,\sinh\Lambda)]} - R_{tc} \tag{44}$$

$$R_{mc} = \frac{8\,\Lambda^5 e^{\Lambda/2}(-2 + 2\,e^{\Lambda} - 2\,\Lambda\,e^{\Lambda} + \Lambda^2 e^{\Lambda})}{[10 + 10\,\Lambda + e^{\Lambda}(-18 + 3\,\Lambda^2 - 2\,\Lambda^3) + e^{2\Lambda}(6 - 6\,\Lambda + 6\,\Lambda^2) + e^{3\Lambda}(2 - 4\,\Lambda + \Lambda^2)]} - R_{tc} \tag{45}$$

where R_{mc} is the critical magnetic Rayleigh number. From the above equations, it is evident that there is a tight coupling between buoyancy and magnetic forces. The case $R_{tc} = 0$ corresponds to the case when the magnetic forces alone are in effect, while $R_{mc} = 0$ corresponds to the case when only the buoyancy forces are in effect. In either of these two cases, the critical stability parameters turn out to be the same. Further we note that heating from above is to increase the value of R_{mc} and thus makes the system more stable.

As $\Lambda \to 0$, Eqs. (43)–(45) respectively, reduce to

$$R_{mc} = 120 - R_{tc} \tag{46a}$$

$$R_{mc} = 720 - R_{tc} \tag{46b}$$

$$R_{mc} = 320 - R_{tc} \tag{46c}$$

The perturbed vertical velocity eigenfunction, $W(z)$, for different boundaries are presented in Figs. (2) and (3) for different values

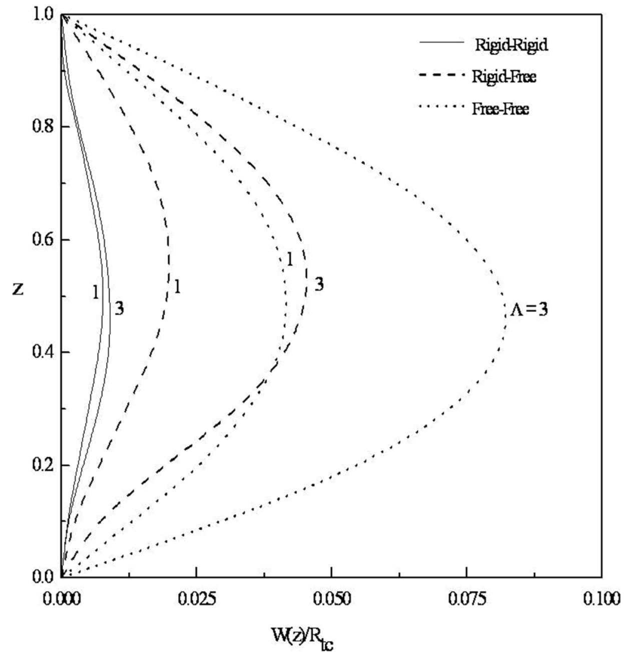

Fig. 2 Perturbed vertical velocity eigenfunction for two values of Λ when $M_1 = 2$

of temperature dependent viscosity, Λ, and the magnetic number, M_1, respectively. As can be seen, the shape of the eigenfunction is parabolic in nature and increasing the values of Λ (see Fig. 2) and M_1 (see Fig. 3) is to increase the vigor of the ferrofluid flow and hence their effects are to hasten the onset of ferroconvection. Also, the vertical velocity is suppressed more in the case of rigid-rigid boundaries when compared to lower rigid and upper free, as well as, to the free-free boundaries.

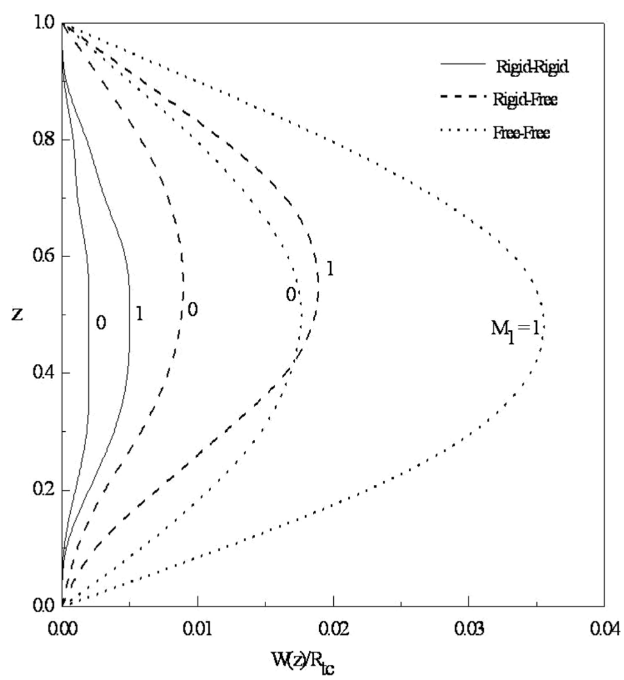

Fig. 3 Perturbed vertical velocity eigenfunction for two values of M_1 when $\Lambda = 2$

5 Conclusions

The following conclusions can be drawn from the foregoing study:

(i) The stability of the system is strongly dependent on the viscosity parameter Λ. Increasing the value of Λ and the magnetic parameter, M_1, is to hasten the onset of thermogravitational ferroconvection.
(ii) The numerically and analytically obtained results complement each other indicating the validity of the methods employed in solving the problem.
(iii) The nonlinearity of fluid magnetization given by M_3 has no effect on the criterion for the onset of thermogravitational ferroconvection.
(iv) As expected on physical grounds, $(R_{tc})_{\text{rigid-rigid}} > (R_{tc})_{\text{rigid-free}} < (R_{tc})_{\text{free-free}}$.
(v) The perturbed vertical velocity eigenfunction increases with an increase in the value of Λ as well as with M_1 and it is suppressed more in the case of rigid-rigid boundaries when compared to lower rigid-upper free and free-free boundaries.
(vi) There is a tight coupling between magnetic and buoyancy forces and in the absence of one or the other, the critical stability parameters remain the same.

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Nomenclature

a = overall horizontal wave number, $\sqrt{\ell^2 + m^2}$
$\vec{B}$ = magnetic induction
C = specific heat
$C_{V,H}$ = specific heat at constant volume and magnetic field
D = differential operator, d/dz
$\vec{H}$ = magnetic field intensity
k_t = thermal conductivity
K = pyromagnetic co-efficient, $-(\partial M/\partial T)_{H_0,T_a}$
$l,\ m$ = wave numbers in the x and y- directions, respectively
$\vec{M}$ = magnetization
M_1 = magnetic number, $\mu_0 K^2 \beta/(1+\chi)\alpha_t \rho_0 g$
M_2 = magnetic parameter, $\mu_0 T_a K^2/(1+\chi)(\rho_0 C)$
M_3 = measure of nonlinearity of magnetization, $(1 + M_0/H_0)/(1+\chi)$
p = pressure
$\vec{q}$ = velocity vector, (u, v, w)
R_t = thermal Rayleigh number, $\alpha_t g \beta\, d^4/\nu\,\kappa$
R_m = magnetic Rayleigh number, $\mu_0 K^2 \beta^2 d^4/(1+\chi)\,\eta_0 \kappa$
t = time
T = temperature
T_a = average temperature, $(T_l + T_u)/2$
$T_i^* s$ = modified Chebyshev polynomials
$x,\ y,\ z$ = Cartesian co-ordinates

Greek Symbols

α_T = thermal expansion coefficient
β = temperature gradient, $(T_l - T_u)/d$
χ = magnetic susceptibility, $(\partial M/\partial H)_{H_0,T_a}$
∇^2 = Laplacian operator, $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$
∇_h^2 = horizontal Laplacian operator, $\partial^2/\partial x^2 + \partial^2/\partial y^2$
κ = thermal diffusivity, $k_t/(\rho_0 C)$

$\Lambda =$ viscosity parameter, $\gamma\beta d$
$\mu_0 =$ magnetic permeability of vacuum
$\eta =$ variable fluid viscosity
$\phi =$ magnetic potential
$\gamma =$ positive constant
$\rho =$ fluid density
$\rho_0 =$ density at $T = T_a$
$v =$ kinematic viscosity, η_0/ρ_0

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