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A Nonparametric Control Chart for Location Based on Sub-Samples of Size Two

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Abstract. Control charts are widely used in statistical process control to detect changes in a production process and for monitoring a process to make sure that it is in control. In conventional statistical process control, the pattern of chance causes is often assumed to follow normal distribution. It is well known that the assumption of normality or any specific parametric form for the process distribution is too restrictive. In such situations, distribution-free or nonparametric control charts can serve the purpose better. In this paper, distribution-free control charts are developed based on a class of one-sample test statistics with sub-samples of size two due to Mehra, Prasad and Madhava Rao (1990). The control charts based on their statistic (T'_a-chart) are easy to understand and to use. The performance of the proposed procedures is studied through the average run length, which is the expected number of samples required by the procedure to signal out of control. It is observed that the performance of the proposed chart is better than the existing charts in the literature.

Keywords. Shewhart Control Charts, Distribution-Free Charts for Location, Average Run Length, One-Sample Location Problem.

2010 Mathematics Subject Classification. 62G10, 62P10.

1 Introduction

Control charts are very useful in detecting a shift in the process average of industrial production processes. The problems is to have the control charts designed adequately. This is difficult because the design depends heavily on the process probability distribution which however is unknown. There are three possibilities to overcome this problem.

(i) The most widespread is to select a standard probability distribution in most cases the normal distribution.

(ii) A second one is to do without a specific process distribution and select a so-called nonparametric approach which is also called distribution-free approach.

(iii) A third hardly known approach would be to derive a stochastic model, which is called Bernoulli space, as described in [5] and base the control chart design on this mode. This third approach represents in some sense in intermediate approach as it utilizes all available information, but nothing else.

In this paper the second alternative is selected and a nonparametric approach is used to design the control charts. Nonparametric control charts have the advantage that they are not based on a possibly wrong and misleading probability distribution, but are independent of the unknown process probability distribution. However, because of the disadvantage of a larger sample size, there are only relatively few published articles and they are hardly used in practice.

Reynolds et al. [14] proposed Shewhart VSI (variable sampling interval) charts, and provided the results that show the advantages of such charts. Many other works on this may be found in Reynolds et al. [13], Rendtel [12], Amin and Hemasinha [1] and several others. It is standard practice to design control charts on the assumption that the distribution of the sample mean is (approximately) normal. When the distribution of the observations is not normal there are potential problems with control charts that are based on the normal approximation. Parent [10] developed control charts based on the signed sequential ranks of the observations. These charts involve the ranking of long sequences of observations, however, the exact properties of these charts are not easy to evaluate. McGilchrist and Woodyer [8] developed a distribution-free cumulative sum CUSUM technique for use in monitoring rainfall amounts. Bakir and Reynolds [4] developed a nonparametric (CUSUM) control chart using the Wilcoxon signed-rank statistic. The ranking is done within groups where the groups are either samples taken at each point or else artificially defined groups. Park [11] suggested the use of nonparametric control charts with asymmetrical distributions, and he included the Shewhart sign chart and the signed rank chart in his study.

Amin and Searcy [3] studied the behavior of the EWMA control chart using the Wilcoxon signed-rank statistic. Hackl and Ledolter [6, 7] also considered nonparametric control charts. Amin, Reynolds and Bakir [2] developed nonparametric quality control charts based on the sign statistic. In this paper, we develop distribution-free control charts based on a class of one-sample test statistics with sub-samples of size two due to Mehra, Prasad and Madhava Rao [9]. The control charts based on Mehra, Prasad and Madhava Rao's statistic are easy to understand and use.

The performance of the proposed procedures will be studied through the average run length (ARL), which is the expected number of samples required by the procedure to signal out of control.

In Section 2, the new distribution-free control chart for location based on Mehra, Prasad and Madhava Rao [9] is proposed and the procedure of detecting the shift using the new chart is described. Section 3 contains the performance of the new control chart in terms of ARL values, which are computed using simulation. In Section 4 some remarks and conclusions are given.

2 Control Charts based on Mehra, Prasad and Madhava Rao's Statistic

Let X be the characteristic of interest that follows a distribution with distribution function F_X. A sample of n observations is taken at a regular time interval from the process. Let $X_i = (X_{i1}, X_{i2}, \dots, X_{in})$ denote the ith sample for $i = 1, 2, \dots$

Define

$$\phi(x_{ij}, x_{ik}) = \begin{cases} 1 & \text{for } \min\{x_{ij}, x_{ik}\} > 0, \\ a\ (-a) & \text{for } x_{ij}x_{ik} < 0 \text{ and } x_{ij} + x_{ij} > 0\ (< 0), \\ -1 & \text{for } \max\{x_{ij}, x_{ik}\} < 0, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where x_{ij} is the jth observation from the ith sample.

Let

$$T'_{ai} = T'_{n,a} = \sum_{j=1}^{n} \sum_{k=j}^{n} \phi(x_{ij}, x_{ik}) \tag{2}$$

be the statistic.

As noted in [9], the distribution of T'_{ai} is symmetric about zero. If the target value μ_0 is not zero, then μ_0 has to be subtracted from each observation x_{ij}, $j = 1, 2, \dots, n$, before computing the statistic T'_{ai}. Since $E[T'_{ai}] = 0$ and the distribution of T'_{ai} is symmetric about zero, the chart based on T'_{ai} (T'_a-chart) signals that the shift has occurred if $|T'_{ai}| > c$, where c is a specified constant.

Since T'_{ai} can be computed using Wilcoxon signed rank statistic as indicated in Section 2, it is easily available. However, T'_a depends on the constant a, which is estimated using sample observations. The procedure described is very useful when the underlying distribution of the process is unknown. For comparing the chart based on T'_{ai} with other procedures in the literature, we need to compute ARL(μ), which is the average number of samples required to signal a process shifts. The ARL for a chart is given by

$$\text{ARL}(\mu) = \frac{1}{\beta}, \tag{3}$$

where β is a detecting power of the considered chart.

For the T_a'-chart, we have

$$\begin{aligned}\beta &= P_\mu(\{t'_{ai} : |t'_{ai}| > c\}) \\ &= 1 - P_\mu(\{t'_{ai} : -c \le t'_{ai} \le c\}),\end{aligned} \tag{4}$$

where t'_{ai} denotes a realization of T_a'.

3 Performance of T_a'-Chart

The performance of the T_a'-control chart for location is evaluated in terms of ARL assuming various underlying process distributions. The performance is analyzed by means of simulation and comparisons with other control charts proposed and investigated in literature.

Table 1 and 2 give the simulated ARL values of the T_a'-chart for samples of size 10 and 11, respectively, when the underlying process distribution is Uniform, Normal, Laplace, Logistic, Triangular and Cauchy.

In Table 3 and 4, we present a comparison of the ARL values for Shewhart charts using sign statistic (SN_i), X-bar, Stephenson's statistic $S_{m,n}$ (Shetty and Nadaf [15]) and T_a'.

4 Remarks and Conclusion

From Table 1 and Table 2, we see that the ARL values of the T_a'-chart decrease as the shift increases, and the decrease is drastic for all considered cases when the shift is 0.5 or larger. In case of light tailed distributions considered here, the ARL values decrease at higher rate for the triangular distribution than for the uniform distribution. However, in case of heavy tailed distributions, the rate of decrease in ARL is highest for the Laplace distribution.

From Table 3 and Table 4, we observe that the out-of-control ARL values of the T_a'-chart are smaller than that of charts based on $S_{m,n}$, sign statistic (SN_i), and X-bar for different shifts considered here except for the relatively small shift 0.25 in the case that the process distribution is assumed to be Normal. However, when the process distribution is Laplace, the out-of-control ARL values of T_a'-chart are smaller than those of charts based on $S_{m,n}$, sign statistic (SN_i), and X-bar for different shifts considered here. But, in case of Cauchy distribution, out-of-control ARL values are less than those of the charts based on $S_{m,n}$, sign statistic (SN_i), and X-bar only for small shifts (less than 0.5). Here, it is to be noted that small values of out-of-control ARL indicate that the chart detects the shifts of the process early. Hence, it can be concluded that the T_a'-chart detects the shifts early when the process distribution is Normal or Laplace or Cauchy.

Shift	Uniform	Normal	Laplace	Logistic	Cauchy	Triangular
0.00	512.00	512.00	512.00	512.00	512.00	512.00
0.25	159.87	96.97	40.89	138.58	124.18	67.13
0.50	37.61	20.53	18.71	37.85	47.53	20.82
0.75	12.67	5.42	3.74	16.53	18.38	4.84
1.00	3.61	3.23	2.38	6.31	9.94	2.38
1.25	2.13	2.00	1.71	3.63	5.89	1.52
1.50	1.29	1.38	1.37	2.85	3.78	1.11
1.75	1.00	1.16	1.14	2.47	2.82	1.03
2.00	1.00	1.04	1.07	1.68	2.66	1.00
2.25	1.00	1.01	1.04	1.32	2.52	1.00
2.50	1.00	1.00	1.00	1.23	1.44	1.00
2.75	1.00	1.00	1.00	1.14	1.23	1.00
3.00	1.00	1.00	1.00	1.08	1.00	1.00

Table 1. ARL values for a control chart based on T'_a when $n = 10$.

Shift	Uniform	Normal	Laplace	Logistic	Cauchy	Triangular
0.00	1024	1024	1024	1024	1024	1024
0.25	138.35	88.52	35.62	100.42	129.83	57.32
0.50	26.28	16.29	6.98	37.18	42.62	12.22
0.75	8.78	4.89	3.48	15.38	17.74	3.97
1.00	2.84	2.16	2.15	5.94	8.57	2.09
1.25	1.69	1.58	1.46	3.86	4.93	1.33
1.50	1.02	1.19	1.23	2.38	3.57	1.08
1.75	1.00	1.06	1.12	1.79	2.01	1.01
2.00	1.00	1.00	1.03	1.36	1.71	1.00
2.25	1.00	1.00	1.02	1.19	1.60	1.00
2.50	1.00	1.00	1.00	1.08	1.58	1.00
2.75	1.00	1.00	1.00	1.04	1.31	1.00
3.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 2. ARL values for a control chart based on T'_a when $n = 11$.

	Shift	$S_{m,m}$ $m = 2$	SN_i $a_2 = 10$	$\overline{X}_i$ $a_1 = 0.95$	T_{ai}
Normal	0.00	512.0	512.0	512.0	512.0
	0.25	103.1	166.0	94.5	96.97
	0.50	22.8	40.0	15.4	20.53
	1.00	3.4	5.6	1.9	3.23
	2.00	1.1	1.3	1.0	1.04
Laplace	0.00	512.0	512.0	512.0	512.0
	0.25	42.7	75.4	135.9	40.89
	0.50	10.0	17.0	24.3	18.71
	1.00	2.5	3.7	2.3	2.38
	2.00	1.1	1.4	1.0	1.07
Cauchy	0.00	512.0	512.0	512.0	512.0
	0.25	136.9	168.9	510.6	124.2
	0.50	45.5	40.0	508.2	47.5
	1.00	11.7	5.6	506.5	9.94
	2.00	3.5	1.3	500.7	2.7

Table 3. ARL values for control charts based on T'_a, Stephenson test ($S_{m,m}$), Shewhart Sign and $\overline{X}_i$ charts for various distributions when $m = 2$, $n = 10$ and $L(\mu_0) = 512$ and $\sigma = 1$.

For future research it would be of interest to use the third approach mentioned in the introduction to design control charts. In contrast to the nonparametric approach they utilize the available information about the underlying distribution and therefore it is to be expected that their performance is better than that of the here proposed chart.

	Shift	$S_{m,m}$ $m = 2$	SN_i $a_2 = 10$	$\overline{X}_i$ $a_1 = 0.95$	T_{ai}
Normal	0.00	1024	1024	1024	1024
	0.25	135.1	265.9	75.9	88.52
	0.50	31.7	81.1	12.4	16.29
	1.00	3.6	10.7	1.8	2.16
	2.00	1.0	1.0	1.0	1.0
Laplace	0.00	1024	1024	1024	1024
	0.25	153.8	169.0	94.9	35.62
	0.50	30.2	40.0	15.4	6.98
	1.00	4.1	5.6	1.9	2.15
	2.00	1.1	1.3	1.0	1.03

Table 4. ARL values for control charts based on T'_a, Stephenson test ($S_{m,m}$), Shewhart Sign and $\overline{X}_i$ charts for various distributions when $m = 2$, $n = 11$ and $L(\mu_0) = 1024$ and $\sigma = 1$.

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