General Mathematics Vol. 13, No. 2 (2005), 105–116

Some explicit values for ratios of Theta-functions

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

In his notebooks [9], Ramanujan recorded several values of theta-functions. B. C. Berndt and L-C. Zhang [6], Berndt and H. H. Chan [5] have proved all these evaluations. The main purpose of this paper is to establish several new explicit evaluations of ratios of theta-functions. We also explicitly determine $a(e^{-2\sqrt{2}\pi})$, $a(e^{-2\sqrt{2/9}\pi})$, $a(e^{-\sqrt{2}\pi})$ and $a(e^{-\sqrt{2/9}\pi})$, where $a(q)$ is the Borweins cubic theta-function.

2000 Mathematical Subject Classification: 33D15 , 33D20.

Key words and phrases: Class invariants, $P-Q$ eta-functions, Theta-functions.

1 Introduction

Ramanujan's general theta-function [2] is defined by

$$f(a,b)=\sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2},\ \ |ab|<1.$$

Let

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty,$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and

$$\chi(-q) = (q;q^2)_\infty,$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \mid q \mid < 1.$$

Let

$$z(r) := z(x,r) = {}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; x\right)$$

and

$$q_r := q(x,r) := exp\left(-\pi csc\left(\frac{\pi}{r}\right)\frac{{}_2F_1(\frac{1}{r},\frac{r-1}{r},1;1-x)}{{}_2F_1(\frac{1}{r},\frac{r-1}{r},1;x)}\right).$$

Let n denote a positive integer and assume that

$$\frac{n\ {}_2F_1(\frac{1}{r},\frac{r-1}{r};1;1-\alpha)}{{}_2F_1(\frac{1}{r},\frac{r-1}{r};1;\alpha)} = \frac{{}_2F_1(\frac{1}{r},\frac{r-1}{r};1;1-\beta)}{{}_2F_1(\frac{1}{r},\frac{r-1}{r};1;\beta)}, \tag{1.1}$$

where $r = 2,3,4,6$ and $0 < x < 1$. Then a modular equation of degree n is a relation between α and β induced by (1.2).

Ramanujan recorded 23 beautiful $P-Q$ eta-function identities involving quotients of eta-functions in Chapter 25 of his notebooks [9]. Three of these $P-Q$ identities have been proved by Berndt and Zhang [6] by classical means on employing various modular equations of Ramanujan [2, pp.156-160]. To establish the remaining $P-Q$ identities Berndt and Zhang

[3, pp.204-236] have used the theory of modular forms. Recently M. S. Mahadeva Naika [8] and S. Bhargava, C. Adiga and Mahadeva Naika [7] have obtained a new class of $P-Q$ identities on employing modular equations belonging to alternative theory of signature 4. These $P-Q$ identities are extremely useful in the computation of class invariants and the values of ratios of theta-functions.

In his notebooks [9], Ramanujan recorded several values of $\varphi(q)$. Each of these values and some new values of $\varphi(q)$ not claimed by Ramanujan have been proved by Berndt and Chan [5] on using Ramanujan's modular equations and class -invariants [4]. Also,they have been able to obtain an explicit evaluation of $a(e^{-2\pi})$, where $a(q)$ is the Borweins' cubic theta-function.

In Section 2, we establish relationships among t_n, s_n and u_n, where

$$t_n := \frac{q^{\frac{n-1}{24}}(-q^n;q^n)_\infty}{(-q;q)_\infty},$$

$$s_n := \frac{\varphi(-q)}{\varphi(-q^n)},$$

and

$$u_n = \frac{s_n}{t_n^3} := \frac{\psi(q)}{q^{\frac{n-1}{8}}\psi(q^n)},$$

where $n > 1$.

We use these relationships, to establish several new explicit evaluations of ratios of theta-functions. Also we obtain some new values for $a(q)$.

2 Evaluations of Ratios of Theta-functions and $a(q)$

Let the Ramanujan -Weber class invariants [4], [8] be defined by

$$G_n := 2^{-1/4} q^{-1/24} (-q; q^2)_\infty$$

and

$$g_n := 2^{-1/4} q^{-1/24} (q; q^2)_\infty,$$

where $q = e^{-\pi\sqrt{n}}$, n is a positive rational number.

In his notebooks [9], Ramanujan recorded the values for 107 class invariants or polynomials satisfied by them. In this section , using the values of g_n, we explicitly evaluate some of the values of ratios of theta-functions and $a(q)$. We also need the following Lemma, Theorems 2.1 and 2.2.

Lemma 1. Let

$$(2.1) \qquad t_n := \frac{q^{\frac{n-1}{24}} (-q^n; q^n)_\infty}{(-q; q)_\infty} \quad \textit{and} \quad s_n := \frac{\varphi(-q)}{\varphi(-q^n)},$$

then

$$(2.2) \qquad \frac{s_n}{t_n} = \frac{f(-q)}{q^{\frac{n-1}{24}} f(-q^n)} \quad \textit{and} \quad u_n = \frac{s_n}{t_n^3} := \frac{\psi(q)}{q^{\frac{n-1}{8}} \psi(q^n)}$$

where $n > 1$.

Proof. We have

$$\begin{aligned} \frac{s_n}{t_n} &= \frac{\varphi(-q)(-q; q)_\infty}{q^{\frac{n-1}{24}} \varphi(-q^n)(-q^n; q^n)_\infty} = \\ &= \frac{(q; q)_\infty (q; q^2)_\infty (-q; q)_\infty}{q^{\frac{n-1}{24}} (q^n; q^n)_\infty (q^n; q^{2n})_\infty (-q^n; q^n)_\infty}. \end{aligned}$$

From the above identity, we can easily obtain first of (2.2).

We have

$$\begin{aligned}\frac{s_n}{t_n^3} &= \frac{\varphi(-q)(-q;q)_\infty^3}{q^{\frac{n-1}{8}}\varphi(-q^n)(-q^n;q^n)_\infty^3} = \\ &= \frac{(q;q)_\infty(q;q^2)_\infty(-q;q)_\infty^3}{q^{\frac{n-1}{8}}(q^n;q^n)_\infty(q^n;q^{2n})_\infty(-q^n;q^n)_\infty^3} = \\ &= \frac{(q^2;q^2)_\infty(-q;q)_\infty}{q^{\frac{n-1}{8}}(q^{2n};q^{2n})_\infty(-q^n;q^n)_\infty}.\end{aligned}$$

From the above identity, we can obtain second of (2.2).

Theorem 2.1. If t_n, s_n, and u_n are defined as in Lemma 1, then

$$s_3^4 + s_3^4 t_3^{12} = 9t_3^{12} + s_3^8 \tag{2.3}$$

and

$$s_5^4 + 5t_5^6 = s_5^2 + s_5^2 t_5^6. \tag{2.4}$$

Proof of (2.3). Using Entries 10(ii) and 11(i) of Chapter 17 of Ramanujan's notebooks [2, pp. 122-123] in Entry 5(vii) of Chapter 19 of Ramanujan's notebooks [2, p.230] and then using (3.1) and (3.2) with $n = 3$ in the resultant identity, we deduce that

$$\left(\frac{s_3}{t_3^3}\right)^4 + s_3^4 = 9 + \left(\frac{s_3}{t_3^3}\right)^4 s_3^4.$$

On simplification, we obtain (2.3).

Proof of (2.4). Using Entries 10(ii) and 11(i) of Chapter 17 of Ramanujan's notebooks [2, pp. 122-123] in Entry 13(xii) of Chapter 19 of Ramanujan's notebooks [2, pp.281-282] and then using (2.1) and (2.2) with $n = 5$ in the resultant identity, we deduce that

$$\left(\frac{s_5}{t_5^3}\right)^2 + s_5^2 = 5 + \left(\frac{s_5}{t_5^3}\right)^2 s_5^2.$$

On simplification, we obtain (2.4).

Theorem 2.2. If t_n, s_n, and u_n are defined as in Lemma 1, then

$$u_3^4 + u_3^4 t_3^{12} = 9 + t_3^{12} u_3^8 \tag{2.5}$$

and

$$u_5^2 + u_5^2 t_5^6 = 5 + t_5^6 u_5^4. \tag{2.6}$$

The proof of Theorem 2.2 is similar to the proof of Theorem 2.1. We omit the details.

Theorem 2.3. We have

$$\frac{\varphi(-e^{-\sqrt{2}\pi})}{\varphi(-e^{-3\sqrt{2}\pi})} = \sqrt{3(\sqrt{3}-\sqrt{2})} \tag{2.7}$$

and

$$\frac{\psi(e^{-\sqrt{2}\pi})}{e^{\frac{-\sqrt{2}\pi}{4}}\psi(e^{-3\sqrt{2}\pi})} = \sqrt{3(\sqrt{3}+\sqrt{2})}. \tag{2.8}$$

Proof. Putting $n = 3$ in (2.1). Then, we see that

$$t_3 = \frac{q^{\frac{1}{12}}(q;q^2)_\infty}{(q^3;q^6)_\infty} = \frac{g_k}{g_{9k}}, \tag{2.9}$$

where $q = e^{\sqrt{k}\pi}$. Putting $k = 2$ in (2.9). Then, we find that

$$t_3^{12} = 49 - 20\sqrt{6}. \tag{2.10}$$

Using (2.10) in (2.3), we deduce that

$$(50 - 20\sqrt{6})s_3^4 = 9(49 - 20\sqrt{6}) + s_3^8. \tag{2.11}$$

Solving (2.11) and noting that $s_3 > 1$, we obtain (2.7).

Using (2.7) and (2.10) in the second of (2.2), we obtain (2.8).

Theorem 2.4. We have

$$\frac{\varphi(-e^{-\sqrt{2/9}\pi})}{\varphi(-e^{-\sqrt{2}\pi})} = \sqrt{\sqrt{3}-\sqrt{2}} \tag{2.12}$$

and

$$\frac{\psi(e^{-\sqrt{2/9}\pi})}{e^{\frac{-\sqrt{2}\pi}{12}}\psi(e^{-\sqrt{2}\pi})} = \sqrt{\sqrt{3}+\sqrt{2}}. \tag{2.13}$$

Proof. Putting $k = 2/9$ in (2.9). Then

$$t_3 = \frac{g_{2/9}}{g_2} = \sqrt[3]{\sqrt{3}-\sqrt{2}}. \tag{2.14}$$

Using (2.14) in (2.3), we obtain (2.11). Solving (2.11) and noting that $0 < s_3 < 1$, we obtain (2.12).

Using (2.12) and (2.14) in (2.2), we deduce (2.13).

Theorem 2.5. We have

$$\frac{\varphi(-e^{-\sqrt{2/3}\pi})}{\varphi(-e^{-\sqrt{6}\pi})} = \sqrt{\sqrt{3}(\sqrt{2}-1)} \tag{2.15}$$

and

$$\frac{\psi(e^{-\sqrt{2/3}\pi})}{e^{\frac{-\sqrt{2}\pi}{4\sqrt{3}}}\psi(e^{-\sqrt{6}\pi})} = \sqrt{\sqrt{3}(\sqrt{2}+1)}. \tag{2.16}$$

Proof. Putting $k = 2/3$ in (2.9). Then, we find that

$$t_3^{12} = 17 - 12\sqrt{2}. \tag{2.17}$$

Using (2.17) in (2.3), we see that

$$s_3^8 - (18 - 12\sqrt{2})s_3^4 + 9(17 - 12\sqrt{2}) = 0. \tag{2.18}$$

Solving (2.18), we obtain

$$s_3 = \frac{\varphi(-e^{\sqrt{2/3}\pi})}{\varphi(-e^{\sqrt{6}\pi})} = \sqrt{\sqrt{3}(\sqrt{2}-1)}.$$

Using (2.15) and (2.17) in (2.2), we deduce (2.16).

Theorem 2.6. We have

$$\frac{\varphi(-e^{-\sqrt{2/5}\pi})}{\varphi(-e^{-\sqrt{10}\pi})} = \sqrt{(5-2\sqrt{5})} \tag{2.19}$$

and

$$\frac{\psi(e^{-\sqrt{2/5}\pi})}{e^{\frac{-\pi}{\sqrt{10}}}\psi(e^{-\sqrt{10}\pi})} = \sqrt{(5+2\sqrt{5})}. \tag{2.20}$$

Proof. Putting $k = 5$ in (2.1). Then

$$t_5 = \frac{g_k}{g_{25k}} \tag{2.21}$$

where $q = e^{\sqrt{k}\pi}$.

Putting $k = 2/5$ in (2.21). Then, we find that

$$t_5 = \frac{g_{2/5}}{g_{10}} = \sqrt[3]{\sqrt{5}-2}. \tag{2.22}$$

Using (2.22) in (2.4), we deduce that

$$s_5^4 - (10-4\sqrt{5})s_5^2 + 5(9-4\sqrt{5}) = 0. \tag{2.23}$$

Solving (2.23) for s_5, we obtain (2.19).

Using (2.19) and (2.22) in the second of (2.2), we obtain (2.20).

Theorem 2.7. We have

(i)

$$a(e^{-2\sqrt{2}\pi}) = \frac{\sqrt{(\sqrt{2}-1)(3\sqrt{3}+2)}\ \Gamma^2(1/8)}{2^{9/4}\sqrt{3}\pi\Gamma(1/4)}, \tag{2.24}$$

(ii)

$$a(e^{\frac{-2\sqrt{2}\pi}{3}}) = \frac{\sqrt{(\sqrt{2}-1)(3\sqrt{3}-2)}\ \Gamma^2(1/8)}{2^{9/4}\pi\Gamma(1/4)}, \tag{2.25}$$

(iii)

$$a(e^{-\sqrt{2}\pi}) = \frac{\sqrt{(\sqrt{2}-1)(3\sqrt{3}-2)}\ \Gamma^2(1/8)}{2^{7/4}\sqrt{3}\pi\Gamma(1/4)}, \tag{2.26}$$

and

(iv)

$$a(e^{\frac{-\sqrt{2}\pi}{3}}) = \frac{\sqrt{(\sqrt{2}-1)(3\sqrt{3}+2)}\ \Gamma^2(1/8)}{2^{7/4}\pi\Gamma(1/4)}. \tag{2.27}$$

Proof of (i). By Theorem 6.1 in [4, p.116], we have

$$a(q^2) = (1/4)\varphi(q)\varphi(q^3)\left(\frac{\varphi^2(q)}{\varphi^2(q^3)} + \frac{3\varphi^2(q^3)}{\varphi^2(q)}\right). \tag{2.28}$$

Let $q = e^{-\sqrt{2}\pi}$. Then, using example (ii) of Chapter 17 of Ramanujan's second notebooks [2], we obtain

$$\varphi(-e^{-\sqrt{2}\pi}) = \frac{(\sqrt{2}-1)^{1/4}\Gamma(1/8)}{2^{7/8}\sqrt{\pi\Gamma(1/4)}}. \tag{2.29}$$

Using (2.28) in (2.7), we obtain

$$\varphi(-e^{-3\sqrt{2}\pi}) = \frac{(\sqrt{2}-1)^{1/4}\Gamma(1/8)}{2^{7/8}\sqrt{3(\sqrt{3}-\sqrt{2})}\sqrt{\pi\Gamma(1/4)}}. \tag{2.30}$$

Using (2.7), (2.29) and (2.30) in (2.28), we obtain (2.24).

Proof of (ii). Let $q = e^{\frac{-\sqrt{2}\pi}{3}}$. Using (2.29) in (2.12), we find that

$$\varphi(-e^{\frac{-\sqrt{2}\pi}{3}}) = \frac{(\sqrt{2}-1)^{1/4}\sqrt{\sqrt{3}-\sqrt{2}}\Gamma(1/8)}{2^{7/8}\sqrt{\pi\Gamma(1/4)}}. \tag{2.31}$$

Using (2.12), (2.29) and (2.31) in (2.28), we obtain (2.25).

Proof of (iii). From Entry 27(ii) of Chapter 16 of Ramanujan's notebooks [2, p.43], we have

$$2\sqrt[4]{\alpha}\psi(e^{-2\alpha}) = \sqrt[4]{\beta}e^{\alpha/4}\varphi(-e^{-\beta}), \alpha\beta = \pi^2. \tag{2.32}$$

Let $\alpha = \frac{\pi}{\sqrt{2}}$ and $\beta = \sqrt{2}\pi$. Substituting these values of α and β in (2.32), we see that

$$2\sqrt[4]{\frac{\pi}{\sqrt{2}}}\psi(e^{-\sqrt{2}\pi}) = \sqrt[4]{\sqrt{2}\pi}e^{\frac{\pi}{4\sqrt{2}}}\varphi(-e^{-\sqrt{2}\pi}).$$

Thus,

$$\psi(e^{-\sqrt{2}\pi}) = 2^{-3/4}e^{\frac{\pi}{4\sqrt{2}}}\varphi(-e^{-\sqrt{2}\pi}).$$

Using (2.29) in the above equation, we deduce that

$$\psi(e^{-\sqrt{2}\pi}) = \frac{e^{\frac{\pi}{4\sqrt{2}}}(\sqrt{2}-1)^{1/4}\Gamma(1/8)}{2^{13/8}\sqrt{\pi\Gamma(1/4)}}. \tag{2.33}$$

Using (2.33) in (2.8), we obtain

$$\psi(e^{-3\sqrt{2}\pi}) = \frac{e^{\frac{3\pi}{4\sqrt{2}}}(\sqrt{2}-1)^{1/4}\Gamma(1/8)}{2^{13/8}\sqrt{3(\sqrt{3}+\sqrt{2})\pi\Gamma(1/4)}}. \tag{2.34}$$

By Theorem 5.4 in [4, p.111], we have

$$a(q) = \psi(q)\psi(q^3)\left(\frac{\psi^2(q)}{\psi^2(q^3)} + 3q\frac{\psi^2(q^3)}{\psi^2(q)}\right). \tag{2.35}$$

Using (2.33), (2.34) and (2.8) in (2.35), we obtain (2.26).

Proof of (iv). From (2.13) and (2.33), we find that

$$\psi(e^{\frac{-\sqrt{2}\pi}{3}}) = \frac{e^{\frac{\pi}{12\sqrt{2}}}(\sqrt{2}-1)^{1/4}\sqrt{\sqrt{3}+\sqrt{2}}\Gamma(1/8)}{2^{13/8}\sqrt{\pi\Gamma(1/4)}}. \tag{2.36}$$

Using (2.33), (2.36) and (2.13) in (2.35), we deduce (2.27).

Remark: Identities (2.24)-(2.27) are new to the literature.

Acknowledgement

The authors are grateful to Prof. G. E. Andrews for his useful suggestions. The authors wish to thank Prof. B. C. Berndt and Prof. C. Adiga for their valuable suggestions during the preparation of the paper. The authors are also thankful to referee for his useful suggestions which improves the quality of the paper.

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