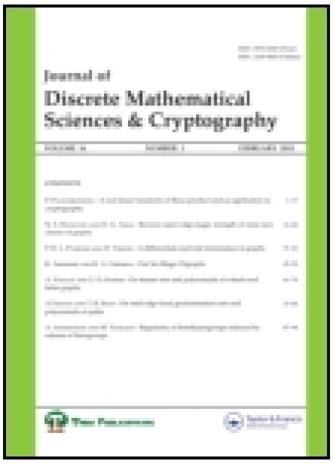

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Complementary total domination in graphs

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Complementary total domination in graphs

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Abstract

Let D be a minimum total dominating set of G. If $V - D$ contains a *total dominating set* (*TDS*) say S of G, then S is called a complementary total dominating set with respect to D. The complementary total domination number $\gamma_{ct}(G)$ of G is the minimum number of vertices in a *complementary total dominating set* (*CTDS*) of G. In this paper, exact values of $\gamma_{ct}(G)$ for some standard graphs are obtained. Also its relationship with other domination related parameters are investigated.

Keywords and phrases : *Graphs, domination, total domination, complementary total domination.*

Introduction

All the graphs considered here are finite, undirected with no loops and multiple edges. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges at a graph G, respectively. In general, we use $\langle X \rangle$ to

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denote the sub graph induced by the set of vertices X and $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v, respectively. For any undefined term in this paper, we refer the reader to Harary [6].

Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is said to be a dominating set of G, if every vertex in $V - D$ is adjacent to some vertex in D. The minimum cardinality of vertices in such a set is called the domination number of G and is denoted by $\gamma(G)$. For complete review on the topic of domination related parameters, see [7], [8] and [13]. Kulli, Soner and Sigarkanti, defined a similar concept, which they called "inverse domination", described here as "Complementary domination". Let D be a minimum dominating set of G. If $V - D$ contains a dominating set say X of G, then X is called a complementary dominating set with respect to D. The complementary domination number $\gamma_c(G)$ of G is the order of a smallest complementary dominating set of G, see [10] and [11]. Cockayne, Dawes and Hedetniemi [1] defined a concept of total domination. A set D is a *total dominating set* (TDS) if each vertex in V has atleast one neighbor in D. The total domination number $\gamma_t(G)$ is the minimum cardinality of a TDS of G. Analogously, we now define a complementary total domination number as follows. Let D be a minimum TDS of G. If $V - D$ contains a TDS say S of G, then S is called a *complementary total dominating set* ($CTDS$) with respect to D. The complementary total domination number $\gamma_{ct}(G)$ of G is the minimum number of vertices in a $CTDS$ of G.

A dominating set D of a graph G with $|D| = \gamma(G)$ is called γ-set. Similarly, the other types of dominating set are defined on the same line.

The following known results are used in the sequel.

Proposition A ([1]). *For any graph with no isolated vertices.*

(i) $\gamma_t(K_{r,s}) = \gamma_t(K_p) = \gamma_t(W_p) = 2$,

(ii) $$\gamma_t(C_p) = \begin{cases} \left(\frac{p}{2}\right) + 1, & \text{if } p \equiv 2 \pmod 4 \\ \left\lceil \frac{p}{2} \right\rceil, & \text{otherwise,} \end{cases}$$

(iii) $$\gamma_t(P_p) = \begin{cases} \left(\frac{p}{2}\right) + 1, & \text{if } p \equiv 2 \pmod 4 \\ \left\lceil \frac{p}{2} \right\rceil, & \text{otherwise,} \end{cases}$$

where $\lceil x \rceil$ is a least integer not less than x.

Theorem B ([1]). *For any graph G with no isolated vertices.*

(i) $\gamma_t(G) \leq p - \Delta + 1$,
(ii) $\gamma_t(G) \leq \frac{2p}{3}$ (*if* $p \geq 3$ *vertices*).

Theorem C [6]. *For any graph G with no isolated vertices,*

$$\alpha_1(G) + \beta_1(G) = p.$$

1. Specific values of complementary total domination numbers

In this section, we illustrate the complementary total domination number by presenting the value of $\gamma_{ct}(G)$ for several classes of graphs.

Proposition 1. *For any complete graph* K_p *with* $p \geq 4$ *vertices,*

$$\gamma_{ct}(K_p) = 2.$$

Proof. Let D be a minimum *TDS* of G. Since every vertex in G is adjacent to every vertex in $V - D$. Thus $V - D$ contains a *TDS* with $S \subseteq V - D$. Hence, $\gamma_{ct}(K_p) = |S| = |D| = 2$. □

Proposition 2. *For any complete bipartite graph* $K_{r,s}$ *with* $2 \leq r \leq s$,

$$\gamma_{ct}(K_{r,s}) = 2.$$

Proof. Let $V = V_1 \cup V_2$ be the vertex set of $K_{r,s}$ with $2 \leq r \leq s$, where $|V_1| = r$ and $|V_2| = s$. Let $D = \{(u,v) : u \in V_1 \text{ and } v \in V_2\}$ be a minimum *TDS* of $K_{r,s}$. Then the induced sub graph $\langle V - D \rangle$ is $K_{r-1,s-1}$. Thus $V - D$ contains a *TDS* with $S \subseteq V - D$. Hence $\gamma_{ct}(K_{r,s}) = \gamma_{ct}(K_{r-1,s-1}) = 2$. □

Proposition 3. *For any cycle graph* C_p *with* $p = 4n$, $n \geq 1$ *vertices,*

$$\gamma_{ct}(C_p) = \frac{p}{2}.$$

Proof. Let D be a minimum *TDS* of C_p. Then by Proposition A, we have $\gamma_t(C_p) = \frac{p}{2}$ and therefore $V - D$ contains a *TDS* of S. Thus $\gamma_{ct}(C_p) = \gamma_{ct}(C_p) = |S| = \frac{p}{2}$ if $p = 4n$, $n \geq 1$ vertices. □

Proposition 4. *Let* $G = C_p + uv$ *where chord* uv *forms two cycles* C_s *and* C_t *sharing edge* uv *(so* $p = s + t - 2$*). Then*

$$\gamma_{ct}(G) \leq \gamma_{ct}(C_s) + \gamma_{ct}(C_t). \quad (1)$$

Further, the bound is attained if and only if the vertices of cycles C_s and C_t are similar with $\{s,t\} = 4n; n \geq 1$.

Proof. First, we prove the upper bound. Since the vertices of a cycle are similar, there exists γ_{ct} -set of S and T of C_s and C_t, respectively, where u, $v \in S$ and u, $v \in T$. Then $S \cup T$ is a γ_{ct} -set of G, so (1) follows.

Now, we prove the second part. Suppose (1) holds. On the contrary, suppose the vertices of cycles C_s and C_t are not similar with $\{s,t\} \neq 4n$; $n \geq 1$. Then there exists cycle graph C_p with at least four vertices x, v, y and u with edges (xv), (vy), (yu) and (ux) such that $C_p + uv$ where chord uv forms two cycles C_s and C_t sharing edge uv (so $p = s + t - 2$). This implies that $\gamma_{ct}(C_p)$ exist. But $\gamma_{ct}(C_s)$ and $\gamma_{ct}(C_t)$ does not exist, a contradiction. Necessity is easy to check. □

Proposition 5. *For any wheel graph W_p with $p \geq 4$ vertices,*

$$\gamma_{ct}(W_p) = \begin{cases} \dfrac{(p+1)}{2}, & \textit{if } p \equiv 3 \pmod 4 \\ \dfrac{p}{2}, & \textit{otherwise.} \end{cases}$$

Proof. Let $W_p = W_p + C_{p-1}$ and u be a vertex of degree $p-1$. Then $D = \{u, v\}$ is a *TDS* of G. The *CTDS* of W_p is the minimum *TDS* of C_{p-1}. Thus,

$$\gamma_{ct}(W_p) = \gamma_t(C_{p-1}) = \begin{cases} \left[\dfrac{(p-1)}{2}\right] + 1, & \text{if } p-1 \equiv 2 \pmod 4 \\ \dfrac{p}{2}, & \text{otherwise.} \end{cases}$$

Therefore

$$\gamma_{ct}(W_p) = \begin{cases} \dfrac{(p+1)}{2}, & \text{if } p \equiv 3 \pmod 4 \\ \dfrac{p}{2}, & \text{otherwise.} \end{cases}$$ □

2. Bounds on complementary total domination number

Proposition 6. For any graph G with $p \geq 4$ vertices,

$$\gamma_t(G) \leq \gamma_{ct}(G) \leq p - \gamma_t(G). \tag{2}$$

Proof. Clearly, every *CTDS* is a *TDS* of G and by definition of $\gamma_{ct}(G)$, (2) follows. □

Proposition 7. *For any connected graph G with $p \geq 4$ vertices,*

(i) $2 \leq \gamma_{ct}(G) \leq p-2$.

(ii) $\left\lceil \frac{(d(G)+1)}{3} \right\rceil \leq \gamma_{ct}(G) \leq p - \left\lceil \frac{(d(G)+1)}{3} \right\rceil$, *where diameter $d(G)$ of a connected graph G is the length of any longest geodesic (a shortest $u-v$ path is called a geodesic).*

(iii) $\rho(G) \leq \gamma_{ct}(G) \leq p - \rho(G)$, *where packing number $\rho(G)$ is the maximum cardinality of packing set in G (a set X of V is a packing set of G if for each pair of vertices u and v in of X, $N[u] \cap N[v] = \varnothing$).*

(iv) $\delta(G) \leq \gamma_{ct}(G) \leq p - \delta(G)$, *if $g(G) \geq 5$.*

(v) $2(\delta(G)-1) \leq \gamma_{ct}(G) \leq p - 2(\delta(G)-1)$, *if $g(G) \geq 6$.*

(vi) $\Delta(G) \leq \gamma_{ct}(G) \leq p - \Delta(G)$, *if $g(G) \geq 7$ and $\delta(G) \geq 2$, where girth $g(G)$ is the length of a shortest cycle in a graph G that contains cycles.*

Proof. (i)-(vi) follows from (2) with the fact $\gamma_t(G) \geq 2$, $\gamma_t(G) \geq \left\lceil \frac{(d(G)+1)}{3} \right\rceil$, $\gamma_t(G) \geq \rho(G)$, $\gamma_t(G) \geq \delta(G)$ (if $g(G) \geq 5$), $\gamma_t(G) \geq 2(\delta(G)-1)$ (if $g(G) \geq 6$) and $\gamma_t(G) \geq \Delta(G)$ (if $g(G) \geq 7$ and $\delta(G) \geq 2$). □

Proposition 8. *For any connected graph G with $p \geq 4$ vertices.*

(i) $\gamma_t(G) + \gamma_{ct}(G) \leq 2p - \Delta(G) - 1$.

(ii) $\gamma_t(G) + \gamma_{ct}(G) \leq \frac{5p}{3} - 2$.

Proof. (i) and (ii) follows from Proposition 7 of (i) and Theorem B. □

To prove our next result we make use of the following definition.

A vertex x in a subset X of vertices of a graph G is redundant if its closed neighborhood is contained in the union of closed neighborhood of vertices of $X - \{x\}$, i.e., x may be removed from X without affecting the totality of accessible vertices. A set of vertices containing no redundant vertex is called irredundant. It is apparent that irredundance is a hereditary property and that any independent set is also irredundant. The irredundance number $\mathrm{ir}(G)$ is the minimum cardinalities taken over all maximal irredundant sets of vertices of G, see [4].

Proposition 9. *If G has no isolated vertex and $X \subseteq V$ is irredundant then $V - X$ is a* CTDS *of G.*

Proof. Let $x \in X \subseteq V$, we define $I(x, X) = N[x] - [X - \{x\}]$. If $I(x, X) \neq \varnothing$, then x is said to be irredundant and suppose x is not adjacent to any vertex of $V - X$. Thus $I(x, X) = \{x\}$ and x is not adjacent to any vertex of $X - \{x\}$. We conclude x is isolated in G, a contradiction. Therefore x is adjacent to some vertex of $V - X$ and this implies $V - X$ complementary total dominates G as required. □

Proposition 10. *For any graph G with $p \geq 4$ vertices,*

$$\gamma_{ct}(G) + \mathrm{ir}(G) \leq p\,. \tag{3}$$

Proof. If X is a largest irredundant set, then by Proposition 9, $V - X$ is a complementary total dominating and $|V - X| \geq \gamma_{ct}(G)$. Thus (3) follows. □

Proposition 11. *For any graph G, the complementary total domination exists if and only if G satisfies.*

(i) $\gamma_t(G) \leq \frac{p}{2}$.
(ii) *For every $u \in V(G), N(u) \not\subset D$.*

Proof. Suppose complementary total domination number exists for G. Now we establish (i) On the contrary, suppose $\frac{p}{2} < \gamma_t(G)$. Then by Proposition 6, $\gamma_t(G) \leq \gamma_{ct}(G)$. Hence $\gamma_t(G) < \frac{p}{2}$. Thus $p < \gamma_t(G) + \gamma_{ct}(G)$, which is a contradiction. Therefore $\gamma_t(G) \leq \frac{p}{2}$. Now we establish (ii) on the contrary, suppose $u \in V(G)$ is such that $N(u) \not\subset D$, where D is a *TDS* in G. Thus u is not adjacent to any vertex in $V - D$ and hence $V - D$ has no *CTDS* of G. This is a contradiction to the fact that $\gamma_{ct}(G)$ exists. Thus (ii) holds.

Conversely, suppose G satisfies (i) and (ii) it is easy to see that $\gamma_{ct}(G)$ exists for G. □

Proposition 12. *Let G be a graph with an end vertex. Then $\gamma_{ct}(G)$ does not exist.*

Proof. Let v be an end vertex of G and $u \in N(v)$. If D is a minimum *TDS* of G. If $v \in D$, then $u \in N(v)$ also belongs to D. By condition (ii) of Proposition 11. It is easy to see that $\gamma_{ct}(G)$ does not exist. If $u \in D$, then in $V - D$, u is not adjacent to any vertex and hence $V - D$ does not contain a *CTDS* of G. Thus $\gamma_{ct}(G)$ does not exist. □

To prove our next result we make use of the following definitions.

A graph G is k-total domination critical or just γ_{tk}-critical if G has a total domination number $\gamma_t(G) = k$ and for any edge $e \in E(\bar{G})$ and

$E(\bar{G}) \neq \phi, \gamma_t(G+e) = k-1$. For more details on this concept, see [2] and [12].

Observation 1. For any graph G and $uv \in E(\bar{G})$ such that $\gamma_t(G+e) < \gamma_t(G)$, every γ_t-set of $G+uv$ contains at least one of u and v.

Observation 2. If u and v are vertices of a graph G with $\text{dist}(u,v) = 2$, then $\gamma_t(G) - 1 \leq \gamma_t(G+e)$. Further, if G is a k-total domination critical graph with *diam* $(G) = 2$, then $\gamma_t(G+e) = k-1$ for any edge $e \in E(\bar{G})$.

Observation 3. For any edge $uv \in E(\bar{G}), \gamma_t(G) - 2 \leq \gamma_t(G+e) \leq \gamma_t(G)$.

Proposition 13. *Let G be a graph with no isolated vertices. Then G is a k-total domination critical graph if and only if $\gamma_t(G) = \alpha_1(G)$ and $\gamma_t(G+e) = \alpha_1(G) - 1$.*

Proof. Let G be a k-total domination critical graph. Then $\gamma_t(G) = k$ and $D = \{v_1, v_2, \ldots, v_k\}$ is a γ_t-set of G. By Theorem C, the set D covers all the vertices of G which is analogous to the edge covering $\alpha_1(G)$. Hence $\gamma_t(G) = k = |D| = \alpha_1(G)$. Suppose if any two non adjacent vertices say w_1 and w_2 are joined. Then by definition of k-total domination critical graph, we have $\gamma_t(G+e) = \alpha_1(G) - 1$.

Conversely, suppose $\gamma_t(G) = k = \alpha_1(G)$ and for every edge e not in G. Then by definition of k-total domination critical graph, G is a k-total domination critical graph. □

The corollary directly follows from Proposition 13 and Theorem C.

Corollary 13. *If G is a k-total domination critical graph, then $k + \beta_1(G) = p$.*

Proposition 14. *Let G be a k-total domination critical graph.*

(i) *If $\gamma_{tk}(G) > \frac{p}{2}$, then complementary γ_{tk}-critical does not exist.*
(ii) *If $\gamma_{tk}(G) = \frac{p}{2}$ and there is a vertex u such that $N(u) \subset D$, then complementary γ_{tk}-critical does not exist.*

Proof. (i) and (ii) follows from Proposition 12 and 13. □

3. Non-complementary total domination number

A total dominating set (*TDS*) D of a graph $G = (V, E)$ is a *non-complementary total dominating set* (*NTDS*), if $V - D$ is not contain a *TDS* of G. The non-complementary total domination number $\gamma_{nt}(G)$ of G is the minimum cardinality of a *NTDS* of G.

Observation 4. Every *NTDS* of G is a *TDS* of G, Clearly, $\gamma_t(G) \leq \gamma_{nt}(G)$. Further, let v be an end vertex of G and D be a γ_t-set of G. Then $v \notin D$ and adjacent to some vertex in D. This implies that v is an isolate in $\langle V - D \rangle$ and hence D is an γ_{nt}-set of G. Thus $\gamma_t(G) = \gamma_{nt}(G)$ follows.

Observation 5. For any graph G with no isolated vertices, $\gamma_t(G) = \gamma_{nt}(G) = p$ if and only if $G = mK_2$.

Proposition 15. *For any non trivial tree T,*

$$p - q + 1 \leq \gamma_{nt}(T) \leq p - \eta + 1\,.$$

Further, the lower bound is attained if and only if T is isomorphic with star or double star (*a double star is a tree with exactly two vertices of degree greater than one*) *and upper bound is attained if and only if T is isomorphic with star, where η is the number of end vertices.*

Proof. The lower bound follows from Observation 7 with the fact that $q = p - 1$ in tree T.

Now, we prove an upper bound. Let A be the set of all end vertices of T with $|A| = \eta$. Then for any end vertex $x \in A$, $(V - A) \cup \{v\}$ is a *NTDS* of G. Thus the upper bound follows.

We shall now show that the lower bound is attained if and only if T is isomorphic with star or double star. Suppose $\gamma_{nt}(T) = p - q + 1$ holds. On contrary, suppose T is neither a star nor a double star, then there exist at least three cut vertices and C is the set of all cut vertices, each cut vertices is adjacent to an end vertex, this implies that C is a *NTDS* of G and hence $\gamma_{nt}(T) > p - q + 1$, a contradiction. This proves necessity, sufficiency is obvious.

Finally, we shall show that the upper bound is attained if and only if T is isomorphic with star. Suppose $\gamma_{nt}(T) = p - \eta + 1$ holds. On contrary, suppose T is not a star, then there exist at least two cut vertices such that every cut vertex is adjacent to end vertex, this implies that $\gamma_{nt},(T) > p - q + 1$, a contradiction. This proves necessity, sufficiency is obvious. □

To prove our next result we make use of the following definitions.

A dominating set (or *TDS*) D of a connected graph G is a split (or, total split) dominating set if the induced sub graph $\langle V - D \rangle$ is disconnected. The split domination number$\gamma_s(G)$ (or, total split domination number $\gamma_{ts}(G)$) of G is the minimum cardinality of a split (or, total split) dominating set of G, see [9].

Proposition 16. *Let G be a graph with $\gamma_{nt}(G) < p-1$. Then one of the following holds,*

(i) $\gamma_{ts}(G) \leq \gamma_{nt}(G)$,

(ii) $\gamma_s(G) + 1 \leq \gamma_{nt}(G)$.

Proof. Let D be a γ_{nt}-set of G. Then either $\langle V - D \rangle$ contains an isolate or D is a *NTDS* of G. Suppose $\langle V - D \rangle$ contains an isolate. Since $|V - D| \geq 2$, $\langle V - D \rangle$ is disconnected and hence D is a total split dominating set of G and (i) hold. Suppose D is a *NTDS* of G. Then there exists a vertex $v \in D$ not adjacent to any vertex in $V - D$ and hence $\langle (V - D) \cup \{v\} \rangle$ is disconnected. This implies that $D - \{v\}$ is a split dominating set of G (ii) holds. □

4. Rank of adjacency matrices in complementary total domination

A graph is said to be reduced if no two vertices have the same set of neighbors. It is well known that for a given natural number r, there are finitely many reduced graphs of rank r. Let $m(r)$ denote the number of vertices of the largest reduced graph of rank r.

Observation 6. Adding isolated vertices to a graph does not change the rank of its adjacency matrix. So we may assume that our graphs have no isolated vertices.

Given any graph G, we define an equivalence relation on the vertices by setting $v \equiv w$ if v and w have the same set of neighbors. Each equivalence class is a coclique (A subset C of $V(G)$ is called a clique if every pair of vertices in C is joined by at least one edge, and no proper super set of C has this property); shrinking each class to a single vertex gives a reduced graph G_r. Conversely, any graph can be constructed from unique reduced graph by replacing the vertices by cocliques of appropriate sizes, and edges by complete bipartite graphs between the corresponding cocliques. We call this process blowing-up.

Proposition 17. $\text{rank}(G_r) = \text{rank}(G)$.

Proof. The adjacency matrix of G is obtained from that of G_r by replacing each 0 by a block of zeros and each 1 by blocks of 1s. Now the result follows by the interlacing theorem. See Chapter I of [5]. □

Observation 7. Let G be a graph with adjacency matrix A of rank r and with no isolated vertices. By Theorem 8.9.1 in Godsil and Royle [3], G contains an induced sub graph H of order r whose adjacency matrix B also has rank r. Note that H must be reduced, since otherwise H_r would be smaller than H and could not have rank r. Further, we have

$$A = \begin{pmatrix} B & BX \\ X^T B & X^T BX \end{pmatrix} \text{ for some (unique) matrix } X.$$

Observation 8. A total dominating set D of G is a minimal *TDS* if and only if for each vertex $v \in D$, $D - \{v\}$ is not a *TDS* of G.

Observation 9. If G has no isolated vertices, then

$$\gamma(G) \leq \{\gamma_t(G) = \gamma_t(G_r)\} \leq \{\gamma_{ct}(G) = \gamma_{ct}(G_r)\}.$$

Since a minimal *TDS* contains at most one point from each equivalence class. Further, a dominating set must either include every vertex in a given equivalence class, or at least one vertex dominating that class. So, if the equivalence classes are sufficiently large (bigger than $\gamma_t(G_r)$ will suffice), then a minimum dominating set contains at most one vertex from each equivalence class, and so is a *TDS*.

Proposition 18. *Let G be a graph with no isolated vertices, having rank r. Then*

$$\gamma(G) \leq \gamma_t(G) \leq \gamma_{ct}(G) \leq r.$$

Further, $\gamma(G) = \gamma_t(G) = \gamma_{ct}(G) = r$ if and only if each component of G is a complete bipartite graph $K_{r,s}$ with $2 \leq r \leq s$.

Proof. After re-ordering if necessary, the adjacency matrix of G has the form

$$A = \begin{pmatrix} B & BX \\ X^T B & X^T BX \end{pmatrix},$$

where B is an $r \times r$ matrix and $\text{rank}(B) = r$. Since there are no isolated vertices, BX has no zero columns, so every vertex outside the set consisting of the first r vertices has a neighbour among the first r vertices, which thus form a total dominating set. Suppose that this dominating set minimal. Then, for any $i \leq r$, there exists $j > r$ such that v_j is joined to v_i and to no other vertex among $v_1, \ldots, v_r$. Choose one such vertex v_j for such v_i

with $i \leq r$, and re-order the vertices so that $j = i + r$ for $i = 1, \ldots, r$. Let Y be the sub matrix consisting of the first r columns of X. Then $BY = I$, so $Y = B^{-1} + Y^T$, and $Y^T BY = B^{-1}$. It is easy to check that, if a graph H has the property that $A(H)^{-1}$ is the adjacency matrix of a graph A, then H is a matching and $A(H)^{-1} = A(H)$. Applying this to the subgraph H on $\{v_1, \ldots, v_r\}$, we see that this subgraph is a matching, and that v_{i+r} is joined to v_{j+r} if and only if v_i is joined to v_j. So the induced subgraph on $\{v_1, \ldots, v_{2r}\}$ is a disjoint union of 4-cycles. Suppose that w is an arbitrary vertex joined to more than one vertex v_i with $i \leq r$, say to $v_1, \ldots, v_k$. It is easy to see that $v_1, \ldots, v_k$ are pair wise non-adjacent. Now replace the neighbour of v_k in H by w to obtain another graph of rank r on r vertices which is not a matching, and hence not a minimal dominating set. Thus no such vertex exists. By observation 6, it follows that G is obtained by blowing up H, as claimed.

The converse is straightforward. Now suppose that G satisfies $\gamma_t(G) = \gamma_{ct}(G) = r$; without loss of generality, G is reduced. Our observations before the proposition show that it is possible to blow up G to a graph H with $\gamma(G) = r$. By previous part of proof, H (and hence G) is obtained by blowing up a matching. Again the converse is clear. □

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