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Some explicit values for ratios of Theta-functions

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

In his notebooks [9], Ramanujan recorded several values of theta-functions. B. C. Berndt and L-C. Zhang [6], Berndt and H. H. Chan [5] have proved all these evaluations. The main purpose of this paper is to establish several new explicit evaluations of ratios of theta-functions. We also explicitly determine $a(e^{-2\sqrt{2}\pi})$, $a(e^{-2\sqrt{2/9}\pi})$, $a(e^{-\sqrt{2}\pi})$ and $a(e^{-\sqrt{2/9}\pi})$, where $a(q)$ is the Borweins cubic theta-function.

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1 Introduction

Ramanujan's general theta-function [2] is defined by

$$f(a,b)=\sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \ \ |ab|<1.$$

Let

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty,$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and

$$\chi(-q) = (q;q^2)_\infty,$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \mid q \mid < 1.$$

Let

$$z(r) := z(x,r) = {}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; x\right)$$

and

$$q_r := q(x,r) := exp\left(-\pi csc\left(\frac{\pi}{r}\right) \frac{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}, 1; 1-x)}{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}, 1; x)}\right).$$

Let n denote a positive integer and assume that

$$n \frac{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\alpha)}{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; \alpha)} = \frac{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\beta)}{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; \beta)}, \tag{1.1}$$

where $r = 2, 3, 4, 6$ and $0 < x < 1$. Then a modular equation of degree n is a relation between α and β induced by (1.2).

Ramanujan recorded 23 beautiful $P-Q$ eta-function identities involving quotients of eta-functions in Chapter 25 of his notebooks [9]. Three of these $P-Q$ identities have been proved by Berndt and Zhang [6] by classical means on employing various modular equations of Ramanujan [2, pp.156-160]. To establish the remaining $P-Q$ identities Berndt and Zhang

[3, pp.204-236] have used the theory of modular forms. Recently M. S. Mahadeva Naika [8] and S. Bhargava, C. Adiga and Mahadeva Naika [7] have obtained a new class of $P-Q$ identities on employing modular equations belonging to alternative theory of signature 4. These $P-Q$ identities are extremely useful in the computation of class invariants and the values of ratios of theta-functions.

In his notebooks [9], Ramanujan recorded several values of $\varphi(q)$. Each of these values and some new values of $\varphi(q)$ not claimed by Ramanujan have been proved by Berndt and Chan [5] on using Ramanujan's modular equations and class -invariants [4]. Also,they have been able to obtain an explicit evaluation of $a(e^{-2\pi})$, where $a(q)$ is the Borweins' cubic theta-function.

In Section 2, we establish relationships among t_n, s_n and u_n, where

$$t_n := \frac{q^{\frac{n-1}{24}}(-q^n;q^n)_\infty}{(-q;q)_\infty},$$

$$s_n := \frac{\varphi(-q)}{\varphi(-q^n)},$$

and

$$u_n = \frac{s_n}{t_n^3} := \frac{\psi(q)}{q^{\frac{n-1}{8}}\psi(q^n)},$$

where $n > 1$.

We use these relationships, to establish several new explicit evaluations of ratios of theta-functions. Also we obtain some new values for $a(q)$.

2 Evaluations of Ratios of Theta-functions and $a(q)$

Let the Ramanujan -Weber class invariants [4], [8] be defined by

$$G_n := 2^{-1/4}q^{-1/24}(-q;q^2)_\infty$$

and

$$g_n := 2^{-1/4}q^{-1/24}(q;q^2)_\infty,$$

where $q = e^{-\pi\sqrt{n}}$, n is a positive rational number.

In his notebooks [9], Ramanujan recorded the values for 107 class invariants or polynomials satisfied by them. In this section , using the values of g_n, we explicitly evaluate some of the values of ratios of theta-functions and $a(q)$. We also need the following Lemma, Theorems 2.1 and 2.2.

Lemma 1. Let

$$t_n := \frac{q^{\frac{n-1}{24}}(-q^n;q^n)_\infty}{(-q;q)_\infty} \quad \textit{and} \quad s_n := \frac{\varphi(-q)}{\varphi(-q^n)}, \tag{2.1}$$

then

$$\frac{s_n}{t_n} = \frac{f(-q)}{q^{\frac{n-1}{24}}f(-q^n)} \quad \textit{and} \quad u_n = \frac{s_n}{t_n^3} := \frac{\psi(q)}{q^{\frac{n-1}{8}}\psi(q^n)} \tag{2.2}$$

where $n > 1$.

Proof. We have

$$\begin{aligned}\frac{s_n}{t_n} &= \frac{\varphi(-q)(-q;q)_\infty}{q^{\frac{n-1}{24}}\varphi(-q^n)(-q^n;q^n)_\infty} = \\ &= \frac{(q;q)_\infty(q;q^2)_\infty(-q;q)_\infty}{q^{\frac{n-1}{24}}(q^n;q^n)_\infty(q^n;q^{2n})_\infty(-q^n;q^n)_\infty}.\end{aligned}$$

From the above identity, we can easily obtain first of (2.2).

We have

$$\begin{aligned}\frac{s_n}{t_n^3} &= \frac{\varphi(-q)(-q;q)_\infty^3}{q^{\frac{n-1}{8}}\varphi(-q^n)(-q^n;q^n)_\infty^3} = \\ &= \frac{(q;q)_\infty(q;q^2)_\infty(-q;q)_\infty^3}{q^{\frac{n-1}{8}}(q^n;q^n)_\infty(q^n;q^{2n})_\infty(-q^n;q^n)_\infty^3} = \\ &= \frac{(q^2;q^2)_\infty(-q;q)_\infty}{q^{\frac{n-1}{8}}(q^{2n};q^{2n})_\infty(-q^n;q^n)_\infty}.\end{aligned}$$

From the above identity, we can obtain second of (2.2).

Theorem 2.1. If t_n, s_n, and u_n are defined as in Lemma 1, then

$$s_3^4 + s_3^4 t_3^{12} = 9t_3^{12} + s_3^8 \tag{2.3}$$

and

$$s_5^4 + 5t_5^6 = s_5^2 + s_5^2 t_5^6. \tag{2.4}$$

Proof of (2.3). Using Entries 10(ii) and 11(i) of Chapter 17 of Ramanujan's notebooks [2, pp. 122-123] in Entry 5(vii) of Chapter 19 of Ramanujan's notebooks [2, p.230] and then using (3.1) and (3.2) with $n = 3$ in the resultant identity, we deduce that

$$\left(\frac{s_3}{t_3^3}\right)^4 + s_3^4 = 9 + \left(\frac{s_3}{t_3^3}\right)^4 s_3^4.$$

On simplification, we obtain (2.3).

Proof of (2.4). Using Entries 10(ii) and 11(i) of Chapter 17 of Ramanujan's notebooks [2, pp. 122-123] in Entry 13(xii) of Chapter 19 of Ramanujan's notebooks [2, pp.281-282] and then using (2.1) and (2.2) with $n = 5$ in the resultant identity, we deduce that

$$\left(\frac{s_5}{t_5^3}\right)^2 + s_5^2 = 5 + \left(\frac{s_5}{t_5^3}\right)^2 s_5^2.$$

On simplification, we obtain (2.4).

Theorem 2.2. If t_n, s_n, and u_n are defined as in Lemma 1, then

$$u_3^4 + u_3^4 t_3^{12} = 9 + t_3^{12} u_3^8 \tag{2.5}$$

and

$$u_5^2 + u_5^2 t_5^6 = 5 + t_5^6 u_5^4. \tag{2.6}$$

The proof of Theorem 2.2 is similar to the proof of Theorem 2.1. We omit the details.

Theorem 2.3. We have

$$\frac{\varphi(-e^{-\sqrt{2}\pi})}{\varphi(-e^{-3\sqrt{2}\pi})} = \sqrt{3(\sqrt{3}-\sqrt{2})} \tag{2.7}$$

and

$$\frac{\psi(e^{-\sqrt{2}\pi})}{e^{\frac{-\sqrt{2}\pi}{4}}\psi(e^{-3\sqrt{2}\pi})} = \sqrt{3(\sqrt{3}+\sqrt{2})}. \tag{2.8}$$

Proof. Putting $n = 3$ in (2.1). Then, we see that

$$t_3 = \frac{q^{\frac{1}{12}}(q;q^2)_\infty}{(q^3;q^6)_\infty} = \frac{g_k}{g_{9k}}, \tag{2.9}$$

where $q = e^{\sqrt{k}\pi}$. Putting $k = 2$ in (2.9). Then, we find that

$$t_3^{12} = 49 - 20\sqrt{6}. \tag{2.10}$$

Using (2.10) in (2.3), we deduce that

$$(50 - 20\sqrt{6})s_3^4 = 9(49 - 20\sqrt{6}) + s_3^8. \tag{2.11}$$

Solving (2.11) and noting that $s_3 > 1$, we obtain (2.7).

Using (2.7) and (2.10) in the second of (2.2), we obtain (2.8).

Theorem 2.4. We have

$$\frac{\varphi(-e^{-\sqrt{2/9}\pi})}{\varphi(-e^{-\sqrt{2}\pi})} = \sqrt{\sqrt{3}-\sqrt{2}} \tag{2.12}$$

and

$$\frac{\psi(e^{-\sqrt{2/9}\pi})}{e^{\frac{-\sqrt{2}\pi}{12}}\psi(e^{-\sqrt{2}\pi})} = \sqrt{\sqrt{3}+\sqrt{2}}. \tag{2.13}$$

Proof. Putting $k = 2/9$ in (2.9). Then

$$t_3 = \frac{g_{2/9}}{g_2} = \sqrt[3]{\sqrt{3}-\sqrt{2}}. \tag{2.14}$$

Using (2.14) in (2.3), we obtain (2.11). Solving (2.11) and noting that $0 < s_3 < 1$, we obtain (2.12).

Using (2.12) and (2.14) in (2.2), we deduce (2.13).

Theorem 2.5. We have

$$\frac{\varphi(-e^{-\sqrt{2/3}\pi})}{\varphi(-e^{-\sqrt{6}\pi})} = \sqrt{\sqrt{3}(\sqrt{2}-1)} \tag{2.15}$$

and

$$\frac{\psi(e^{-\sqrt{2/3}\pi})}{e^{\frac{-\sqrt{2}\pi}{4\sqrt{3}}}\psi(e^{-\sqrt{6}\pi})} = \sqrt{\sqrt{3}(\sqrt{2}+1)}. \tag{2.16}$$

Proof. Putting $k = 2/3$ in (2.9). Then, we find that

$$t_3^{12} = 17 - 12\sqrt{2}. \tag{2.17}$$

Using (2.17) in (2.3), we see that

$$s_3^8 - (18 - 12\sqrt{2})s_3^4 + 9(17 - 12\sqrt{2}) = 0. \tag{2.18}$$

Solving (2.18), we obtain

$$s_3 = \frac{\varphi(-e^{\sqrt{2/3}\pi})}{\varphi(-e^{\sqrt{6}\pi})} = \sqrt{\sqrt{3}(\sqrt{2}-1)}.$$

Using (2.15) and (2.17) in (2.2), we deduce (2.16).

Theorem 2.6. We have

$$\frac{\varphi(-e^{-\sqrt{2/5}\pi})}{\varphi(-e^{-\sqrt{10}\pi})} = \sqrt{(5-2\sqrt{5})} \tag{2.19}$$

and

$$\frac{\psi(e^{-\sqrt{2/5}\pi})}{e^{\frac{-\pi}{\sqrt{10}}}\psi(e^{-\sqrt{10}\pi})} = \sqrt{(5+2\sqrt{5})}. \tag{2.20}$$

Proof. Putting $k=5$ in (2.1). Then

$$t_5 = \frac{g_k}{g_{25k}} \tag{2.21}$$

where $q = e^{\sqrt{k}\pi}$.

Putting $k = 2/5$ in (2.21). Then, we find that

$$t_5 = \frac{g_{2/5}}{g_{10}} = \sqrt[3]{\sqrt{5}-2}. \tag{2.22}$$

Using (2.22) in (2.4), we deduce that

$$s_5^4 - (10-4\sqrt{5})s_5^2 + 5(9-4\sqrt{5}) = 0. \tag{2.23}$$

Solving (2.23) for s_5, we obtain (2.19).

Using (2.19) and (2.22) in the second of (2.2), we obtain (2.20).

Theorem 2.7. We have

(i)

$$a(e^{-2\sqrt{2}\pi}) = \frac{\sqrt{(\sqrt{2}-1)(3\sqrt{3}+2)}\;\Gamma^2(1/8)}{2^{9/4}\sqrt{3}\pi\Gamma(1/4)}, \tag{2.24}$$

(ii)

$$a(e^{\frac{-2\sqrt{2}\pi}{3}}) = \frac{\sqrt{(\sqrt{2}-1)(3\sqrt{3}-2)}\ \Gamma^2(1/8)}{2^{9/4}\pi\Gamma(1/4)}, \tag{2.25}$$

(iii)

$$a(e^{-\sqrt{2}\pi}) = \frac{\sqrt{(\sqrt{2}-1)(3\sqrt{3}-2)}\ \Gamma^2(1/8)}{2^{7/4}\sqrt{3}\pi\Gamma(1/4)}, \tag{2.26}$$

and

(iv)

$$a(e^{\frac{-\sqrt{2}\pi}{3}}) = \frac{\sqrt{(\sqrt{2}-1)(3\sqrt{3}+2)}\ \Gamma^2(1/8)}{2^{7/4}\pi\Gamma(1/4)}. \tag{2.27}$$

Proof of (i). By Theorem 6.1 in [4, p.116], we have

$$a(q^2) = (1/4)\varphi(q)\varphi(q^3)\left(\frac{\varphi^2(q)}{\varphi^2(q^3)} + \frac{3\varphi^2(q^3)}{\varphi^2(q)}\right). \tag{2.28}$$

Let $q = e^{-\sqrt{2}\pi}$. Then, using example (ii) of Chapter 17 of Ramanujan's second notebooks [2], we obtain

$$\varphi(-e^{-\sqrt{2}\pi}) = \frac{(\sqrt{2}-1)^{1/4}\Gamma(1/8)}{2^{7/8}\sqrt{\pi\Gamma(1/4)}}. \tag{2.29}$$

Using (2.28) in (2.7), we obtain

$$\varphi(-e^{-3\sqrt{2}\pi}) = \frac{(\sqrt{2}-1)^{1/4}\Gamma(1/8)}{2^{7/8}\sqrt{3(\sqrt{3}-\sqrt{2})}\sqrt{\pi\Gamma(1/4)}}. \tag{2.30}$$

Using (2.7), (2.29) and (2.30) in (2.28), we obtain (2.24).

Proof of (ii). Let $q = e^{\frac{-\sqrt{2}\pi}{3}}$. Using (2.29) in (2.12), we find that

$$\varphi(-e^{\frac{-\sqrt{2}\pi}{3}}) = \frac{(\sqrt{2}-1)^{1/4}\sqrt{\sqrt{3}-\sqrt{2}}\Gamma(1/8)}{2^{7/8}\sqrt{\pi\Gamma(1/4)}}. \tag{2.31}$$

Using (2.12), (2.29) and (2.31) in (2.28), we obtain (2.25).

Proof of (iii). From Entry 27(ii) of Chapter 16 of Ramanujan's notebooks [2, p.43], we have

$$2\sqrt[4]{\alpha}\psi(e^{-2\alpha}) = \sqrt[4]{\beta}e^{\alpha/4}\varphi(-e^{-\beta}), \alpha\beta = \pi^2. \tag{2.32}$$

Let $\alpha = \frac{\pi}{\sqrt{2}}$ and $\beta = \sqrt{2}\pi$. Substituting these values of α and β in (2.32), we see that

$$2\sqrt[4]{\frac{\pi}{\sqrt{2}}}\psi(e^{-\sqrt{2}\pi}) = \sqrt[4]{\sqrt{2}\pi}e^{\frac{\pi}{4\sqrt{2}}}\varphi(-e^{-\sqrt{2}\pi}).$$

Thus,

$$\psi(e^{-\sqrt{2}\pi}) = 2^{-3/4}e^{\frac{\pi}{4\sqrt{2}}}\varphi(-e^{-\sqrt{2}\pi}).$$

Using (2.29) in the above equation, we deduce that

$$\psi(e^{-\sqrt{2}\pi}) = \frac{e^{\frac{\pi}{4\sqrt{2}}}(\sqrt{2}-1)^{1/4}\Gamma(1/8)}{2^{13/8}\sqrt{\pi\Gamma(1/4)}}. \tag{2.33}$$

Using (2.33) in (2.8), we obtain

$$\psi(e^{-3\sqrt{2}\pi}) = \frac{e^{\frac{3\pi}{4\sqrt{2}}}(\sqrt{2}-1)^{1/4}\Gamma(1/8)}{2^{13/8}\sqrt{3(\sqrt{3}+\sqrt{2})\pi\Gamma(1/4)}}. \tag{2.34}$$

By Theorem 5.4 in [4, p.111], we have

$$a(q) = \psi(q)\psi(q^3)\left(\frac{\psi^2(q)}{\psi^2(q^3)} + 3q\frac{\psi^2(q^3)}{\psi^2(q)}\right). \tag{2.35}$$

Using (2.33), (2.34) and (2.8) in (2.35), we obtain (2.26).

Proof of (iv). From (2.13) and (2.33), we find that

$$\psi(e^{\frac{-\sqrt{2}\pi}{3}}) = \frac{e^{\frac{\pi}{12\sqrt{2}}}(\sqrt{2}-1)^{1/4}\sqrt{\sqrt{3}+\sqrt{2}}\Gamma(1/8)}{2^{13/8}\sqrt{\pi\Gamma(1/4)}}. \tag{2.36}$$

Using (2.33), (2.36) and (2.13) in (2.35), we deduce (2.27).

Remark: Identities (2.24)-(2.27) are new to the literature.

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