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A Note on Cubic Modular Equations of Degree Two*

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Abstract

On Page 259 of his second notebook [3], Ramanujan recorded many cubic modular equations of degree 2. In this paper we establish several cubic modular equations of degree 2 akin to those in Ramanujan's work. As an application of our results, we also establish some new $P-Q$ eta-function identities.

Keywords and Phrases: *Cubic modular equations, Eta-function identities.*

1. A Family of Cubic Modular Equations

The ordinary hypergeometric series ${}_2F_1(a,b;c;x)$ is defined by

$$ {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n x^n}{(c)_n}, $$

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where

$$(a)_0 = 1, (a)_n = a(a+1)(a+2)...(a+n-1), for\ n \geq 1, \mid x \mid < 1.$$

Let

$$Z(r) := Z(r;x) :=_2 F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; x\right)$$

and

$$q_r := q_r(x) := \exp\left(-\pi \csc\left(\frac{\pi}{r}\right) \frac{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; 1-x)}{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; x)}\right).$$

where $r = 2, 3, 4, 6$ and $0 < x < 1$.

Let n denote a fixed natural number, and assume that

$$n\frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; \beta\right)}, \tag{1.1}$$

where $r = 2, 3, 4$ or 6. Then a modular equation of degree n in the theory of elliptic functions of signature r is a relation between α and β induced by (1.1). On Pages 257-262 of his second notebook [3, pp. 257-262], Ramanujan gives an outline of the theories of elliptic functions to alternate bases corresponding to the classical theory by way of statements of some results. Venkatachaliengar [4] examined some of these results. Proofs of all these identities can be found in [2, pp.122-123]. Recently, Adiga, Kim and Naika [1] also established some cubic modular equations in the theory of signature 3. Now we state a transformation formula which is useful in establishing several cubic equations of degree 2 in the theory of signature 3.

Lemma 1.1. (see [3, p. 258]). *If*

$$\alpha := \alpha(q) = \frac{p(3+p)^2}{2(1+p)^3} \text{ and } \beta := \beta(q) = \frac{p^2(3+p)}{4}, \tag{1.2}$$

then for $0 \leq p \leq 1$,

$${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \alpha\right) = (1+p){}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \beta\right). \tag{1.3}$$

For a proof of Lemma 1.1, see the work of Berndt [2, p. 112].

Theorem 1.1. *If* β *is of degree 2 over* α *in the theory of signature 3, then*

(i)

$$m^3 = 3\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{\frac{1}{3}}\left(\left(\frac{1-\beta}{\alpha}\right)^{\frac{1}{3}} - \left(\frac{\beta}{1-\alpha}\right)^{\frac{1}{3}}\right) + \frac{8}{m^3}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right), \tag{1.4}$$

(ii)

$$m^2\left(\frac{\alpha(1-\alpha)}{\beta^2(1-\beta)^2}\right)^{\frac{1}{3}} = m^6\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right) + \frac{4}{3}, \tag{1.5}$$

(iii)

$$m^4\left(\frac{\beta(1-\beta)}{\alpha^2(1-\alpha)^2}\right)^{\frac{1}{3}} = 16\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right) + \frac{m^6}{3}, \tag{1.6}$$

(iv)

$$\frac{8}{m^3} = \frac{\alpha}{\beta} - 3\left(\frac{\alpha(1-\alpha)^2}{\beta^2(1-\beta)}\right)^{\frac{1}{3}}, \tag{1.7}$$

(v)

$$m^3 = \frac{1-\beta}{1-\alpha} - 3\left(\frac{\beta^2(1-\beta)}{\alpha(1-\alpha)^2}\right)^{\frac{1}{3}}, \tag{1.8}$$

(vi)

$$m^3 = 3\left(\frac{\beta(1-\beta)^2}{\alpha^2(1-\alpha)}\right)^{\frac{1}{3}} - \frac{\beta}{\alpha}, \tag{1.9}$$

(vii)

$$\frac{8}{m^3} = 3\left(\frac{\alpha^2(1-\alpha)}{\beta(1-\beta)^2}\right)^{\frac{1}{3}} - \frac{1-\alpha}{\beta}, \tag{1.10}$$

(viii)

$$m = 3\left(\frac{\beta}{\alpha^2}\right)^{\frac{1}{3}} - \frac{4}{m^2}\frac{\beta}{\alpha} \tag{1.11}$$

(ix)

$$m^2 = 3\left(\frac{1-\alpha}{(1-\beta)^2}\right) - \frac{2}{m}\left(\frac{1-\beta}{1-\alpha}\right), \tag{1.12}$$

(x)

$$\left(\frac{\beta(1-\alpha)^2}{\alpha^2(1-\beta)}\right)^{\frac{1}{3}} = \left(\frac{(\alpha(1-\beta)^2)^{\frac{1}{3}} - 3(\beta^2(1-\alpha))^{\frac{1}{3}}}{3(\alpha(1-\beta)^2)^{\frac{1}{3}} - (\beta^2(1-\alpha))^{\frac{1}{3}}}\right) \tag{1.13}$$

and

(xi)

$$\left(\frac{\alpha(1-\beta)^2}{\beta^2(1-\alpha)}\right)^{\frac{1}{3}} = \left(\frac{(\beta(1-\alpha)^2)^{\frac{1}{3}} - 3(\alpha^2(1-\beta))^{\frac{1}{3}}}{3(\beta(1-\alpha)^2)^{\frac{1}{3}} - (\alpha^2(1-\beta))^{\frac{1}{3}}}\right). \tag{1.14}$$

Proof of (1.4). From (1.2), by elementary calculations, we have

$$1-\alpha = \frac{(1-p)^2(1+p)}{2(1+p)^3} \text{ and } 1-\beta = \frac{(1-p)(2+p)^2}{4} \tag{1.15}$$

Using (1.2) and (1.15) in (1.4), we find that

$$3\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{\frac{1}{3}}\left(\left(\frac{1-\beta}{\alpha}\right)^{\frac{1}{3}} - \left(\frac{\beta}{1-\alpha}\right)^{\frac{1}{3}}\right) + \frac{8}{m^3}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)$$

$$= \quad (1+p)^3 = m^3.$$

This completes the proof of (1.4).

The proofs of the identities (1.5) to (1.15) are similar to the proof of (1.4). We omit the details.

2. P-Q Eta-Function Identities

Following Ramanujan's work, we define

$$\varphi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2},$$

$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}$$

and

$$f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \mid q \mid < 1.$$

In this section we obtain some new $P-Q$ eta-function identities on employing modular equations in Section 2 and the following lemma:

Lemma 2.1. *For* $0 < x < 1$,

$$b(q) = (1-x)^{\frac{1}{3}} z = \frac{f^3(-q)}{f(-q^3)} \quad \textit{and} \quad c(q) = x^{\frac{1}{3}} z = \frac{3q^{\frac{1}{3}} f^3(-q^3)}{f(-q)}. \tag{2.1}$$

For a proof of Lemma 2.1, see [2, p.109].

Theorem 2.1. (see [3, p. 327]).*Let*

$$P = \frac{f(-q^2)}{q^{\frac{1}{24}} f(-q^3)} \quad and \quad Q = \frac{f(-q)}{q^{\frac{5}{24}}}. \tag{2.2}$$

Then

$$(PQ)^2 - \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3. \tag{2.3}$$

Proof. Using (2.1) in (1.4) and then using (2.2), we obtain

$$1 = \frac{P^5}{Q} + \frac{9P}{Q^5} + \frac{8P^6}{Q^6}. \tag{2.4}$$

On simplification, we obtain (2.3).

Theorem 2.2. Let

$$P = \frac{\psi^4(q)}{q\psi^4(q^3)} \quad and \quad Q = \frac{\psi^4(q^2)}{q^2\psi^4(q^6)}. \tag{2.5}$$

Then

$$P^2 \left(\frac{P-9}{P-1}\right) = Q \left(\frac{Q-9}{Q-1}\right)^2. \tag{2.6}$$

Proof. Using (2.1) in (1.13), we find that

$$\frac{\varphi^4(-q)}{\varphi^4(-q^3)} = \frac{\psi^4(q) - 9q\psi^4(q^3)}{\psi^4(q) - q\psi^4(q^3)}. \tag{2.7}$$

Using Entry 24(ii) and (iv) of Chapter 16 of Ramanujan's second notebook [3, p. 198] in (2.7), we obtain

$$\frac{f^6(-q)}{q^{\frac{1}{2}}f^6(-q^3)} = \frac{\psi^2(q)}{q^{\frac{1}{2}}\psi^2(q^3)}\frac{\psi^4(q)-9q\psi^4(q^3)}{\psi^4(q)-q\psi^4(q^3)} \tag{2.8}$$

and

$$\frac{f^{12}(-q^2)}{f^{12}(-q^6)} = \frac{\psi^8(q)}{\psi^8(q^3)}\left(\frac{\psi^4(q)-9q\psi^4(q^3)}{\psi^4(q)-q\psi^4(q^3)}\right) \tag{2.9}$$

Using (2.5) in (2.8) and (2.9), we obtain the required result.

Theorem 2.3. *Let*

$$P = \frac{\varphi(-q)}{\varphi(-q^3)} \quad and \quad Q = \frac{\varphi(-q^2)}{\varphi(-q^6)}. \tag{2.10}$$

Then

$$P\left(\frac{P-9}{P-1}\right)^2 = Q^2\left(\frac{Q-9}{Q-1}\right). \tag{2.11}$$

The proof of Theorem 2.3 is similar to the proof of Theorem 2.2, so we omit the details.

Remark. *The $P-Q$ eta-function identities (2.6) and (2.12) appear to be new in the literature.*

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References

[1] C. Adiga, T. Kim and M. S. Mahadeva Naika, Modular equations in the theory of signature 3 and $P-Q$ identities, *Adv. Stud. Contemp. Math.* **7** (2003), 33-40.

[2] B. C. Berndt, *Ramanujan's Notebooks*, Part V, Springer-Verlag, New York, 1994.

[3] S. Ramanujan, *Notebooks* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.

[4] K. Venkatachaliengar, *Development of Elliptic Functions According to Ramanujan*, *Technical Report* **2**, Madurai Kamaraj University, Madurai,1988.