Journal of Discrete Mathematical Sciences and Cryptography
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tdmc20

Products of distance degree regular and distance degree injective graphs

Medha Itagi Huilgol [a], M. Rajeshwari [b] & S. Syed Asif Ulla [a]

[a] Department of Mathematics, Bangalore University, Central College Campus, Bangalore, 560 001, India

[b] Department of Mathematics, Acharya Institute of Technology, Bangalore, 560 090, India

Published online: 03 Jun 2013.

Products of distance degree regular and distance degree injective graphs

Medha Itagi Huilgol [1,*]
M. Rajeshwari [2,†]
S. Syed Asif Ulla [1,§]

[1] *Department of Mathematics*
Bangalore University
Central College Campus
Bangalore 560 001
India

[2] *Department of Mathematics*
Acharya Institute of Technology
Bangalore 560 090
India

Abstract

The eccentricity $e(u)$ of a vertex u is the maximum distance of u to any other vertex in G. The distance degree sequence (dds) of a vertex v in a graph $G = (V, E)$ is a list of the number of vertices at distance $1, 2, \ldots.., e(u)$ in that order, where $e(u)$ denotes the eccentricity of u in G. Thus the sequence $\left(d_{i_0}, d_{i_1}, d_{i_2} \ldots, d_{i_j}, \ldots\right)$ is the dds of the vertex v_i in G where d_{i_j} denotes number of vertices at distance j from v_i. A graph is distance degree regular (DDR) graph if all vertices have the same dds. A graph is distance degree injective (DDI) graph if no two vertices have same dds.

In this paper we consider Cartesian and normal products of DDR and DDI graphs. Some structural results have been obtained along with some characterizations.

Keywords: *Distance degree sequence, Distance degree regular (DDR) graphs, Distance degree injective (DDI) graphs, Cartesian and Normal product of graphs.*

E-mail: medha@bub.ernet.in
†*E-mail:* rajeshwarim@acharya.ac.in
§*E-mail:* syedasif.ulla84@gmail.com

Journal of Discrete Mathematical Sciences & Cryptography
Vol. 15 (2012), No. 4 & 5, pp. 303–314

1. Introduction

Unless mentioned otherwise for terminology and notation the reader may refer Buckley and Harary [4], new ones will be introduced as and when found necessary.

Products in graphs have always given more generalized results compared to the graphs involved in the product itself. It is also a powerful tool to construct bigger graphs given smaller ordered (sized) graphs. Many parameters are tested in the products in literature [6], [7], [8], etc. Among many products defined between graphs the cartesian product is the most used one. Recently a whole monograph by Imrich *et. al.*, [10] is dedicated to graphs and their cartesian product. The cartesian product is defined as,

The cartesian product of two graphs G and H, denoted $G\Box H$, is a graph with vertex set $V(G\Box H) = V(G)\times V(H)$, that is, the set $g \in V(G)$ & $h \in V(H)$.

The edge set of $G\Box H$ consists of all pairs $[(g_1,h_1),(g_2,h_2)]$ of vertices with $[g_1,g_2] \in E(G)$ and $h_1 = h_2$ or $g_1 = g_2$ and $[h_1,h_2] \in E(H)$.

Also the normal product is defined as,

The normal product of two graphs G and H, denoted $G\oplus H$, is a graph with vertex set $V(G\oplus H) = V(G)\times V(H)$, that is, the set $g \in V(G), h \in V(H)$ and an edge $[(g_1,h_1),(g_2,h_2)]$ exists whenever any of the following conditions hold good:

(i) $[g_1,g_2] \in E(G)$ and $h_1 = h_2$,

(ii) $g_1 = g_2$ and $[h_1,h_2] \in E(H)$,

(iii) $[g_1,g_2] \in E(G)$ & $[h_1,h_2] \in E(H)$.

The distance $d(u,v)$ from a vertex u of G to a vertex v is the length of a shortest u to v path. The eccentricity $e(v)$ of v is the distance to a farthest vertex from v. If, $dist(u,v) = e(u), (v \neq u)$, we say that v is an eccentric vertex of u. The radius $r(G)$ is the minimum eccentricity of the vertices, whereas the diameter $d(G)$ is the maximum eccentricity. A vertex v is a central vertex if $e(v) = r(G)$, and a vertex is an antipodal vertex if $e(v) = d(G)$. A graph is self-centered if every vertex has the same eccentricity, i.e., $r(G) = d(G)$.

The distance degree sequence (*dds*) of a vertex v in a graph $G = (V,E)$ is a list of the number of vertices at distance $1,2,\ldots,e(v)$ in that order, where $e(v)$ denotes the eccentricity of v in G. Thus, the sequence $(d_{i_0}, d_{i_1}, d_{i_2}\ldots, d_{i_j},\ldots)$ is the dds of the vertex v_i in G where, d_{i_j} denotes number of vertices at distance j from v_i. The concept of distance degree regular (DDR) graphs was introduced by G. S. Bloom *et. al.*, [1], as the graphs for which all vertices have the same dds. For example, the three dimensional cube $Q_3 = K_2 \times K_2 \times K_2$ is a DDR graph with each vertex having its dds as (1,3,3,1). Clearly, d_{i_1} denotes the degree of the vertex v_i in G and hence, in general, a DDR graph must be a regular graph; but, it is easy to verify that a regular graph may not be DDR. In Bloom [1], detailed study of DDR graphs can be found and one of the fundamental results therein states that *"Every regular graph with diameter at most two is DDR"*. Bloom, Quintas and Kennedy [2] have dealt many problems concerning distance and path degree sequences in graphs. Halberstam *et. al.*, [9] have dealt in particular the distance and path degree sequences for cubic graphs. It is worth to mention that computer investigation and generation of cubic graphs is done by Brinkmann [3] and Bussemaker *et. al.*, [5]. In [12], Itagi Huilgol *et. al.*, have listed all DDR graphs of diameter three with extremal degree regularity. In the same paper they have shown the existence of a diameter three DDR graph of any arbitrary degree regularity. In [13], Itagi Huilgol *et. al.*, have constructed DDR graphs of arbitrary diameter. Also, they have studied the DDR graphs with respect to other parameters.

A graph is distance degree injective (DDI) graph if no two vertices have same dds. These graphs were defined by G.S. Bloom *et. al.*, in [2]. DDI graphs being highly irregular, in comparison with the DDR graphs, at least the degree regularity is looked into by Jiri volf in [11]. A particular case of cubic DDI graphs is considered by Martenez and Quintas in [14]. There are very few examples of DDI graphs, so it is important to get DDI graphs from smaller (sized/ordered) DDI or other graphs as products.

In this paper we consider cartesian and normal products of DDR and DDI graphs. For some products both necessary and sufficient conditions have been obtained.

2. Cartesian product of DDR and DDI graph

In this section we consider cartesian products of DDR and DDI graphs.

Theorem 2.1: *Cartesian product of two graphs G_1 and G_2 is a DDR graph if and only if both G_1 and G_2 are DDR graphs.*

Proof. Let G_1 and G_2 be two DDR graphs having the dds of each vertex $(d_0, d_1, d_2, \ldots\ldots d_{r_1})$ and $(d'_0, d'_1, d'_2, \ldots\ldots d'_{r_2})$ respectively, where r_1 and r_2 are radii of G_1 and G_2 respectively. In the cartesian product of any two graphs, the distance between any two vertices (u_1, v_1) and (u_2, v_2) is given by $d_{G_1 \Box G_2}((u_1, v_1), (u_2, v_2)) = d_{G_1}(u_1, u_2) + d_{G_2}(v_1, v_2)$ as in [15]. Now let u be any vertex in G_1 and v be any vertex in G_2. Then, it is very clear that the number of vertices at distance i from (u, v) in $G_1 \Box G_2$

$$d_{i_{G_1 \Box G_2}}(u, v) = d_i(u) + d'_i(v) + \sum_{j=1}^{i-1} d_j(u) d'_{i-j}(v).$$

Since the graphs G_1 and G_2 are DDR graphs $d_i(u) = d_i(x)$, $0 \le i \le diam(G_1)$ for all $x \in G_1$ and $d'_i(v) = d'_i(y), 0 \le i \le diam(G_2)$ for all $y \in G_2, d_{i_{G_1 \Box G_2}}(u, v) = d_{i_{G_1 \Box G_2}}(s, t), 0 \le i \le diam(G_1) + diam(G_2)$ for all $(s, t) \in G_1 \Box G_2$. Hence the graph $G_1 \Box G_2$ is a DDR graph.

Now let $G_1 \Box G_2$, the cartesian product of G_1 and G_2 be a DDR graph. Suppose G_1 is not a DDR graph, then there exist at least two vertices u and v having different dds i.e., $(d_0(u), d_1(u), d_2(u), \ldots\ldots d_{e(u)}(u))$ and $(d_0(v), d_1(v), d_2(v), \ldots\ldots d_{e(v)}(v))$ are dds of u and v, respectively in G_1 and k, the minimum value of $i, 1 \le i \le d(G_1)$, such that $d_k(u) \ne d_k(v)$. Let w be any vertex in G_2, having the dds $(d'_0(w), d'_1(w), d'_2(w), \ldots\ldots, d'_{e(w)}(w))$ and $d'_k(w)$ be the number of vertices at distance k from w in G_2. The number of vertices at distance k from (u, w) in $G_1 \Box G_2$ is given by $d_{k_{G_1 \Box G_2}}(u, w) = d_k(u) + d'_k(w) + d_1(u)\, d'_{(k-1)}(w) + d_2(u)\, d'_{(k-2)}(w) + d_3(u)\ d'_{(k-3)}(w) + \ldots + d_{(k-1)}(u) d'_1(w)$ and the number of vertices at distance k from (v, w) in $G_1 \Box G_2$ is given by $d_{k_{G_1 \Box G_2}}(v, w) = d_k(v) + d'_k(w) + d_1(v)\, d'_{(k-1)}(w) + d_2(v)\, d'_{(k-2)}(w) + d_3(v)\ d'_{(k-3)}(w) + \ldots + d_{(k-1)}(v)\, d'_1(w)$. Hence $d_{k_{G_1 \Box G_2}}(u, w) \ne d_{k_{G_1 \Box G_2}}(v, w)$, since $d_k(u) \ne d_k(v)$ and $d_j(u) = d_j(v)$, for all $j, 0 \le j \le k$. Hence $G_1 \Box G_2$ is non-DDR graph, a contradiction. Hence G_1 should be a DDR graph. Similarly we can prove G_2 is also a DDR graph. Hence, the result. □

Theorem 2.2: *If the cartesian product of two graphs G_1 and G_2 is DDI then both G_1 and G_2 are DDI graphs.*

Proof. Let $G_1 \Box G_2$ be a DDI graph. Suppose G_1 is not a DDI graph, then there exist at least two vertices u_1, u_2 in G_1 having the same dds, i.e., $dds(u_1) = dds(u_2)$. Let v_1 be any vertex in G_2. Then the number of vertices at distance $l, 0 \le l \le e(u_1) + e(v_1)$ from (u_1, v_1) is given by

$$d_{l_{G_1 \square G_2}}(u_1, v_1) = d_i(u_1) + d'_i(v_1) + \sum_{j=1}^{i-1} d_j(u_1) d'_{i-j}(v_1).$$

and the number of vertices at distance $l, 0 \leq l \leq e(u_2) + e(v_1)$ from (u_2, v_1) is given by

$$d_{l_{G_1 \square G_2}}(u_2, v_1) = d_i(u_2) + d'_i(v_1) + \sum_{j=1}^{i-1} d_j(u_2)\, d'_{i-j}(v_1).$$

Since, $dds(u_1) = dds(u_2)$, we get $d_{l\,G_1 \square G_2}(u_1, v_1) = d_{l\,G_1 \square G_2}(u_2, v_1)$, for all $l, 0 \leq l \leq e(u_2) + e(v_1)$, *i.e.*, $dds(u_1, v_1) = dds(u_2, v_1)$, hence $G_1 \square G_2$ is not DDI, a contradiction. Hence G_1 should be DDI. Similarly, we can prove G_2 is also DDI. Hence, the proof. □

Remark 1. Car*tesian product of two DDI graphs need not be DDI. The following are the two DDI graphs whose cartesian product is not DDI.*

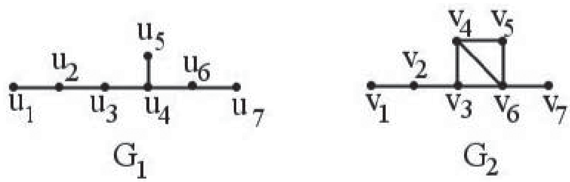

Figure 1

Two DDI graphs whose cartesian product is not DDI

Lemma 2.1: *Let G_1 and G_2 be two DDI graphs. Let $A = \{dds(u_i) \,/\, u_i \in V(G_1)\}$ and $B = \{dds(v_i) \,/\, v_i \in V(G_2)\}$. If $|A \cap B| \geq 2$ then $G_1 \square G_2$ is not DDI.*

Proof. Let $|A \cap B| \geq 2$ Then there exist u_k, u_l in G_1 and v_m, v_n in G_2 such that $dds(u_k) = dds(v_m)$ and $dds(u_l) = dds(v_n)$. Hence in $G_1 \square G_2$, $dds(u_k, v_n) = dds(u_l, v_m)$,making $G_1 \square G_2$ non DDI. Hence, the proof □

Theorem 2.3. *Let G_1 and G_2 be any two graphs. Let u be any vertex in G_1 and S be a subset of $V(G_2)$ such that no two vertices of S have same dds, then no two vertices of $\{u\} \times S$ have same dds in $G_1 \square G_2$.*

Proof. Let u be any vertex in G_1 and S be a subset of $V(G_2)$ such that no two vertices of S have same *dds*. Suppose there exist at least two vertices (u, v) and (u, w) in $\{u\} \times S$ having same *dds*. Hence $d_{l_{G_1 \square G_2}}(u, v) = d_{l_{G_1 \square G_2}}(u, w)$, for all $l, 0 \leq l \leq e(u) + e(v)$, here $e(v) = e(w)$.

Hence $d_l(u) + d'_l(v) + \sum_{j=1}^{l-1} d_j(u)\, d'_{l-j}(v) = d_l(u) + d'_l(w) + \sum_{j=1}^{l-1} d_j(u)\, d'_{l-j}(w)$,

implies $d'_l(v) + \sum_{j=1}^{l-1} d_j(u)\, d'_{l-j}(v) = d'_l(w) + \sum_{j=1}^{l-1} d_j(u)\, d'_{l-j}(w) \to (1)$

For $l = 1$, $eq.(1)$ implies $d'_l(v) = d'_l(w)$.

For $l = 2$, $eq.(1)$ implies $d'_2(v) + d_1(u)\, d'_1(v) = d'_2(w) + d_1(u) d'_1(w)$

$\Rightarrow d'_2(v) = d'_2(w)$ and so on

For, $l = e(v) = e(w), eq.(1)$ implies $d'_{e(v)}(v) = d'_{e(w)}(w)$. Hence, $dds(v) = dds(w)$, a contradiction. Hence no two vertices of $\{u\} \times S$ have same dds in $G_1 \square G_2$. Hence, the proof. □

Remark 2. *Let S_1 and S_2 be two subsets of $V(G_1)$ such that every pair (x, y), $x \in S_1$, $y \in S_2$ satisfies $dds(x) \neq dds(y)$ and let $z \in V(G_2)$ be any vertex in G_2, then in $G_1 \square G_2$, the subsets $\{(x,z) / x \in S_1\}$ and $\{(y,z) / y \in S_2\}$ are such that every pair $((x,z),(y,z))$, $x \in S_1$ and $y \in S_2$ satisfies $dds(x,z) \neq dds(y,z)$.*

3. Normal product of DDR and DDI graphs

In this section we consider normal product of DDR and DDI graphs. Stevanović in [16] has considered the distance between any pair of vertices in normal product. Given two vertices (u_i, v_j) and (u_k, v_m) the distance between these two vertices in the normal product is given by;

$$d_{G_1 \oplus G_2}((u_i, v_j), (u_k, v_m)) = \max\{d_{G_1}(u_i, v_k), d_{G_2}(u_j, v_m)\}$$

Immediate conclusion we can draw as follows:

Lemma 3.1: *Let $S = \{G_i / i \geq 2\}$. If there exist k such that G_k is self centred and $diam(G_k) \geq diam(G_i)$ for all $i \geq 2$ then normal product of all the graphs in S is a self centred graph with diameter equal to $diam(G_k)$.*

Theorem 3.1: *Normal product $G_1 \oplus G_2$ of two graphs G_1 and G_2 is DDR if and only if both G_1 and G_2 are DDR graphs.*

Proof. Let G_1 and G_2 be two DDR graphs having the dds $(d_0, d_1, d_2, \ldots\ldots d_{d(G_1)})$ and $(d'_0, d'_1, d'_2, \ldots\ldots, d_{d(G_2)})$ respectively, then the number of vertices at distance $i, 0 \le i \le \max\{diam(G_1), diam(G_2)\}$ from any vertex u_l, v_m in $G_1 \oplus G_2$ is given by

$$d_{i_{G_1 \oplus G_2}}(u_l, v_m) = d_i(u_l) \cdot d'_i(v_m) + d_i(u_l) \sum_{j=0}^{i-1} d'_j(v_m) + d'_i(v_m) \sum_{j=0}^{i-1} d_j(u_l).$$

Hence, the Normal product $G_1 \oplus G_2$ is DDR.

Conversely, let $G_1 \oplus G_2$ be DDR. Suppose G_1 is not DDR, then there exist at least two vertices u_1 and u_2 in G_1 such that $dds(u_1) \ne dds(u_2)$. Let k be the minimum value such that $d_k(u_1) \ne d_k(u_2)$ and v_1 be any arbitrary vertex in G_2, then the number of vertices at distance k from (u_1, v_1) is given by

$$d_{k_{G_1 \oplus G_2}}(u_1, v_1) = d_k(u_1) \cdot d'_k(v_1) + d_k(u_1) \sum_{j=0}^{i-1} d'_j(v_1) + d'_k(v_1) \sum_{j=0}^{k-1} d_j(u_1)$$

and the number of vertices at distance k from (u_2, v_1) is given by

$$d_{k_{G_1 \oplus G_2}}(u_2, v_1) = d_k(u_2)\, d'_k(v_1) + d_k(u_2) \sum_{j=0}^{i-1} d'_j(v_1) + d'_k(v_1) \sum_{j=0}^{k-1} d_j(u_2),$$

implies $d_{k_{G_1 \oplus G_2}}(u_1, v_1) \ne d_{k_{G_1 \oplus G_2}}(u_2, v_1)$, since $d_k(u_1) \ne d_k(u_2)$. So $G_1 \oplus G_2$ is not DDR, a contradiction. Hence G_1 is DDR. Similarly we can prove G_2 is also DDR. Hence, the proof. □

Proposition 3.1: *If the normal product $G_1 \oplus G_2$ of two graphs is DDI then both G_1 and G_2 are DDI.*

Proof. Let the normal product $G_1 \oplus G_2$ of two graphs be DDI. Suppose G_1 is not DDI, then there exist two vertices u_1 and u_2 such that $dds(u_1) = dds(u_2)$. Let v_1 be any vertex in G_2. The number of vertices at distance $k, 0 \le k \le \max\{e(u_1), e(v_1)\}$ from (u_1, v_1) is given by

$$d_{k_{G_1 + G_2}}(u_1, v_1) = d_k(u_1)\, d'_k(v_1) + d_k(u_1) \sum_{j=0}^{k-1} d'_j(v_1) + d'_k(v_1) \sum_{j=0}^{k-1} d_j(u_1).$$

and the number of vertices at distance $k, 0 \leq k \leq \max\{e(u_2), e(v_1)\}$ from (u_2, v_1) is given by

$$d_{kG_1 \oplus G_2}(u_2, v_1) = d_k(u_2)\, d'_k(v_1) + d_k(u_2) \sum_{j=0}^{k-1} d'_j(v_1) + d'_k(v_1) \sum_{j=0}^{k-1} d_j(u_2).$$

Hence, $d_{kG_1 \oplus G_2}(u_1, v_1) = d_{kG_1 \oplus G_2}(u_2, v_1), 0 \leq k \leq \max\{e(u_1) = e(u_2), e(v_1)\}$, implies $dds(u_1, v_1) = dds(u_2, v_1)$, a contradiction. Hence G_1 is DDI. Similarly we can prove G_2 is also DDI. Hence, the proof. □

Remark 3 : *Normal product of two DDI graphs need not be DDI. The graphs in Figure 1 are the two DDI graphs whose normal product is not DDI.*

Lemma 3.2: *Let G_1 and G_2 be two DDI graphs. Let $A = \{dds(u_i)/u_i \in G_1\}$ and $B = \{dds(v_i)/v_i \in V(G_2)\}$. If $|A \cap B| \geq 2$ then $G_1 \oplus G_2$ is not DDI.*

Proof. Let $|A \cap B| \geq 2$. Then there exist u_k, u_l in G_1 and v_m, v_n in G_2 such that $dds(u_k) = dds(v_m)$ and $dds(u_l) = dds(v_n)$. Hence in $G_1 \oplus G_2$, $dds(u_k, v_n) = dds(u_l, v_m)$, making $G_1 \oplus G_2$ non DDI. Hence the proof. □

Proposition 3.2: *Let G_1 and G_2 be any two graphs. Let u be any vertex in G_1 and S be a subset of $V(G_2)$ such that no two vertices of S have same dds, then no two vertices of $\{u\} \times S$ have same dds in $G_1 \oplus G_2$.*

Proof. Suppose there exist two vertices (u, v) and (u, w) in $\{u\} \times S$ having same dds, i.e., $dds(u, v) = dds(u, w)$, this condition is satisfied only if $e(v) = e(w)$. Here three subcases arise, viz. **Case(a):** $e(u) < e(v) = e(w)$, **Case(b):** $e(u) = e(v) = e(w)$ and **Case(c):** $e(u) > e(v) = e(w)$.

Case(a): $dds(u, v) = dds(u, w), e(u) < e(v) = e(w), e(u, v) = e(v) = e(w)$.

$$d_{mG_1 \oplus G_2}(u, v) = d_m(u)\, d'_m(v) + d_m(u) \sum_{j=0}^{m-1} d'_j(v) + d'_m(v) \sum_{j=0}^{m-1} d_j(u) \text{ and} \tag{1}$$

$$d_{mG_1 \oplus G_2}(u, w) = d_m(u)\, d'_m(w) + d_m(u) \sum_{j=0}^{m-1} d'_j(w) + d'_m(w) \sum_{j=0}^{m-1} d_j(u), \tag{2}$$

$d_{mG_1 \oplus G_2}(u, v) = d_{mG_1 \oplus G_2}(u, w)$, for all $m, 0 \leq m \leq e(v) = e(w)$.

First taking, $d_{mG_1 \oplus G_2}(u,v) = d_{mG_1 \oplus G_2}(u,w)$, for all $m, 0 \le m \le e(u)$, we get $d_m(u)[d'_m(v) - d'_m(w)] + d_m(u)\left[\sum_{j=0}^{m-1} d'_j(v) - \sum_{j=0}^{m-1} d'_j(w)\right]$

$$+\left[\sum_{j=0}^{m-1} d_j(u)\,[d'_m(v) - d'_m(w)]\right] = 0$$

$$\text{i.e.,}\left[d_m(u) + \sum_{j=0}^{m-1} d_j(u)\right][d'_m(v) - d'_m(w)] + d_m(u)\left[\sum_{j=0}^{m-1} d'_j(v) - \sum_{j=0}^{m-1} d'_j(w)\right] = 0$$

$$\text{i.e.,}\left[\sum_{j=0}^{m} d_j(u)\right][d'_m(v) - d'_m(w)] + d_m(u)\left[\sum_{j=0}^{m-1} d'_j(v) - \sum_{j=0}^{m-1} d'_j(w)\right] = 0 \qquad (3)$$

Put $m = 1$ in Eqn (3). $\Longrightarrow [d_0(u) + d_1(u)][d'_1(v) - d'_1(w)]$

$$+ d_1(u)[d'_0(v) - d'_0(w)] = 0$$

i.e., $d'_1(v) = d'_1(w)$

Put $m = 2$ in Eqn. (3). $\Longrightarrow \left[\sum_{j=0}^{2} d_j(u)\right][d'_2(v) - d'_2(w)] + d_2(u)$

$$\left[\sum_{j=0}^{1} d'_j(v) - \sum_{j=0}^{1} d'_j(w)\right] = 0$$

i.e., $d'_2(v) = d'_2(w)$, And so on,

Put $m = e(u)$ in Eqn. (1). $\Longrightarrow \left[\sum_{j=0}^{e(u)} d_j(u)\right][d'_{e(u)}(v) - d'_{e(u)}(w)]$

$$+ d_{e(u)}(u)\left[\sum_{j=0}^{e(u)-1} d'_j(v) - \sum_{j=0}^{e(u)-1} d'_j(w)\right] = 0$$

i.e., $d'_{e(u)}(v) = d'_{e(u)}(w)$.

Hence $d'_m(v) = d'_m(w)$ for all $m, 0 \le m \le e(u)$. (4)

Now for all $m, e(u) < m \le e(v) = e(u,w)$, we have

$$d_{mG_1 \oplus G_2}(u, v) = d'_m(v) \sum_{j=0}^{e(u)} d_j(u) \text{ and } d_{mG_1 \oplus G_2}(u, w) = d'_m(w) \sum_{j=0}^{e(u)} d_j(u).$$

Hence $d_{mG_1 \oplus G_2}(u, v) = d_{mG_1 \oplus G_2}(u, w)$, for all $m, e(u) < m \leq e(v)$

$= e(u,w)$ gives $d'_m(v) \sum_{j=0}^{e(u)} d_j(u) - d'_m(w) \sum_{j=0}^{e(u)} d_j(u) = 0$, i.e..,

$[d'_m(v) - d'_m(w)] \sum_{j=0}^{e(u)} d_j(u) = 0$ i.e., $d'_m(v) - d'_m(w) = 0$.

$$\text{Hence } d'_m(v) = d'_m(w), \text{ for all } m, e(u) < m \leq e(v) = e(u,w). \tag{5}$$

Combining (4) and (5), we get $d'_m(v) = d'_m(w)$, for all $m, 0 \leq m \leq e(v) = e(w) = e(u,w)$, i.e., $dds(v) = dds(w)$, a contradiction. Hence $dds(u,v) \neq dds(u,w)$.

Case (b): $dds(u,v) = dds(u,w), e(u) = e(v) = e(w) = e(u,v)$.

Substituting these conditions in (1) and (2), we get

$$\left[\sum_{j=0}^{m} d_j(u)\right][d'_m(v) - d'_m(w)] + d_m(u)\left[\sum_{j=0}^{m-1} d'_j(v) - \sum_{j=0}^{m-1} d'_j(w)\right] = 0 \tag{6}$$

Substituting values of m in Eq.(6) we get $d'_m(v) = d'_m(w)$ for all $m, 0 \leq m \leq e(u) = e(v) = e(w) = e(u,v)$.

i.e., $dds(v) = dds(w)$, a contradiction. Hence $dds(u,v) \neq dds(u,w)$.

Case (c): $e(u) > e(v) = e(w), e(u,v) = e(u)$.

Substituting the values in (1) and (2) we get

$$\left[\sum_{j=0}^{m} d_j(u)\right][d'_m(v) - d'_m(w)] + d_m(u)\left[\sum_{j=0}^{m-1} d'_j(v) - \sum_{j=0}^{m-1} d'_j(w)\right] = 0 \tag{7}$$

Substituting the values of m in (7) we get

$$d'_m(v) = d'_m(w) \text{ for all } m, 0 \leq m \leq e(v) = e(w).$$

i.e., $dds(v) = dds(w)$, a contradiction. Hence, $dds(u,v) \neq dds(u,w)$. □

Remark 4: *Let S_1 and S_2 be two subsets of $V(G_1)$ such that every pair $(x,y), x \in S_1, y \in S_2$ satisfies $dds(x) \neq dds(y)$ and let $z \in V(G_2)$ be any vertex in G_2, then in $G_1 \oplus G_2$, the subsets $\{(x,z) / x \in S_1\}$ and $\{(y,z) / y \in S_2\}$ are such that every pair $((x,z),(y,z)), x \in S_1$ and $y \in S_2$ satisfies $dds(x,z) \neq dds(y,z)$.*

4. Acknowledgement

The third author (SAUS) thanks the UGC, New Delhi for granting fellowship under Special Assistance Program CAS Phase-II.

References

[1] G. S. Bloom, L. V. Quintas and J. W. Kennedy, "Distance Degree Regular Graphs", The Theory and Applications of Graphs, 4th International conference, Western Michigan University, Kalamazoo, MI, May (1980), John Wiley and Sons, New York, 1981, pp. 95–108.

[2] G. S. Bloom, L. V. Quintas and J. W. Kennedy, "Some problems concerning distance and path degree sequences", *Lecture Notes in Math.* 1018, 1983, pp. 179–190.

[3] G. Brinkmann," Fast generation of Cubic graphs", *J. Graph Theory*, Vol. 23(2), 1996, pp. 139–149.

[4] F. Buckley and F. Harary, "Distance in Graphs", Addison Wesley 1990.

[5] F. C. Bussemaker, S. Cobeljic, D. B. Cvetkovic, J. J. Seidel, "Computer investigation of cube Graphs", T.H. Report 76-WSK-01, Technological University Eindhoven, Netherlands, 1976.

[6] T. Feder, "Stable Network and Product Graphs", *Mem. Amer. Math. Soc.*, 116, pp. xii+223. 1995,

[7] R. L. Graham and P. M. Winkler, "On isomorphic embeddings of graphs", *Trans. Amer. Math. Soc.*, Vol. 288, 1985, pp. 527–536.

[8] A. Graovac and T. Pisanski, "On the wiener index of a graph", *J. Math. Chem*, Vol. 8, 1991, pp. 53–62.

[9] F. Y. Halberstam and L. V. Quintas, "Distance and path degree sequences for cubic graphs", Mathematics Department, Pace University, New York, NY, 1982.

[10] W. Imrich, S. Klavžar, D. F. Rall, "Topics in Graph Theory: Graphs and Their Cartesian Product", A K Peters, Ltd, 2008.

[11] Jiri volf, " A small distance degree injective cubic graphs", Notes from New York Graph Theory Day XIII, New York Academy of Sciences, 1987, pp. 31–32.

[12] Medha Itagi Huilgol, H. B. Walikar, B. D. Acharya, " On diameter three distance degree regular graphs", *Advances and Applications in Discrete Mathematics*, Vol. 7(1), 2011, pp. 39–61.

[13] Medha Itagi Huilgol, Rajeshwari M. and Syed Asif Ulla S.,"Distance degree regular graphs and their eccentric digraphs", *International J. of Math. Sci. and Engg*. Appls.(IJMSEA), Vol. 5(VI), 2011, pp. 405–416

[14] P. Martinez and L. V. Quintas, "Distance degree injective regular graphs", Notes from New York Graph Theory Day VIII, *New York Academy of Sciences*, 1984, pp. 19–21.

[15] J. Nieminen, "The center problem in the product of graphs", *Recent Studies in Graph Theory*, 1989, pp. 201–205.

[16] D. Stevanović, "Distance regularity of compositions of graphs", *Applied Mathematics Letters*, Vol. 17(3), March 2004, pp. 337–343.

Received December, 2011