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CERTAIN NEW MODULAR IDENTITIES FOR RAMANUJAN'S CUBIC CONTINUED FRACTION

M. S. Mahadeva Naika, S. Chandankumar and N. P. Suman

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Abstract. In this paper, first we establish some new relations for ratios of Ramanujan's theta functions. We establish some new general formulas for explicit evaluations of Ramanujan's theta functions. We also establish new relations connecting Ramanujan's cubic continued fraction $V(q)$ with four other continued fractions $V(q^{15})$, $V(q^{5/3})$, $V(q^{21})$ and $V(q^{7/3})$.

1. Introduction

On page 366 of his 'lost' notebook [**14**], S. Ramanujan gave the cubic continued fraction

$$V(q) := \frac{q^{1/3}}{1} \, + \, \frac{q+q^2}{1} \, + \, \frac{q^2+q^4}{1} \, + \, \frac{q^3+q^6}{1} \, + \, \cdots, \qquad |q|<1 \tag{1.1}$$

and claimed that there are many results of $V(q)$ analogous to those for the Rogers-Ramanujan continued fraction. H. H. Chan [**5**] has established all the claims made by Ramanujan. Subsequently, many mathematicians have contributed to the theory of Ramanujan's cubic continued fraction. Some of them are N. D. Baruah [**2**], C. Adiga, T. Kim, M. S. Mahadeva Naika and H. S. Madhusudhan [**1**], Mahadeva Naika [**7**], B. Cho, J. K. Koo and Y. K. Park [**6**]. Recently, Mahadeva Naika, S. Chandankumar and K. Sushan Bairy [**10**], [**11**] have established new modular identities connecting $V(q)$ with $V(q^n)$, for $n = 4$, 6, 8, 9, 10, 12, 14, 16, 17, 19 and 22.

In [**15**], J. Yi introduced two parameters $h_{k,n}$ and $h'_{k,n}$ as follows:

$$h_{k,n} := \frac{\varphi(e^{-\pi\sqrt{n/k}})}{k^{1/4}\varphi(e^{-\pi\sqrt{nk}})}, \tag{1.2}$$

$$h'_{k,n} := \frac{\varphi(-e^{-\pi\sqrt{n/k}})}{k^{1/4}\varphi(-e^{-\pi\sqrt{nk}})} \tag{1.3}$$

and established several properties as well as explicit evaluations of $h_{k,n}$ and $h'_{k,n}$ for different positive rational values of n and k. Recently, Mahadeva Naika and Chandankumar [**9**] have established several new modular equations of degree 2 and

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established general formulas for explicit evaluations of $h_{2,n}$. They have also established several new explicit evaluations for the Ramanujan-Göllnitz-Gordon continued fraction, the Ramanujan-Selberg continued fraction and a continued fraction of Eisenstein. In [**8**], Mahadeva Naika, Bairy and M. Manjunatha have established several new modular equations of degree 4 and established general formulas for explicit evaluations of $h_{4,n}$.

In [**3**], Baruah and Nipen Saikia and Yi, Y. Lee and D. H. Paek [**16**] have defined two parameters $l_{k,n}$ and $l'_{k,n}$ as follows:

$$l_{k,n} := \frac{\psi(-e^{-\pi\sqrt{n/k}})}{k^{1/4}e^{-\frac{(k-1)\pi}{8}\sqrt{n/k}}\psi(-e^{-\pi\sqrt{nk}})}, \tag{1.4}$$

and

$$l'_{k,n} := \frac{\psi(e^{-\pi\sqrt{n/k}})}{k^{1/4}e^{-\frac{(k-1)\pi}{8}\sqrt{n/k}}\psi(e^{-\pi\sqrt{nk}})}. \tag{1.5}$$

They have also established several properties as well as explicit evaluations of $l_{k,n}$ and $l'_{k,n}$ for different positive rational values of n and k. In [**12**], Mahadeva Naika, Chandankumar and Bairy have established several new modular equations of degree 9 and also established several general formulas for explicit evaluations for the ratios of Ramanujan's theta function ψ.

Consider

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{1.6}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2}, \tag{1.7}$$

which are special cases of Ramanujan's general theta function

$$f(a,b) \quad := \quad \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \;\; |ab| < 1.$$

Let

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2}k^{2n} = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right), \tag{1.8}$$

where $0 < k < 1$ and ${}_2F_1$ is the ordinary or Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n\,n!}z^n, \quad 0 \le |z| < 1,$$

where $(a)_0 = 1$, $(a)_n = a(a+1)\cdots(a+n-1)$ for n a positive integer with a, b and c are complex numbers such that $c \neq 0,-1,-2,\ldots$. The number k is called the modulus of K and $k' := \sqrt{1-k^2}$ is called the complementary modulus. Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l and l', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{1.9}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is induced by (1.9). Following Ramanujan, set $\alpha = k^2$ and $\beta = l^2$. Then we say β is of degree n over α. The multiplier m is defined by

$$m = \frac{K}{L}. \tag{1.10}$$

In Section 2, we collect some identities which are useful to prove our main results. In Section 3, we establish some new modular equations of degree 3 for the ratios of Ramanujan's theta–functions. In Section 4, we establish some general formulas for the explicit evaluations of $h_{3,n}$ and $l_{3,n}$, for positive rational values of n. In section 5, we establish modular relations for $V(q)$ with four other continued fractions $V(q^{15})$, $V(q^{5/3})$, $V(q^{21})$ and $V(q^{7/3})$.

2. Preliminary Results

In this section, we record identities which are useful in proving our main results.

Lemma 2.1. [**4**, Ch. 17, Entry 10 (i), Entry 11 (ii), pp. 122–123] *For* $0 < \alpha < 1$,

$$\varphi(q) = \sqrt{z}, \tag{2.1}$$

$$\sqrt{2}q^{1/8}\psi(-q) = \sqrt{z}\{\alpha(1-\alpha)\}^{1/8}, \tag{2.2}$$

where $z := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)$.

Lemma 2.2. [**4**, Ch. 20, Entry 11 (viii), (ix), p. 384] *Let* α, β, γ *and* δ *be of the first, third, fifth and fifteenth degrees respectively. Let* m *denote the multiplier connecting* α *and* β *and* m' *be the multiplier relating* γ *and* δ. *Then*

$$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8} = \sqrt{\frac{m'}{m}}, \tag{2.3}$$

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8} = -\sqrt{\frac{m}{m'}}. \tag{2.4}$$

Lemma 2.3. [**4**, Ch. 20, Entry 13 (i), (ii), p. 401] *Let* α, β, γ *and* δ *be of the first, third, seventh and twenty first degrees respectively. Let* m *denote the multiplier connecting* α *and* β *and* m' *be the multiplier relating* γ *and* δ, *then*

$$\begin{aligned}\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/4} &+ \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/4} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/4} \\ &+ 4\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/6} = \frac{m}{m'},\end{aligned} \tag{2.5}$$

$$\begin{aligned}\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/4} &+ \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/4} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/4} \\ &+ 4\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/6} = \frac{m'}{m}.\end{aligned} \tag{2.6}$$

Lemma 2.4. [**4**, Ch. 20, Entry 1 (i), p. 345] *We have*

$$1+\frac{1}{V(q)}=\frac{\psi^4(q)}{q\psi^4(q^3)}, \tag{2.7}$$

where $V(q)$ *is defined as in the equation* (1.1).

Lemma 2.5. [**4**, Ch. 20, Entry 1(iii), p. 345] *We have*

$$\frac{\varphi(q^{1/3})}{\varphi(q^3)}=1+\left(\frac{\varphi^4(q)}{\varphi^4(q^3)}-1\right)^{1/3}, \tag{2.8}$$

$$\frac{3\varphi(q^9)}{\varphi(q)}=1+\left(\frac{9\varphi^4(q^3)}{\varphi^4(q)}-1\right)^{1/3}. \tag{2.9}$$

Lemma 2.6. [**1**] *If* $M:=\dfrac{\psi(-q)}{q^{1/4}\psi(-q^3)}$ *and* $N:=\dfrac{\varphi(q)}{\varphi(q^3)}$, *then*

$$N^4+M^4N^4=9+M^4. \tag{2.10}$$

3. Modular Equations of Degree Three

In this section, we establish several new modular equations for the ratios of Ramanujan's theta functions φ and ψ.

Theorem 3.1. *If* $P:=\dfrac{\varphi(q)}{\varphi(q^3)}$ *and* $Q:=\dfrac{\varphi(q^{15})}{\varphi(q^{45})}$, *then*

$$\begin{aligned}
\frac{Q^9}{P^9}&=1860\frac{Q^5}{P^5}-3^2\left(20\frac{Q^7}{P^7}-3^3\frac{P^7}{Q^7}\right)-216\left(5^2\frac{Q^3}{P^3}-21\frac{P^3}{Q^3}\right)\\
&+18\left(1121\frac{Q}{P}-330\frac{P}{Q}\right)+3\left(PQ+\frac{3}{PQ}\right)\left[-15\left(98\frac{P}{Q}-179\frac{Q}{P}\right)\right.\\
&\left.+5\left(126\frac{P^3}{Q^3}-241\frac{Q^3}{P^3}\right)-\left(135\frac{P^5}{Q^5}-296\frac{Q^5}{P^5}\right)\right]+5\left(P^2Q^2+\frac{3^2}{P^2Q^2}\right)\\
&\times\left[-3\left(180\frac{P}{Q}-107\frac{Q}{P}\right)+3\left(3^3\frac{P^3}{Q^3}-154\frac{Q^3}{P^3}\right)-2\left(3^3\frac{P^5}{Q^5}-31\frac{Q^5}{P^5}\right)\right]\\
&+3\left(P^3Q^3+\frac{3^3}{P^3Q^3}\right)\left[-\left(311\frac{P}{Q}-140\frac{Q}{P}\right)+75\left(\frac{P^3}{Q^3}-2^2\frac{Q^3}{P^3}\right)\right.\\
&\left.-\left(3^2\frac{P^5}{Q^5}-20\frac{Q^5}{P^5}\right)\right]+2\left(P^4Q^4+\frac{3^4}{P^4Q^4}\right)\left[-5\left(12\frac{P}{Q}-23\frac{Q}{P}\right)\right.\\
&\left.+3\left(15\frac{P^3}{Q^3}-28\frac{Q^3}{P^3}\right)\right]+5\left(P^5Q^5+\frac{3^5}{P^5Q^5}\right)\left[-\left(\frac{P}{Q}-18\frac{Q}{P}\right)\right.\\
&\left.+3\left(\frac{P^3}{Q^3}-\frac{Q^3}{P^3}\right)\right]+\left(P^6Q^8+\frac{3^7}{P^8Q^6}\right)+\left(P^9Q^3+\frac{3^6}{P^3Q^9}\right)\\
&+15\left(P^5Q^7+\frac{3^6}{P^7Q^5}\right)-50\left(3\frac{Q^6}{P^8}+\frac{Q^8}{P^6}\right).
\end{aligned} \tag{3.1}$$

Proof. Transcribing the equations (2.3) and (2.4) by using the equations (2.1) and (2.2) into theta function, we deduce that

$$a(P+q_1)=b(P-q_1), \tag{3.2}$$

where

$$P:=\frac{\varphi(q)}{\varphi(q^3)},\ \ q_1:=\frac{\varphi(q^5)}{\varphi(q^{15})},\ \ a:=\frac{\psi(-q)}{q^{1/4}\psi(-q^3)},\ \ b:=\frac{\psi(-q^5)}{q^{5/4}\psi(-q^{15})}.$$

Using the equation (2.10) in the above equation (3.2) along with the equation (2.9), we deduce that

$$\begin{aligned}&9Q^6+6Q^6P^4A^2+15Q^6P^4A+15Q^5P^3A^2+45Q^4P^2A-27QP^3A^2-81P^2\\&+15Q^5P^7A^2-90Q^2P^4A-Q^6P^8A+Q^6P^8A^2-27QP^7A^2+30Q^4P^6A^2\\&+48Q^5P^3A+15Q^4P^6A-45Q^2P^4A^2+18Q^6P^4+180Q^4P^2-135QP^3\\&+123Q^5P^3-15Q^4P^6-45Q^2P^4+27QP^7-2Q^6P^8-15Q^5P^7=0,\end{aligned} \tag{3.3}$$

where

$$A:=\sqrt[3]{\frac{9}{P^4}-1}.$$

Solving the above equation (3.3) for A and then cubing both sides, we arrive at (3.1). □

Remark 1. The equation (3.1) holds for

$$P:=\frac{\psi(q)}{q^{1/4}\psi(q^3)}\ \text{ and }\ Q:=\frac{\psi(q^{15})}{q^{15/4}\psi(q^{45})}.$$

Eliminating P and q_1 in the equation (3.2), we arrive at the equation involving a and b. Replacing q by $-q$ in the resulting equation, the Remark 1 holds.

Theorem 3.2. *If* $P := \dfrac{\varphi(q)}{\varphi(q^3)}$ *and* $Q := \dfrac{\varphi(q^{5/3})}{\varphi(q^5)}$, *then*

$$\begin{aligned}
&\frac{P^9}{Q^9} + 9\left(20\frac{P^7}{Q^7} - 27\frac{Q^7}{P^7}\right) - 1860\frac{P^5}{Q^5} + 216\left(5^2\frac{P^3}{Q^3} - 21\frac{Q^3}{P^3}\right) - 18\left(1121\frac{P}{Q}\right.\\
&\left.-330\frac{Q}{P}\right) - \left(PQ + \frac{3}{PQ}\right)\left[-45\left(179\frac{P}{Q} - 98\frac{Q}{P}\right) + 15\left(241\frac{P^3}{Q^3} - 126\frac{Q^3}{P^3}\right)\right.\\
&\left.-3\left(296\frac{P^5}{Q^5} - 135\frac{Q^5}{P^5}\right)\right] - \left(P^2Q^2 + \frac{3^2}{P^2Q^2}\right)\left[15\left(107\frac{P}{Q} - 180\frac{Q}{P}\right)\right.\\
&\left.-15\left(154\frac{P^3}{Q^3} - 3^3\frac{Q^3}{P^3}\right) + 10\left(31\frac{P^5}{Q^5} - 3^3\frac{Q^5}{P^5}\right)\right] - \left(P^3Q^3 + \frac{3^3}{P^3Q^3}\right)\\
&\times\left[-3\left(140\frac{P}{Q} - 311\frac{Q}{P}\right) + 225\left(4\frac{P^3}{Q^3} - \frac{Q^3}{P^3}\right) - 3\left(20\frac{P^5}{Q^5} - 3^2\frac{Q^5}{P^5}\right)\right]\\
&-\left(P^4Q^4 + \frac{3^4}{P^4Q^4}\right)\left[10\left(23\frac{P}{Q} - 12\frac{Q}{P}\right) - 6\left(28\frac{P^3}{Q^3} - 15\frac{Q^3}{P^3}\right)\right]\\
&-\left(P^5Q^5 + \frac{3^5}{P^5Q^5}\right)\left[-5\left(18\frac{P}{Q} - \frac{Q}{P}\right) + 15\left(\frac{P^3}{Q^3} - \frac{Q^3}{P^3}\right)\right] - 50\left(\frac{P^8}{Q^6} + 3\frac{P^6}{Q^8}\right)\\
&+\left(P^8Q^6 + \frac{3^7}{P^6Q^8}\right) - \left(P^3Q^9 + \frac{3^6}{P^9Q^3}\right) - 15\left(P^7Q^5 + \frac{3^6}{P^5Q^7}\right) = 0.
\end{aligned} \tag{3.4}$$

Proof. The proof of the equation (3.4) is similar to the proof of the equation (3.1) except that in the place of the equation (2.9); equation (2.8) is used. □

Remark 2. The equation (3.4) holds for

$$P := \frac{\psi(q)}{q^{1/4}\psi(q^3)} \quad \text{and} \quad Q := \frac{\psi(q^{5/3})}{q^{5/12}\psi(q^5)}.$$

Theorem 3.3. *If* $P := \dfrac{\varphi(q)}{\varphi(q^3)}$ *and* $Q := \dfrac{\varphi(q^{21})}{\varphi(q^{63})}$, *then*

$$\begin{aligned}
&\frac{Q^{12}}{P^{12}} = -63\frac{Q^{11}}{P^{11}} + 3\left(3^6\frac{P^{10}}{Q^{10}} - 413\frac{Q^{10}}{P^{10}}\right) - 9849\frac{Q^9}{P^9} + 21\left(3^5\frac{P^8}{Q^8} - 2041\frac{Q^8}{P^8}\right)\\
&- 567\left(3^3\frac{P^7}{Q^7} + 200\frac{Q^7}{P^7}\right) + 5103\left(13\frac{P^6}{Q^6} - 41\frac{Q^6}{P^6}\right) + 189\left(279\frac{P^5}{Q^5} - 2210\frac{Q^5}{P^5}\right)\\
&+ 3^3\left(1815\frac{P^4}{Q^4} - 33754\frac{Q^4}{P^4}\right) - 378\left(963\frac{P^3}{Q^3} + 4621\frac{Q^3}{P^3}\right)\\
&- 1404\left(315\frac{P^2}{Q^2} + 1541\frac{Q^2}{P^2}\right) - 2646\left(183\frac{P}{Q} + 5^4\frac{Q}{P}\right) + \left(P^{12}Q^4 + \frac{3^8}{P^4Q^{12}}\right)\\
&+\left(P^9Q^{11} + \frac{3^{10}}{P^{11}Q^9}\right) - 21\left(P^5Q^{11} + \frac{3^8}{P^{11}Q^5}\right) + 700\left(P^3Q^9 + \frac{3^6}{P^9Q^3}\right)\\
&+ 315\left(\frac{Q^{11}}{P^7} + 3^2\frac{Q^7}{P^{11}}\right) + \left(P^2Q^2 + \frac{3^2}{P^2Q^2}\right)\left[252\left(993\frac{P}{Q} + 1628\frac{Q}{P}\right)\right.
\end{aligned}$$

$$+3024\left(5\frac{P^2}{Q^2}+98\frac{Q^2}{P^2}\right)-63\left(123\frac{P^3}{Q^3}-3949\frac{Q^3}{P^3}\right)-378\left(19\frac{P^4}{Q^4}-563\frac{Q^4}{P^4}\right)$$
$$+42\left(459\frac{P^5}{Q^5}+3551\frac{Q^5}{P^5}\right)-630\left(3^2\frac{P^6}{Q^6}-107\frac{Q^6}{P^6}\right)-3^2\left(378\frac{P^7}{Q^7}-2315\frac{Q^7}{P^7}\right)$$
$$-14\left(3^5\frac{P^8}{Q^8}-269\frac{Q^8}{P^8}\right)+409752\Bigg]+\left(P^4Q^4+\frac{3^4}{P^4Q^4}\right)$$
$$\times\Bigg[-3\left(8709\frac{P}{Q}+10633\frac{Q}{P}\right)-189\left(113\frac{P^2}{Q^2}+239\frac{Q^2}{P^2}\right)$$
$$-42\left(267\frac{P^3}{Q^3}+1039\frac{Q^3}{P^3}\right)+126\left(29\frac{P^4}{Q^4}-213\frac{Q^4}{P^4}\right)+42\left(63\frac{P^5}{Q^5}-283\frac{Q^5}{P^5}\right)$$
$$+189\left(7\frac{P^6}{Q^6}-15\frac{Q^6}{P^6}\right)-\left(9^2\frac{P^7}{Q^7}+385\frac{Q^7}{P^7}\right)-17640\Bigg]+\left(P^6Q^6+\frac{3^6}{P^6Q^6}\right)$$
$$\times\Bigg[147\left(7\frac{P}{Q}+22\frac{Q}{P}\right)-546\left(\frac{P^2}{Q^2}-3\frac{Q^2}{P^2}\right)+6\left(7\frac{P^4}{Q^4}+45\frac{Q^4}{P^4}\right)$$
$$+21\left(3\frac{P^5}{Q^5}+7\frac{Q^5}{P^5}\right)+2660\Bigg]+\left(P^8Q^8+\frac{3^8}{P^8Q^8}\right)\left[-7\left(\frac{P}{Q}-6\frac{Q}{P}\right)+63\right]$$
$$-701568. \tag{3.5}$$

Proof. The proof of the equation (3.5) is similar to the proof of the equation (3.1) except that in the place of the equations (2.3) and (2.4); equations (2.5) and (2.6) are used. □

Remark 3. The equation (3.5) holds for

$$P:=\frac{\psi(q)}{q^{1/4}\psi(q^3)}\quad\text{and}\quad Q:=\frac{\psi(q^{21})}{q^{21/4}\psi(q^{63})}.$$

Theorem 3.4. *If* $P:=\dfrac{\varphi(q)}{\varphi(q^3)}$ *and* $Q:=\dfrac{\varphi(q^{7/3})}{\varphi(q^7)}$, *then*

$$\frac{P^{12}}{Q^{12}}=63\frac{P^{11}}{Q^{11}}-3\left(413\frac{P^{10}}{Q^{10}}-3^6\frac{Q^{10}}{P^{10}}\right)+9849\frac{P^9}{Q^9}-21\left(2041\frac{P^8}{Q^8}-3^5\frac{Q^8}{P^8}\right)$$
$$+567\left(200\frac{P^7}{Q^7}+3^3\frac{Q^7}{P^7}\right)-5103\left(41\frac{P^6}{Q^6}-13\frac{Q^6}{P^6}\right)+189\left(2210\frac{P^5}{Q^5}-279\frac{Q^5}{P^5}\right)$$
$$-3^3\left(33754\frac{P^4}{Q^4}-1815\frac{Q^4}{P^4}\right)+378\left(4621\frac{P^3}{Q^3}+963\frac{Q^3}{P^3}\right)-1404\left(1541\frac{P^2}{Q^2}\right.$$
$$\left.+315\frac{Q^2}{P^2}\right)+2646\left(5^4\frac{P}{Q}+183\frac{Q}{P}\right)+21\left(Q^5P^{11}+\frac{3^8}{P^5Q^{11}}\right)-315\left(\frac{P^{11}}{Q^7}\right.$$
$$\left.+9\frac{P^7}{Q^{11}}\right)+385\left(\frac{P^{11}}{Q^3}+3^4\frac{P^3}{Q^{11}}\right)+3^4\left(3^4\frac{Q^3}{P^{11}}+\frac{Q^{11}}{P^3}\right)+\left(P^4Q^{12}+\frac{3^8}{P^{12}Q^4}\right)$$
$$-700\left(P^9Q^3+\frac{3^6}{P^3Q^9}\right)-\left(P^{11}Q^9+\frac{3^{10}}{P^9Q^{11}}\right)+\left(P^2Q^2+\frac{9}{P^2Q^2}\right)$$

$$\times\left[-252\left(1628\frac{P}{Q}+993\frac{Q}{P}\right)+3024\left(98\frac{P^2}{Q^2}+5\frac{Q^2}{P^2}\right)+63\left(-3949\frac{P^3}{Q^3}+123\frac{Q^3}{P^3}\right)\right.$$
$$+378\left(563\frac{P^4}{Q^4}-19\frac{Q^4}{P^4}\right)-42\left(3551\frac{P^5}{Q^5}+459\frac{Q^5}{P^5}\right)+630\left(107\frac{P^6}{Q^6}-3^2\frac{Q^6}{P^6}\right)$$
$$\left.+3^2\left(-2315\frac{P^7}{Q^7}+378\frac{Q^7}{P^7}\right)+14\left(269\frac{P^8}{Q^8}-3^5\frac{Q^8}{P^8}\right)+409752\right]-701568$$
$$+\left(P^4Q^4+\frac{3^4}{P^4Q^4}\right)\left[3\left(10633\frac{P}{Q}+8709\frac{Q}{P}\right)-189\left(239\frac{P^2}{Q^2}+113\frac{Q^2}{P^2}\right)\right.$$
$$+42\left(1039\frac{P^3}{Q^3}+267\frac{Q^3}{P^3}\right)+126\left(-213\frac{P^4}{Q^4}+29\frac{Q^4}{P^4}\right)+42\left(283\frac{P^5}{Q^5}-63\frac{Q^5}{P^5}\right)$$
$$\left.+189\left(-15\frac{P^6}{Q^6}+7\frac{Q^6}{P^6}\right)-17640\right]+\left(P^6Q^6+\frac{3^6}{P^6Q^6}\right)\left[-147\left(22\frac{P}{Q}+7\frac{Q}{P}\right)\right.$$
$$\left.+546\left(3\frac{P^2}{Q^2}-\frac{Q^2}{P^2}\right)+6\left(45\frac{P^4}{Q^4}+7\frac{Q^4}{P^4}\right)-21\left(7\frac{P^5}{Q^5}+3\frac{Q^5}{P^5}\right)+2660\right]$$
$$+\left(P^8Q^8+\frac{3^8}{P^8Q^8}\right)\left[7\left(-6\frac{P}{Q}+\frac{Q}{P}\right)+63\right]. \tag{3.6}$$

Proof. The proof of the equation (3.6) is similar to the proof of the equation (3.5) except that in the place of the equation (2.9); equation (2.8) is used. □

Remark 4. The equation (3.6) holds for

$$P:=\frac{\psi(q)}{q^{1/4}\psi(q^3)}\quad\text{and}\quad Q:=\frac{\psi(q^{7/3})}{q^{7/12}\psi(q^7)}.$$

4. Formulas for Explicit Evaluations of Ramanujan's Theta Functions

In this section, we establish several new general formulas for explicit evaluations of ratios of Ramanujan's theta functions by using the modular equations of degree 3 established in the previous section.

Theorem 4.1. *If $X := h_{3,n}$ and $Y := h_{3,225n}$, then*

$$\frac{Y^9}{X^9}=1860\frac{Y^5}{X^5}-3^2\left(20\frac{Y^7}{X^7}-3^3\frac{X^7}{Y^7}\right)-216\left(5^2\frac{Y^3}{X^3}-21\frac{X^3}{Y^3}\right)$$
$$+18\left(1121\frac{Y}{X}-330\frac{X}{Y}\right)+\sqrt{3}\left(XY+\frac{1}{XY}\right)\left[-45\left(98\frac{X}{Y}-179\frac{Y}{X}\right)\right.$$
$$\left.+15\left(126\frac{X^3}{Y^3}-241\frac{Y^3}{X^3}\right)-3\left(135\frac{X^5}{Y^5}-296\frac{Y^5}{X^5}\right)\right]+3\left(X^2Y^2+\frac{1}{X^2Y^2}\right)$$
$$\times\left[-15\left(180\frac{X}{Y}-107\frac{Y}{X}\right)+15\left(3^3\frac{X^3}{Y^3}-154\frac{Y^3}{X^3}\right)-10\left(3^3\frac{X^5}{Y^5}-31\frac{Y^5}{X^5}\right)\right]$$
$$+3\sqrt{3}\left(X^3Y^3+\frac{1}{X^3Y^3}\right)\left[-3\left(311\frac{X}{Y}-140\frac{Y}{X}\right)+225\left(\frac{X^3}{Y^3}-2^2\frac{Y^3}{X^3}\right)\right.$$

$$
\begin{aligned}
&-3\left(3^2\frac{X^5}{Y^5}-20\frac{Y^5}{X^5}\right)\Bigg]+3^2\left(X^4Y^4+\frac{1}{X^4Y^4}\right)\Bigg[-10\left(12\frac{X}{Y}-23\frac{Y}{X}\right)\\
&+6\left(15\frac{X^3}{Y^3}-28\frac{Y^3}{X^3}\right)\Bigg]+9\sqrt{3}\left(X^5Y^5+\frac{1}{X^5Y^5}\right)\Bigg[-5\left(\frac{X}{Y}-18\frac{Y}{X}\right)\\
&+15\left(\frac{X^3}{Y^3}-\frac{Y^3}{X^3}\right)\Bigg]+27\sqrt{3}\left(X^6Y^8+\frac{1}{X^8Y^6}\right)+27\left(X^9Y^3+\frac{1}{X^3Y^9}\right)\\
&+405\left(X^5Y^7+\frac{1}{X^7Y^5}\right)-50\sqrt{3}\left(\frac{Y^6}{X^8}+\frac{Y^8}{X^6}\right).
\end{aligned}
\tag{4.1}
$$

Proof. Using the equations (3.1) and (1.2), we arrive at (4.1). □

Lemma 4.1. *We have*

$$h_{3,15}=(6\sqrt{5}-6\sqrt{3}-3)^{1/4}\sqrt{\sqrt{3}+2}, \tag{4.2}$$

$$h_{3,1/15}=\left(\frac{1+2\sqrt{3}+2\sqrt{5}}{3}\right)^{1/4}, \tag{4.3}$$

$$l_{3,15}=(6\sqrt{5}+6\sqrt{3}+3)^{1/4}\sqrt{\sqrt{3}+2}, \tag{4.4}$$

$$l_{3,1/15}=\left(\frac{2\sqrt{5}-1-2\sqrt{3}}{3}\right)^{1/4}. \tag{4.5}$$

Proofs of (4.2) *and* (4.3)**.** Putting $n=1/15$ in the equation (4.1) and using the fact that $h_{3,15}h_{3,1/15}=1$, we deduce that

$$
\begin{aligned}
&(h_{3,15}^8+186h_{3,15}^4+108\sqrt{3}h_{3,15}^4-63-36\sqrt{3})(h_{3,15}^8-6h_{3,15}^4\\
&-2\sqrt{3}h_{3,15}^4+3)^2(h_{3,15}^4+3-2\sqrt{3})^2=0.
\end{aligned}
\tag{4.6}
$$

Since the first factor of the equation (4.6) is zero for the specific value of $q=e^{-\pi/3\sqrt{5}}$, whereas the other factors are not zero, we find that

$$h_{3,15}^8+186h_{3,15}^4+108\sqrt{3}h_{3,15}^4-63-36\sqrt{3}=0. \tag{4.7}$$

On solving (4.7) and using $0<h_{3,15}<1$, we arrive at (4.2).
Since $h_{3,15}h_{3,1/15}=1$, we arrive at (4.3). □

Proofs of (4.4) *and* (4.5)**.** Using the equations (2.10), (4.2) and (4.3), we arrive at (4.4) and (4.5). □

Theorem 4.2. *If* $X := h_{3,n}$ *and* $Y := h_{3,25n/9}$*, then*

$$\begin{aligned}
&\frac{X^9}{Y^9} + 9\left(20\frac{X^7}{Y^7} - 27\frac{Y^7}{X^7}\right) - 1860\frac{X^5}{Y^5} + 216\left(5^2\frac{X^3}{Y^3} - 21\frac{Y^3}{X^3}\right) - 18\left(1121\frac{X}{Y}\right.\\
&\left.-330\frac{Y}{X}\right) - \sqrt{3}\left(XY + \frac{1}{XY}\right)\left[-45\left(179\frac{X}{Y} - 98\frac{Y}{X}\right) + 15\left(241\frac{X^3}{Y^3} - 126\frac{Y^3}{X^3}\right)\right.\\
&\left.-3\left(296\frac{X^5}{Y^5} - 135\frac{Y^5}{X^5}\right)\right] - 3\left(X^2Y^2 + \frac{1}{X^2Y^2}\right)\left[15\left(107\frac{X}{Y} - 180\frac{Y}{X}\right)\right.\\
&\left.-15\left(154\frac{X^3}{Y^3} - 3^3\frac{Y^3}{X^3}\right) + 10\left(31\frac{X^5}{Y^5} - 3^3\frac{Y^5}{X^5}\right)\right] - 3\sqrt{3}\left(X^3Y^3 + \frac{1}{X^3Y^3}\right)\\
&\times\left[-3\left(140\frac{X}{Y} - 311\frac{Y}{X}\right) + 225\left(4\frac{X^3}{Y^3} - \frac{Y^3}{X^3}\right) - 3\left(20\frac{X^5}{Y^5} - 3^2\frac{Y^5}{X^5}\right)\right]\\
&- 3^2\left(X^4Y^4 + \frac{1}{X^4Y^4}\right)\left[10\left(23\frac{X}{Y} - 12\frac{Y}{X}\right) - 6\left(28\frac{X^3}{Y^3} - 15\frac{Y^3}{X^3}\right)\right]\\
&- 9\sqrt{3}\left(X^5Y^5 + \frac{1}{X^5Y^5}\right)\left[-5\left(18\frac{X}{Y} - \frac{Y}{X}\right) + 15\left(\frac{X^3}{Y^3} - \frac{Y^3}{X^3}\right)\right]\\
&- 50\sqrt{3}\left(\frac{X^8}{Y^6} + \frac{X^6}{Y^8}\right) + 27\sqrt{3}\left(X^8Y^6 + \frac{1}{X^6Y^8}\right) - 3^3\left(X^3Y^9 + \frac{1}{X^9Y^3}\right)\\
&- 405\left(X^7Y^5 + \frac{1}{X^5Y^7}\right) = 0.
\end{aligned} \tag{4.8}$$

Proof. Using the equations (3.4) and (1.2), we arrive at (4.8). □

Lemma 4.2. *We have*

$$h_{3,5/3} = \left(\frac{1 - 2\sqrt{3} + 2\sqrt{5}}{3}\right)^{1/4}, \tag{4.9}$$

$$h_{3,3/5} = (6\sqrt{5} + 6\sqrt{3} - 3)^{1/4}\sqrt{2 - \sqrt{3}}, \tag{4.10}$$

$$l_{3,5/3} = \left(\frac{-1 + 2\sqrt{3} + 2\sqrt{5}}{3}\right)^{1/4}, \tag{4.11}$$

$$l_{3,3/5} = (6\sqrt{5} - 6\sqrt{3} + 3)^{1/4}\sqrt{2 - \sqrt{3}}. \tag{4.12}$$

Proofs of (4.9) *and* (4.10)**.** Putting $n = 3/5$ in the equation (4.8) and using the fact that $h_{3,5/3}h_{3,3/5} = 1$, we deduce that

$$\begin{aligned}
&(3h_{3,5/3}^8 - 6h_{3,5/3}^4 + 2\sqrt{3}h_{3,5/3}^4 + 1)^2(3h_{3,5/3}^4 - 3 + 2\sqrt{3})^2\\
&(9h_{3,5/3}^8 - 6h_{3,5/3}^4 + 12\sqrt{3}h_{3,5/3}^4 - 7 - 4\sqrt{3}) = 0.
\end{aligned} \tag{4.13}$$

Since the last factor of the above equation (4.13) is zero for the specific value of $q = e^{-\pi/\sqrt{5}}$ and other factors are not zero, we find that

$$9h_{3,5/3}^8 - 6h_{3,5/3}^4 + 12\sqrt{3}h_{3,5/3}^4 - 7 - 4\sqrt{3} = 0. \tag{4.14}$$

Solving the above equation (4.14) and $h_{3,5/3} < 1$, we arrive at (4.9) and (4.10). □

Proofs of (4.11) *and* (4.12)**.** Using the equation (2.10) along with the equations (4.9) and (4.10) respectively, we arrive at (4.11) and (4.12). □

Theorem 4.3. *If* $X := h_{3,n}$ *and* $Y := h_{3,441n}$, *then*

$$
\begin{aligned}
\frac{Y^{12}}{X^{12}} &= -63\frac{Y^{11}}{X^{11}} + 3\left(3^6\frac{X^{10}}{Y^{10}} - 413\frac{Y^{10}}{X^{10}}\right) - 9849\frac{Y^9}{X^9} + 21\left(3^5\frac{X^8}{Y^8} - 2041\frac{Y^8}{X^8}\right) \\
&- 567\left(3^3\frac{X^7}{Y^7} + 200\frac{Y^7}{X^7}\right) + 5103\left(13\frac{X^6}{Y^6} - 41\frac{Y^6}{X^6}\right) + 189\left(279\frac{X^5}{Y^5} - 2210\frac{Y^5}{X^5}\right) \\
&+ 3^3\left(1815\frac{X^4}{Y^4} - 33754\frac{Y^4}{X^4}\right) - 378\left(963\frac{X^3}{Y^3} + 4621\frac{Y^3}{X^3}\right) \\
&- 1404\left(315\frac{X^2}{Y^2} + 1541\frac{Y^2}{X^2}\right) - 2646\left(183\frac{X}{Y} + 5^4\frac{Y}{X}\right) + 3^4\left(X^{12}Y^4 + \frac{1}{X^4Y^{12}}\right) \\
&+ 3^5\left(X^9Y^{11} + \frac{1}{X^{11}Y^9}\right) - 1701\left(X^5Y^{11} + \frac{1}{X^{11}Y^5}\right) + 18900\left(X^3Y^9 + \frac{1}{X^9Y^3}\right) \\
&+ 945\left(\frac{Y^{11}}{X^7} + \frac{Y^7}{X^{11}}\right) + 3\left(X^2Y^2 + \frac{1}{X^2Y^2}\right)\left[252\left(993\frac{X}{Y} + 1628\frac{Y}{X}\right)\right. \\
&+3024\left(5\frac{X^2}{Y^2} + 98\frac{Y^2}{X^2}\right) - 63\left(123\frac{X^3}{Y^3} - 3949\frac{Y^3}{X^3}\right) - 378\left(19\frac{X^4}{Y^4} - 563\frac{Y^4}{X^4}\right) \\
&+42\left(459\frac{X^5}{Y^5} + 3551\frac{Y^5}{X^5}\right) - 630\left(3^2\frac{X^6}{Y^6} - 107\frac{Y^6}{X^6}\right) - 3^2\left(378\frac{X^7}{Y^7} - 2315\frac{Y^7}{X^7}\right) \\
&\left.-14\left(3^5\frac{X^8}{Y^8} - 269\frac{Y^8}{X^8}\right) + 409752\right] + 3^2\left(X^4Y^4 + \frac{1}{X^4Y^4}\right) \\
&\times\left[-3\left(8709\frac{X}{Y} + 10633\frac{Y}{X}\right) - 189\left(113\frac{X^2}{Y^2} + 239\frac{Y^2}{X^2}\right)\right. \\
&-42\left(267\frac{X^3}{Y^3} + 1039\frac{Y^3}{X^3}\right) + 126\left(29\frac{X^4}{Y^4} - 213\frac{Y^4}{X^4}\right) + 42\left(63\frac{X^5}{Y^5} - 283\frac{Y^5}{X^5}\right) \\
&\left.+189\left(7\frac{X^6}{Y^6} - 15\frac{Y^6}{X^6}\right) - \left(9^2\frac{X^7}{Y^7} + 385\frac{Y^7}{X^7}\right) - 17640\right] + 3^3\left(X^6Y^6 + \frac{1}{X^6Y^6}\right) \\
&\times\left[147\left(7\frac{X}{Y} + 22\frac{Y}{X}\right) - 546\left(\frac{X^2}{Y^2} - 3\frac{Y^2}{X^2}\right) + 6\left(7\frac{X^4}{Y^4} + 45\frac{Y^4}{X^4}\right)\right. \\
&\left.+21\left(3\frac{X^5}{Y^5} + 7\frac{Y^5}{X^5}\right) + 2660\right] + 3^4\left(X^8Y^8 + \frac{1}{X^8Y^8}\right)\left[-7\left(\frac{X}{Y} - 6\frac{Y}{X}\right) + 63\right] \\
&- 701568.
\end{aligned}
\tag{4.15}
$$

Proof. Using (1.2) and (3.5), we arrive at (4.15). □

Lemma 4.3. *We have*

$$h_{3,21} = \frac{\sqrt{3-\sqrt{21}+\sqrt{6+2\sqrt{21}}}}{2}, \tag{4.16}$$

$$h_{3,1/21} = \frac{\sqrt{18+6\sqrt{27+6\sqrt{21}}}}{6}, \tag{4.17}$$

$$l_{3,21} = \frac{\sqrt{(36+8\sqrt{21})+(10+2\sqrt{21})\sqrt{6+2\sqrt{21}}}}{2}, \tag{4.18}$$

$$l_{3,1/21} = \frac{\sqrt{-9+(15-3\sqrt{21})\sqrt{27+6\sqrt{21}}}}{3}. \tag{4.19}$$

Proof of (4.16) *and* (4.17)**.** Putting $n = 1/21$ in the equation (4.15) and using the fact that $h_{3,21}h_{3,1/21} = 1$, we deduce that

$$\begin{aligned}&(h_{3,21}^{12}+30h_{3,21}^{10}+81h_{3,21}^{8}+60h_{3,21}^{6}-9h_{3,21}^{4}-18h_{3,21}^{2}-9)^2\\&(h_{3,21}^{6}+3h_{3,21}^{4}+3h_{3,21}^{2}-3)^2(h_{3,21}^{8}-3h_{3,21}^{6}+6h_{3,21}^{2}-3)=0.\end{aligned} \tag{4.20}$$

Since the last factor of the equation (4.20) is zero for the specific value of $q = e^{-\pi/3\sqrt{7}}$, we find that

$$h_{3,21}^{8}-3h_{3,21}^{6}+6h_{3,21}^{2}-3=0. \tag{4.21}$$

On solving (4.21) and using $0 < h_{3,21} < 1$, we arrive at (4.16) and (4.17). □

Proofs of (4.18) *and* (4.19)**.** Using the equation (2.10) along with the equations (4.16) and (4.17) respectively, we arrive at (4.18) and (4.19). □

Theorem 4.4. *If* $X := h_{3,n}$ *and* $Y := h_{3,49n/9}$*, then*

$$\begin{aligned}\frac{X^{12}}{Y^{12}} &= 63\frac{X^{11}}{Y^{11}} - 3\left(413\frac{X^{10}}{Y^{10}} - 3^6\frac{Y^{10}}{X^{10}}\right) + 9849\frac{X^9}{Y^9} - 21\left(2041\frac{X^8}{Y^8} - 3^5\frac{Y^8}{X^8}\right)\\
&+ 567\left(200\frac{X^7}{Y^7} + 3^3\frac{Y^7}{X^7}\right) - 5103\left(41\frac{X^6}{Y^6} - 13\frac{Y^6}{X^6}\right) + 189\left(2210\frac{X^5}{Y^5} - 279\frac{Y^5}{X^5}\right)\\
&- 3^3\left(33754\frac{X^4}{Y^4} - 1815\frac{Y^4}{X^4}\right) + 378\left(4621\frac{X^3}{Y^3} + 963\frac{Y^3}{X^3}\right) - 1404\left(1541\frac{X^2}{Y^2}\right.\\
&\left.+315\frac{Y^2}{X^2}\right) + 2646\left(5^4\frac{X}{Y} + 183\frac{Y}{X}\right) + 1701\left(Y^5X^{11} + \frac{1}{X^5Y^{11}}\right) - 945\left(\frac{X^{11}}{Y^7}\right.\\
&\left.+\frac{X^7}{Y^{11}}\right) + 3465\left(\frac{X^{11}}{Y^3} + \frac{X^3}{Y^{11}}\right) + 3^6\left(\frac{Y^3}{X^{11}} + \frac{Y^{11}}{X^3}\right) + 3^4\left(X^4Y^{12} + \frac{1}{X^{12}Y^4}\right)\\
&- 18900\left(X^9Y^3 + \frac{1}{X^3Y^9}\right) - 3^5\left(X^{11}Y^9 + \frac{1}{X^9Y^{11}}\right) + 3\left(X^2Y^2 + \frac{1}{X^2Y^2}\right)\\
&\times\left[-252\left(1628\frac{X}{Y} + 993\frac{Y}{X}\right) + 3024\left(98\frac{X^2}{Y^2} + 5\frac{Y^2}{X^2}\right) + 63\left(-3949\frac{X^3}{Y^3} + 123\frac{Y^3}{X^3}\right)\right.\end{aligned} \tag{4.22}$$

$$+378\left(563\frac{X^4}{Y^4}-19\frac{Y^4}{X^4}\right)-42\left(3551\frac{X^5}{Y^5}+459\frac{Y^5}{X^5}\right)+630\left(107\frac{X^6}{Y^6}-3^2\frac{Y^6}{X^6}\right)$$
$$+3^2\left(-2315\frac{X^7}{Y^7}+378\frac{Y^7}{X^7}\right)+14\left(269\frac{X^8}{Y^8}-3^5\frac{Y^8}{X^8}\right)+409752\Bigg]-701568$$
$$+3^2\left(X^4Y^4+\frac{1}{X^4Y^4}\right)\left[3\left(10633\frac{X}{Y}+8709\frac{Y}{X}\right)-189\left(239\frac{X^2}{Y^2}+113\frac{Y^2}{X^2}\right)\right.$$
$$+42\left(1039\frac{X^3}{Y^3}+267\frac{Y^3}{X^3}\right)+126\left(-213\frac{X^4}{Y^4}+29\frac{Y^4}{X^4}\right)+42\left(283\frac{X^5}{Y^5}-63\frac{Y^5}{X^5}\right)$$
$$\left.+189\left(-15\frac{X^6}{Y^6}+7\frac{Y^6}{X^6}\right)-17640\right]+3^3\left(X^6Y^6+\frac{1}{X^6Y^6}\right)\left[-147\left(22\frac{X}{Y}+7\frac{Y}{X}\right)\right.$$
$$\left.+546\left(3\frac{X^2}{Y^2}-\frac{Y^2}{X^2}\right)+6\left(45\frac{X^4}{Y^4}+7\frac{Y^4}{X^4}\right)-21\left(7\frac{X^5}{Y^5}+3\frac{Y^5}{X^5}\right)+2660\right]$$
$$+3^4\left(X^8Y^8+\frac{1}{X^8Y^8}\right)\left[7\left(-6\frac{X}{Y}+\frac{Y}{X}\right)+63\right].$$

Proof. Using (3.6) and (1.2), we arrive at (4.22). □

Lemma 4.4. *We have*

$$h_{3,7/3}=\frac{\sqrt{-18+6\sqrt{27+6\sqrt{21}}}}{6},\tag{4.23}$$
$$h_{3,3/7}=\frac{\sqrt{-3+\sqrt{21}+\sqrt{6+2\sqrt{21}}}}{2},\tag{4.24}$$

$$l_{3,7/3}=\frac{\sqrt{9+(15-3\sqrt{21})\sqrt{27+6\sqrt{21}}}}{3},\tag{4.25}$$
$$l_{3,3/7}=\frac{\sqrt{(10+2\sqrt{21})\sqrt{6+2\sqrt{21}}-(36+8\sqrt{21})}}{2}.\tag{4.26}$$

Proofs of (4.23) *and* (4.24)**.** Putting $n=3/7$ in the equation (4.22) and using the fact that $h_{3,7/3}h_{3,3/7}=1$, we deduce that

$$\begin{aligned}&(3h_{3,7/3}^8+6h_{3,7/3}^6-3h_{3,7/3}^2-1)(3h_{3,7/3}^6+3h_{3,7/3}^4-3h_{3,7/3}^2+1)^2\\&(9h_{3,7/3}^{12}-18h_{3,7/3}^{10}+9h_{3,7/3}^8+60h_{3,7/3}^6-81h_{3,7/3}^4+30h_{3,7/3}^2-1)^2=0.\end{aligned}\tag{4.27}$$

Since the first factor of the equation (4.27) is zero for the specific value of $q=e^{-\pi/\sqrt{7}}$ and other factors are not zero, we find that

$$3h_{3,7/3}^8+6h_{3,7/3}^6-3h_{3,7/3}^2-1=0.\tag{4.28}$$

Solving (4.28) and using $0<h_{3,7/3}<1$, we arrive at (4.23) and (4.24). □

Proofs of (4.25) *and* (4.26)**.** Using the equation (2.10) along with the equations (4.23) and (4.24) respectively, we arrive at (4.25) and (4.26). □

5. Modular Relations for Ramanujan's Cubic Continued Fraction

In this section, we establish modular relation connecting Ramanujan's cubic continued fraction $V(q)$ with each of $V(q^{15})$, $V(q^{5/3})$, $V(q^{7/3})$ and $V(q^{21})$.

Theorem 5.1. *If $v = V(q)$ and $w = V(q^{15})$, then*

$$\begin{aligned}
& w^{18} + 2\left(180v^3 - 8192v^{15} - 15360v^{12} - 320v^6 - 3 - 7680v^9\right)w^{17} - \left(133120v^{12}\right.\\
& \left.+1300v^3 + 49152v^{15} - 21360v^6 + 61440v^9 - 21\right)w^{16} - 2\left(25 + 105v^3 + 49920v^{12}\right.\\
& \left.-840v^6 + 6400v^9 + 43008v^{15}\right)w^{15} - 10\left(2582v^6 - 9 - 5760v^9 + 2304v^{12}\right.\\
& \left.+11264v^{15} - 321v^3\right)w^{14} - 2\left(61440v^{15} + 31680v^9 + 1597v^3 + 63 + 104320v^{12}\right.\\
& \left.-13950v^6\right)w^{13} - \left(56960v^9 + 109056v^{15} - 564v^3 - 4096v^{18} - 14955v^6\right.\\
& \left.+166080v^{12} - 141\right)w^{12} - 2\left(15410v^6 - 6144v^{18} + 63 - 31200v^9 + 30464v^{15}\right.\\
& \left.-26880v^{12} - 903v^3\right)w^{11} - 2\left(365v^3 - 45 - 10140v^9 - 10752v^{18} + 1880v^{12}\right.\\
& \left.-4305v^6 + 18432v^{15}\right)w^{10} - 10\left(5472v^{12} + 4170v^9 - 2560v^{18} + 2880v^{15} + 5\right.\\
& \left.+45v^3 - 684v^6\right)w^9 + \left(23040v^{18} + 288v^3 - 235v^6 - 10140v^9 + 21 + 34440v^{12}\right.\\
& \left.+23360v^{15}\right)w^8 + 2\left(30820v^{12} - 840v^6 + 8064v^{18} + 14448v^{15} - 119v^3 + 7800v^9\right.\\
& \left.-3\right)w^7 + \left(14955v^{12} + 9024v^{18} + 1 + 213v^3 - 4512v^{15} + 7120v^9 - 2595v^6\right)w^6\\
& + 2\left(815v^6 - 6388v^{15} - 1980v^9 - 6975v^{12} + 2016v^{18} - 60v^3\right)w^5 + 5\left(11v^3\right.\\
& \left.-360v^9 - 18v^6 - 1291v^{12} - 1284v^{15} + 288v^{18}\right)w^4 + \left(195v^6 - 200v^9 - 210v^{12}\right.\\
& \left.+400v^{18} - 210v^{15} - 21v^3\right)w^3 + \left(650v^{15} + 84v^{18} - 130v^6 + 1335v^{12} + 480v^9\right.\\
& \left.+6v^3\right)w^2 - \left(v^3 - 15v^6 - 90v^{15} + 60v^9 - 12v^{18} - 20v^{12}\right)w + v^{18} = 0.
\end{aligned} \tag{5.1}$$

Proof. Using the Remark 1 along with the equations (3.1) and (2.7), we deduce that

$$\begin{aligned}
& 27w^9v^3 - 159426w^{12}v^9 + 225v^9w^3 - 45v^6w^3 + 4347v^{12}w^3 - 1728w^{12}v^3\\
& - 4650v^9w^6 - 108w^9v^6 + 360v^6w^6 + 27216v^{12}w^6 + 43254v^9w^9 + 24372w^{12}v^6\\
& - w^9 - 6w^{12} - 967329v^{12}w^9 + 1572102w^{12}v^{12} - 25344w^{12}v^{27} - 126x^2w^{24}v^3\\
& - 25280x^2w^{24}v^6 + 771330x^2w^{15}v^{12} - 265945x^2w^{15}v^9 + 3230x^2w^{15}v^6\\
& - 45x^2w^{15}v^3 + 186x^2w^3v^{12} - 1065624x^2w^{12}v^{12} + 294543x^2w^9v^{12}\\
& - 21432x^2w^6v^{12} - 29349x^2w^9v^9 - 514x^2w^{12}v^6 - 27x^2w^9v^6 + 45x^2w^{12}v^3\\
& + 1936x^2w^6v^9 - 2x^2w^6v^6 - 4860576w^{18}v^{12}x^2 + 341390b^2w^{12}v^{15}x^2\\
& + 8490b^2w^6v^{15}x^2 - 4721600b^2w^{18}v^{15}x^2 - 108420b^2w^{15}v^{15}x^2 - 387w^{18}v^3x^2\\
& - 67165w^9v^{15}x^2 + 518766w^{18}v^9x^2 + 144725w^{12}v^9x^2 - 19290w^{18}v^6x^2\\
& - 23w^3v^9x^2 + w^3v^6x^2 - 590w^3v^{15}x^2 - 6975040x^2v^{18}w^{12}\\
& - 2841310x^2v^{18}w^9 + 286710x^2v^{18}w^6 + 415x^2v^{18}w^3 + 2292328x^2w^{21}v^9
\end{aligned}$$

$$\begin{aligned}
&+1346368x^2w^{24}v^9+19660800x^2w^{24}v^{21}+32876544x^2w^{24}v^{18}\\
&+5045760x^2w^{24}v^{15}-9660288x^2w^{24}v^{12}-7701696x^2w^{12}v^{21}\\
&+101606400x^2w^{21}v^{21}+158138880x^2w^{18}v^{21}+69691200x^2w^{15}v^{21}\\
&+161839360x^2w^{18}v^{18}-401600x^2w^{21}v^{15}-16598208x^2w^{21}v^{12}\\
&+68051520x^2w^{15}v^{18}-49250w^{21}x^2v^6-441w^{21}x^2v^3-2672952x^2v^{21}w^9\\
&+273234x^2v^{21}w^6+675x^2v^{21}w^3-505728x^2v^{24}w^{12}+53312x^2v^{24}w^9\\
&-9952x^2v^{24}w^6-210x^2v^{24}w^3+2637824x^2v^{24}w^{24}+344064x^2v^{24}w^{21}\\
&-1832960x^2v^{24}w^{18}+1139200x^2v^{24}w^{15}+119414272x^2v^{18}w^{21}-15w^{15}\\
&-6w^{24}-15w^{21}-20w^{18}+15255w^{15}v^3+48615w^{15}v^9+14191740w^{15}v^{12}\\
&-197325w^{15}v^6-5008104v^{15}w^{12}-19644672v^{18}w^{12}+12861510v^{18}w^9\\
&-312540v^{18}w^6-62835v^{18}w^3+3201435v^{15}w^9-62100v^{15}w^6-17595v^{15}w^3\\
&-26142720w^{24}v^{18}+467040w^{24}v^9+521208w^{21}v^9+265986w^{18}v^9\\
&-13092192w^{12}v^{21}-21565440w^{24}v^{21}-112112640w^{21}v^{21}-205562880w^{18}v^{21}\\
&-137419200w^{15}v^{21}-656640w^{24}v^{15}-159774720w^{21}v^{18}-32541120w^{21}v^{15}\\
&-303694080w^{18}v^{18}-75503520w^{18}v^{15}-202231200w^{15}v^{18}-51214140w^{15}v^{15}\\
&+7655040w^{24}v^{12}+17203104w^{21}v^{12}+20949408w^{18}v^{12}+8654040v^{21}w^9\\
&-212490v^{21}w^6-43605v^{21}w^3-262080w^{24}v^6-50670w^{21}v^6-15120w^{18}v^6\\
&+6993w^{24}v^3-25785w^{21}v^3-15930w^{18}v^3-238464v^{24}w^{12}-105120v^{24}w^9\\
&-10080v^{24}w^6-2835v^{24}w^3-2801664v^{24}w^{24}-1658880v^{24}w^{21}\\
&+1972224v^{24}w^{18}-315648v^{24}w^{15}+196608w^{24}v^{27}+18x^2w^{27}v^3\\
&-30x^2v^{27}w^3+960x^2w^{27}v^6+14784x^2w^{27}v^9+101376x^2w^{27}v^{12}\\
&+524288x^2w^{27}v^{24}+860160x^2w^{27}v^{21}+745472x^2w^{27}v^{18}\\
&+366080x^2w^{27}v^{15}+140800x^2v^{27}w^{15}-101376x^2v^{27}w^{12}\\
&-50752x^2v^{27}w^9-3136x^2v^{27}w^6+131072x^2v^{27}w^{27}\\
&-262144a^2v^{27}w^{24}b^2+270336a^2v^{27}w^{21}b^2-204800a^2v^{27}w^{18}b^2\\
&-658944w^{27}v^{15}-4320w^{27}v^6-44352w^{27}v^9-1118208w^{27}v^{18}\\
&-1105920w^{27}v^{21}-589824w^{27}v^{24}-131072w^{27}v^{27}+16896w^{15}v^{27}\\
&-15840v^{27}w^6+28672w^{18}v^{27}-w^{27}-v^{27}-228096w^{27}v^{12}-162w^{27}v^3\\
&-402v^{27}w^3-109888v^{27}w^9-122880w^{21}v^{27}=0,
\end{aligned} \tag{5.2}$$

where

$$x:=\frac{\psi(q)}{q^4\psi(q^3)}\frac{\psi(q^{15})}{\psi(q^{45})}.$$

Collecting the terms containing x^2 on one side of the equation (5.2) and then squaring both sides, we arrive at (5.1). This completes the proof. □

Theorem 5.2. *If $v = V(q)$ and $w = V(q^{5/3})$, then*

$$\begin{aligned}
&v^{18} - 2\left(15360w^{12} + 3 + 320w^6 + 8192w^{15} + 7680w^9 - 180w^3\right)v^{17} - \left(1300w^3\right.\\
&\left.-21 + 133120w^{12} + 61440w^9 - 21360w^6 + 49152w^{15}\right)v^{16} - 2\left(105w^3 - 840w^6\right.\\
&\left.+49920w^{12} + 43008w^{15} + 5^2 + 6400w^9\right)v^{15} - 10\left(2582w^6 - 5760w^9 + 2304w^{12}\right.\\
&\left.-9 + 11264w^{15} - 321w^3\right)v^{14} + 2\left(13950w^6 - 31680w^9 - 61440w^{15} - 1597w^3\right.\\
&\left.-104320w^{12} - 63\right)v^{13} - \left(166080w^{12} + 109056w^{15} + 56960w^9 - 2^{12}w^{18} - 141\right.\\
&\left.-14955w^6 - 564w^3\right)v^{12} + 2\left(26880w^{12} - 30464w^{15} - 63 - 15410w^6 + 6144w^{18}\right.\\
&\left.+903w^3 + 31200w^9\right)v^{11} - 2\left(1880w^{12} - 45 + 18432w^{15} - 10752w^{18} - 4305w^6\right.\\
&\left.+365w^3 - 10140w^9\right)v^{10} - 10\left(5472w^{12} + 4170w^9 - 2560w^{18} - 684w^6 + 45w^3\right.\\
&\left.+5 + 2880w^{15}\right)v^9 + \left(34440w^{12} + 23360w^{15} + 23040w^{18} + 288w^3 - 10140w^9\right.\\
&\left.+21 - 235w^6\right)v^8 + \left(16128w^{18} - 1680w^6 + 61640w^{12} + 15600w^9 + 28896w^{15}\right.\\
&\left.-238w^3 - 6\right)v^7 + \left(9024w^{18} - 2595w^6 + 7120w^9 + 14955w^{12} - 4512w^{15} + 1\right.\\
&\left.+213w^3\right)v^6 + 2\left(2016w^{18} - 60w^3 - 6975w^{12} + 815w^6 - 6388w^{15} - 1980w^9\right)v^5\\
&+ 5\left(288w^{18} - 1291w^{12} - 1284w^{15} + 11w^3 - 18w^6 - 360w^9\right)v^4 - \left(200w^9\right.\\
&\left.+21w^3 - 195w^6 + 210w^{15} + 210w^{12} - 400w^{18}\right)v^3 + \left(1335w^{12} - 130w^6 + 6w^3\right.\\
&\left.+84w^{18} + 650w^{15} + 480w^9\right)v^2 + \left(12w^{18} + 90w^{15} + 15w^6 - w^3 - 60w^9\right.\\
&\left.+20w^{12}\right)v + w^{18} = 0.
\end{aligned} \tag{5.3}$$

Proof. The proof of the equation (5.3) is similar to the proof of the equation (5.1); except that in the place of the equation (3.1), the equation (3.4) is used. □

Theorem 5.3. *If $v = V(q)$ and $w = V(q^{21})$, then*

$$\begin{aligned}
&v^{24} + w^{24} + 8\left(2352v^6 - 344064v^{18} - 87808v^{12} - 301056v^{15} - 131072v^{21}\right.\\
&\left.+2688v^9 - 1 - 84v^3\right)w^{23} + 4\left(9 - 1260v^3 + 688128v^{18} - 1048576v^{21}\right.\\
&\left.-844032v^9 + 88144v^6 + 1286656v^{12} + 3956736v^{15}\right)w^{22} - 8\left(171808v^9\right.\\
&-915264v^{12} - 3257856v^{15} + 1179648v^{21} + 36316v^6 + 14 - 1032192v^{18}\\
&\left.-4473v^3\right)w^{21} + 14\left(53040v^6 - 204352v^9 - 2411520v^{15} - 1114112v^{21}\right.\\
&\left.+24448v^{12} + 19 - 2548v^3 - 3809280v^{18}\right)w^{20} - 56\left(928v^3 + 877312v^{15}\right.\\
&\left.+9 + 1296384v^{18} + 56400v^9 + 30368v^{12} + 376832v^{21} - 16048v^6\right)w^{19}\\
&- 28\left(42844v^6 - 28 - 209808v^{12} + 884736v^{21} + 51200v^{18} - 3825v^3\right.\\
&\left.-81944v^9 - 909952v^{15}\right)w^{18} - 8\left(47600v^{12} + 682136v^9 + 8372224v^{18}\right.\\
&\left.-136080v^6 + 6598v^3 + 4427136v^{15} + 127 + 3211264v^{21}\right)w^{17}
\end{aligned}$$

$$\begin{aligned}
&+\left(6480432v^9+289240v^6-23541280v^{12}+1107-41492v^3-130842880v^{15}\right.\\
&\left.-123863040v^{18}+65536v^{24}-23379968v^{21}\right)w^{16}-4\left(865872v^{12}+285509v^6\right.\\
&+6912640v^{18}-1366330v^9-65536v^{24}+4941664v^{15}-14391v^3+254\\
&\left.+4764672v^{21}\right)w^{15}+4\left(147456v^{24}+5390112v^{15}+1919148v^{12}+196\right.\\
&\left.-1474816v^{18}-3403776v^{21}-1981v^3+203763v^6-1678922v^9\right)w^{14}\\
&-28\left(533532v^{12}+133v^3+13469v^6+18-32768v^{24}+1184640v^{18}\right.\\
&\left.+162816v^{21}+1825856v^{15}-200766v^9\right)w^{13}+7\left(124416v^{21}-243v^3-16799v^6\right.\\
&\left.-1904192v^{15}+238024v^9+155648v^{24}-1075136v^{18}-219438v^{12}+38\right)w^{12}\\
&+28\left(36864v^{24}-4+9255v^6+803064v^{15}+266766v^{12}-34048v^{21}-114116v^9\right.\\
&\left.+431008v^{18}-159v^3\right)w^{11}+4\left(479787v^{12}+3260208v^{18}+1662v^3+200704v^{24}\right.\\
&\left.-5761v^6-168441v^9+9+3357844v^{15}+253568v^{21}\right)w^{10}+\left(9136288v^{18}\right.\\
&+432936v^{12}+5465320v^{15}-8-308854v^9+54005v^6+3684096v^{21}-4653v^3\\
&\left.+520192v^{24}\right)w^9+\left(1022210v^9+283392v^{24}+1-120960v^6-3240216v^{15}\right.\\
&\left.-1471330v^{12}+2854v^3+1327744v^{21}+1156960v^{18}\right)w^8-4\left(392v^3+34587v^9\right.\\
&\left.-32512v^{24}-8176v^6+211136v^{21}-2975v^{12}+341068v^{15}+544320v^{18}\right)w^7\\
&-7\left(122400v^{21}+7109v^9-13113v^{12}-108v^3+171376v^{18}+40972v^{15}+50v^6\right.\\
&\left.-7168v^{24}\right)w^6-14\left(3427v^9+14848v^{21}+32096v^{18}+23v^3-949v^{12}-633v^6\right.\\
&\left.+14100v^{15}-1152v^{24}\right)w^5+7\left(10192v^{21}+608v^{24}+26520v^{18}+2355v^9+17v^3\right.\\
&\left.-465v^6+191v^{12}+12772v^{15}\right)w^4-\left(21476v^{15}+14301v^{12}-6363v^9-35784v^{21}\right.\\
&\left.+36v^3-896v^{24}+252v^6-36316v^{18}\right)w^3+2\left(11018v^{18}+72v^{24}+13188v^{15}\right.\\
&\left.-966v^9+21v^6+2513v^{12}+4v^3+1260v^{21}\right)w^2-\left(147v^9-84v^{15}+168v^{21}\right.\\
&\left.-16v^{24}+588v^{18}-21v^6-343v^{12}+v^3\right)w=0.
\end{aligned}\tag{5.4}$$

Proof. The proof of the equation (5.4) is similar to the proof of the equation (5.1); except that in the place of the equation (3.1), the equation (3.5) is used. □

Theorem 5.4. *If $v=V(q)$ and $w=V(q^{7/3})$, then*

$$\begin{aligned}
&w^{24}+v^{24}-8\left(131072w^{21}+344064w^{18}+301056w^{15}+87808w^{12}-2688w^9+84w^3\right.\\
&\left.-2352w^6+1\right)v^{23}+4\left(688128w^{18}+88144w^6-844032w^9+3956736w^{15}-1260w^3\right.\\
&\left.-2^{20}w^{21}+1286656w^{12}+9\right)v^{22}+8\left(915264w^{12}-171808w^9-1179648w^{21}-14\right.\\
&\left.+3257856w^{15}+4473w^3-36316w^6+1032192w^{18}\right)v^{21}+14\left(53040w^6-2548w^3\right.\\
&\left.-3809280w^{18}+19-2411520w^{15}+24448w^{12}-204352w^9-1114112w^{21}\right)v^{20}
\end{aligned}$$

$$+56\left(16048w^{6}-30368w^{12}-56400w^{9}-1296384w^{18}-376832w^{21}-877312w^{15}\right.$$
$$\left.-9-928w^{3}\right)v^{19}+28\left(3825w^{3}+209808w^{12}+81944w^{9}-884736w^{21}-42844w^{6}\right.$$
$$\left.+909952w^{15}-51200w^{18}+28\right)v^{18}+8\left(136080w^{6}-682136w^{9}-4427136w^{15}\right.$$
$$\left.-3211264w^{21}-47600w^{12}-6598w^{3}-127-8372224w^{18}\right)v^{17}+\left(65536w^{24}+1107\right.$$
$$-130842880w^{15}+6480432w^{9}-23379968w^{21}-123863040w^{18}-23541280w^{12}$$
$$\left.-41492w^{3}+289240w^{6}\right)v^{16}+4\left(65536w^{24}-254-4941664w^{15}+1366330w^{9}\right.$$
$$\left.+14391w^{3}-6912640w^{18}-4764672w^{21}-865872w^{12}-285509w^{6}\right)v^{15}+4\,(196$$
$$+5390112w^{15}-1474816w^{18}+147456w^{24}+1919148w^{12}-3403776w^{21}-1981w^{3}$$
$$\left.-1678922w^{9}+203763w^{6}\right)v^{14}+28\left(-13469w^{6}-533532w^{12}-18+32768w^{24}\right.$$
$$\left.+200766w^{9}-1825856w^{15}-133w^{3}-1184640w^{18}-162816w^{21}\right)v^{13}+7\left(-243w^{3}\right.$$
$$+238024w^{9}-219438w^{12}-1904192w^{15}+38+124416w^{21}+155648w^{24}-16799w^{6}$$
$$\left.-1075136w^{18}\right)v^{12}+28\left(803064w^{15}-34048w^{21}-4+36864w^{24}+431008w^{18}\right.$$
$$\left.-159w^{3}+9255w^{6}+266766w^{12}-114116w^{9}\right)v^{11}+4\left(200704w^{24}+3357844w^{15}\right.$$
$$\left.+3260208w^{18}+479787w^{12}-168441w^{9}+253568w^{21}+1662w^{3}-5761w^{6}+9\right)v^{10}$$
$$+\left(-4653w^{3}+5465320w^{15}+54005w^{6}-308854w^{9}+9136288w^{18}+432936w^{12}\right.$$
$$\left.-8+3684096w^{21}+520192w^{24}\right)v^{9}+\left(1327744w^{21}-3240216w^{15}+1022210w^{9}\right.$$
$$\left.+1+2854w^{3}+283392w^{24}-1471330w^{12}+1156960w^{18}-120960w^{6}\right)v^{8}$$
$$+4\left(-211136w^{21}+2975w^{12}-544320w^{18}+32512w^{24}+8176w^{6}-341068w^{15}\right.$$
$$\left.-392w^{3}-34587w^{9}\right)v^{7}+7\left(13113w^{12}-171376w^{18}-122400w^{21}+7168w^{24}\right.$$
$$\left.-50w^{6}-7109w^{9}+108w^{3}-40972w^{15}\right)v^{6}+14\left(-32096w^{18}+633w^{6}+1152w^{24}\right.$$
$$\left.-3427w^{9}+949w^{12}-14100w^{15}-23w^{3}-14848w^{21}\right)v^{5}+7\left(191w^{12}+10192w^{21}\right.$$
$$\left.+2355w^{9}+608w^{24}-465w^{6}+12772w^{15}+17w^{3}+26520w^{18}\right)v^{4}+\left(896w^{24}\right.$$
$$\left.-14301w^{12}-252w^{6}-36w^{3}+6363w^{9}+35784w^{21}-21476w^{15}+36316w^{18}\right)v^{3}$$
$$+2\left(13188w^{15}-966w^{9}+2513w^{12}+21w^{6}+1260w^{21}+11018w^{18}+72w^{24}\right.$$
$$\left.+4w^{3}\right)v^{2}+\left(343w^{12}-w^{3}-147w^{9}-588w^{18}-168w^{21}+21w^{6}+84w^{15}\right.$$
$$\left.+16w^{24}\right)v=0. \tag{5.5}$$

Proof. The proof of the equation (5.5) is similar to the proof of the equation (5.3); except that in the place of the equation (3.4), the equation (3.6) is used. □

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M. S. Mahadeva Naika, S. Chandankumar and N. P. Suman
Department of Mathematics,
Bangalore University,
Central College Campus,
Bangalore-560 001,
Karnataka,
India
msmnaika@rediffmail.com, chandan.s17@gmail.com, npsuman.13@gmail.com