

ON SOME NEW MIXED MODULAR EQUATIONS INVOLVING RAMANUJAN'S THETA-FUNCTIONS

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Abstract. In his second notebook, Ramanujan recorded altogether 23 P - Q modular equations involving his theta functions. In this paper, we establish several new mixed modular equations involving Ramanujan's theta-functions φ and ψ which are akin to those recorded in his notebook.

1. Introduction

Following Ramanujan, let $(a; q)_\infty$ denote the infinite product $\prod_{n=0}^{\infty} (1 - aq^n)$ (a, q are complex numbers, $|q| < 1$) and

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1,$$

denote the Ramanujan theta-function. The product representation of $f(a, b)$ follows from Jacobi's triple product identity and is given by:

$$f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty.$$

The following definitions of theta-functions φ , ψ and f with $|q| < 1$ are classical:

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_\infty (q^2; q^2)_\infty, \quad (1.1)$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \quad (1.2)$$

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty. \quad (1.3)$$

The ordinary or Gaussian hypergeometric function is defined by

$${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad 0 \leq |z| < 1,$$

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where a, b, c are complex numbers such that $c \neq 0, -1, -2, \dots$, and

$$(a)_0 = 1, \quad (a)_n = a(a+1) \cdots (a+n-1) \quad \text{for any positive integer } n.$$

Now we recall the notion of modular equation. Let $K(k)$ be the complete elliptic integral of the first kind of modulus k . Recall that

$$K(k) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}} = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2} k^{2n} = \frac{\pi}{2} \varphi^2(q), \quad (0 < k < 1), \quad (1.4)$$

and set $K' = K(k')$, where $k' = \sqrt{1-k^2}$ is the so called complementary modulus of k . It is classical to set $q(k) = e^{-\pi K(k')/K(k)}$ so that q is one-to-one, increasing from 0 to 1.

In the same manner introduce $L_1 = K(\ell_1), L'_1 = K(\ell'_1)$ and suppose that the following equality

$$n_1 \frac{K'}{K} = \frac{L'_1}{L_1} \quad (1.5)$$

holds for some positive integer n_1 . Then a modular equation of degree n_1 is an algebraic relation between the moduli k and ℓ_1 which is induced by (1.5). Following Ramanujan, set $\alpha = k^2$ and $\beta = \ell_1^2$. Then we say that β is of degree n_1 over α . The multiplier m is defined by

$$m = \frac{K}{L_1} = \frac{\varphi^2(q)}{\varphi^2(q^{n_1})}, \quad (1.6)$$

for $q = e^{-\pi K(k')/K(k)}$.

Let $K, K', L_1, L'_1, L_2, L'_2, L_3$ and L'_3 denote complete elliptic integrals of the first kind corresponding, in pairs, to the moduli $\sqrt{\alpha}, \sqrt{\beta}, \sqrt{\gamma}$ and $\sqrt{\delta}$, and their complementary moduli, respectively. Let n_1, n_2 and n_3 be positive integers such that $n_3 = n_1 n_2$. Suppose that the equalities

$$n_1 \frac{K'}{K} = \frac{L'_1}{L_1}, \quad n_2 \frac{K'}{K} = \frac{L'_2}{L_2} \quad \text{and} \quad n_3 \frac{K'}{K} = \frac{L'_3}{L_3} \quad (1.7)$$

hold. Then a ‘‘mixed’’ modular equation is a relation between the moduli $\sqrt{\alpha}, \sqrt{\beta}, \sqrt{\gamma}$ and $\sqrt{\delta}$ that is induced by (1.7). We say that β, γ and δ are of degrees n_1, n_2 and n_3 respectively over α . The multipliers $m = K/L_1$ and $m' = L_2/L_3$ are algebraic relation involving α, β, γ and δ .

At scattered places of his second notebook [5], Ramanujan recorded a total of nine P - Q mixed modular relations of degrees 1, 3, 5 and 15. For example, if

$$P := \frac{f(-q^3)f(-q^5)}{q^{1/3}f(-q)f(-q^{15})} \quad \text{and} \quad Q := \frac{f(-q^6)f(-q^{10})}{q^{2/3}f(-q^2)f(-q^{30})},$$

then

$$PQ + \frac{1}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3 + 4. \quad (1.8)$$

The proofs of these nine P - Q mixed modular relations can be found in the book by Berndt [2, pp. 214–235]. In [3,4], S. Bhargava, C. Adiga and M. S. Mahadeva

Naika have established several new P - Q mixed modular relations with four moduli. These mixed modular equations are employed to evaluate explicit evaluations of Ramanujan-Weber class invariants, ratios of Ramanujan's theta functions and several continued fractions.

Motivated by all these works, in this article, we establish several new mixed modular equations involving Ramanujan's theta-functions.

In Section 2, we collect some identities which are useful in proofs of our main results. In Section 3, we establish several new P - Q mixed modular equations akin to those recorded by Ramanujan in his notebooks involving his theta-functions $\varphi(q)$ and $\psi(q)$.

2. Preliminary results

In this section, we collect some identities which are useful in establishing our main results.

LEMMA 2.1. [1, Ch. 16, Entry 24 (ii), p. 39] *We have*

$$f^3(-q) = \varphi^2(-q)\psi(q). \quad (2.1)$$

LEMMA 2.2. [1, Ch. 16, Entry 24 (iv), p. 39] *We have*

$$f^3(-q^2) = \varphi(-q)\psi^2(q). \quad (2.2)$$

LEMMA 2.3. [1, Ch. 17, Entry 10 (i) and Entry 11(ii), pp. 122–123] *For $0 < \alpha < 1$, we have*

$$\varphi(q) = \sqrt{z}, \quad (2.3)$$

$$\sqrt{2}q^{1/8}\psi(-q) = \sqrt{z}\{\alpha(1-\alpha)\}^{1/8}. \quad (2.4)$$

LEMMA 2.4. [1, Ch. 20, Entry 3 (xii), (xiii), pp. 352–353] *Let α , β and γ be of the first, third and ninth degrees respectively. Let m denote the multiplier relating α , β and m' be the multiplier relating γ , δ . Then*

$$\left(\frac{\beta^2}{\alpha\gamma}\right)^{1/4} + \left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{1/4} - \left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4} = -3\frac{m}{m'}, \quad (2.5)$$

$$\left(\frac{\alpha\gamma}{\beta^2}\right)^{1/4} + \left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right)^{1/4} - \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/4} = \frac{m'}{m}. \quad (2.6)$$

LEMMA 2.5. [1, Ch. 20, Entry 11 (viii), (ix), p. 384] *Let α , β , γ and δ be of the first, third, fifth and fifteenth degrees respectively. Let m denote the multiplier*

relating α and β and m' be the multiplier relating γ and δ . Then

$$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8} = \sqrt{\frac{m'}{m}}, \tag{2.7}$$

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8} = -\sqrt{\frac{m}{m'}}. \tag{2.8}$$

LEMMA 2.6. [3] If $L := \frac{q^{1/6}f(-q)f(-q^9)}{f^2(-q^3)}$ and $M := \frac{q^{1/3}f(-q^2)f(-q^{18})}{f^2(-q^6)}$, then

$$\frac{L^9}{M^9} + \frac{M^9}{L^9} + 12\left[\frac{L^3}{M^3} + \frac{M^3}{L^3}\right] + 27L^3M^3 - \frac{1}{L^3M^3} = 0. \tag{2.9}$$

LEMMA 2.7. [2,3] If $L := \frac{q^{1/3}f(-q)f(-q^{15})}{f(-q^3)f(-q^5)}$ and $M := \frac{q^{2/3}f(-q^2)f(-q^{30})}{f(-q^6)f(-q^{10})}$, then

$$L^3M^3 + \frac{1}{L^3M^3} - 48\left[\frac{L^3}{M^3} + \frac{M^3}{L^3}\right] - 12\left[\frac{L^6}{M^6} + \frac{M^6}{L^6}\right] - \left[\frac{L^9}{M^9} + \frac{M^9}{L^9}\right] = 76. \tag{2.10}$$

3. New mixed modular equations

In this section, we establish several new mixed modular equations involving Ramanujan’s theta-functions $\varphi(q)$ and $\psi(q)$.

THEOREM 3.1. If $L := \frac{\varphi(q)\varphi(q^9)}{\varphi^2(q^3)}$ and $M := \frac{\sqrt{q}\psi(-q)\psi(-q^9)}{\psi^2(-q^3)}$, then

$$L^2 = \frac{M^2 + 1}{1 - 3M^2}. \tag{3.1}$$

Proof. The equations (2.5) and (2.6) can be rewritten as

$$\frac{m'}{m} + \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/4} = 1 - 3\frac{m}{m'}\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/4}. \tag{3.2}$$

Transforming the above equation (3.2) by using the equations (2.3) and (2.4), we arrive at the equation (3.1). ■

THEOREM 3.2. If $P := \frac{\sqrt{q}\psi(-q)\psi(-q^9)}{\psi^2(-q^3)}$ and $Q := \frac{q\psi(-q^2)\psi(-q^{18})}{\psi^2(-q^6)}$, then

$$\begin{aligned} &\frac{P^2}{Q^2} + \frac{Q^2}{P^2} + \left(\frac{P^2}{Q^2} + \frac{Q^2}{P^2}\right)^2 \\ &= \left(3PQ - \frac{1}{PQ}\right)\left(3PQ - \frac{1}{PQ} + \frac{3P}{Q} + \frac{3Q}{P} + \frac{P^3}{Q^3} + \frac{Q^3}{P^3}\right) + 14. \end{aligned} \tag{3.3}$$

Proof. Using the equation (2.2) in the equation (2.9), we deduce

$$a^3P^{12} - aP^4bQ^4 + 27a^2P^8b^2Q^8 + 12a^2P^8bQ^4 + 12aP^4b^2Q^8 + b^3Q^{12} = 0, \tag{3.4}$$

where $a := a(q) := \frac{\varphi^2(q)\varphi^2(q^9)}{\varphi^4(q^3)}$ and $b := a(q^2)$.

Using the equation (3.1), we have

$$a := \frac{P^2 + 1}{1 - 3P^2} \quad \text{and} \quad b := \frac{Q^2 + 1}{1 - 3Q^2}. \tag{3.5}$$

Employing the equations (3.5) in the equation (3.4), we find

$$A(P, Q)B(P, Q) = 0, \tag{3.6}$$

where

$$\begin{aligned} A(P, Q) := & (-2P^2Q^4 - 13P^6Q^2 + 25P^6Q^4 + 9P^4Q^4 - 13P^2Q^6 - 16P^8Q^2 \\ & + 39P^8Q^4 - 6P^{10}Q^2 - 27P^6Q^6 - 18P^8Q^6 + 25P^4Q^6 - 16P^2Q^8 + 39P^4Q^8 \\ & + 9P^{10}Q^4 - 18P^6Q^8 - 27P^8Q^8 - 6P^2Q^{10} + 9P^4Q^{10} + P^2Q^2 + P^6 + Q^6 \\ & + P^{10} + 2P^8 + 2Q^8 + Q^{10} - 2P^4Q^2) \end{aligned}$$

and

$$\begin{aligned} B(P, Q) := & (-P^8 + 3P^8Q^2 - P^6Q^2 - P^6 + 9P^6Q^4 + 9P^6Q^6 - Q^6 + 9P^4Q^6 \\ & + 6P^4Q^4 - 3P^4Q^2 + 3P^2Q^8 - P^2Q^6 - 3P^2Q^4 + P^2Q^2 - Q^8). \end{aligned}$$

Expanding in powers of q , the first and second factor of the equation (3.6), one gets respectively,

$$A(P, Q) = -q^3(-1 - q + q^2 + 4q^3 - 10q^4 - 4q^5 + 37q^6 - 28q^7 - 48q^8 + \dots)$$

and

$$B(P, Q) = q^5(4 - 12q - 28q^2 + 144q^3 - 32q^4 - 764q^5 + 1424q^6 + 1400q^7 + \dots).$$

As $q \rightarrow 0$, the factor $q^{-3}B(P, Q)$ of the equation (3.6) vanishes whereas the other factor $q^{-3}A(P, Q)$ do not vanish. Hence, we arrive at the equation (3.3) for $q \in (0, 1)$. By analytic continuation the equation (3.3) is true for $|q| < 1$. ■

THEOREM 3.3. *If $P := \frac{\sqrt{q}\psi(q)\psi(q^9)}{\psi^2(q^3)}$ and $Q := \frac{q\psi(q^2)\psi(q^{18})}{\psi^2(q^6)}$, then*

$$\frac{P^2}{Q^2} + \frac{Q^2}{P^2} = 4 - 3P^2 + \frac{1}{P^2}. \tag{3.7}$$

Proof. Using the equation (2.1) in the equation (2.9), we deduce

$$27a^2P^4b^2Q^4 + a^3P^6 + 12a^2P^4bQ^2 + 12aP^2b^2Q^4 + b^3Q^6 = aP^2bQ^2, \tag{3.8}$$

where $a := a(q) = \frac{\varphi^2(-q)\varphi^2(-q^9)}{\varphi^4(-q^3)}$ and $b := a(q^2)$.

Using the equation (3.1) after changing q by $-q$, we find

$$a = \frac{1 - P^2}{1 + 3P^2} \quad \text{and} \quad b = \frac{1 - Q^2}{1 + 3Q^2}. \tag{3.9}$$

Employing the equations (3.9) in the equation (3.8), we find

$$\begin{aligned} & (3P^2Q^2 + P^2 - 1 + Q^2)(3P^4Q^2 + P^4 - 4P^2Q^2 - Q^2 + Q^4) \\ & (1 - 2P^2 - 2Q^2 + P^4 + 44P^2Q^2 + 6P^4Q^2 + 9P^4Q^4 + 16PQ \\ & + 48P^3Q^3 + 6P^2Q^4 + Q^4) = 0. \end{aligned} \tag{3.10}$$

Expanding in powers of q , factors of the equation (3.10) becomes respectively,

$$\begin{aligned} & (-1 + q + 3q^2 + 2q^3 + 2q^4 - 2q^5 - q^6 - 8q^7 - 24q^8 + \dots), \\ & q^4(-4 - 8q - 8q^2 + 12q^3 + 52q^4 + 24q^5 - 76q^6 - 120q^7 + 32q^8 + \dots) \end{aligned}$$

and

$$(1 - 2q - 5q^2 + 50q^3 + 105q^4 + 68q^5 - 82q^6 - 374q^7 - 362q^8 + \dots).$$

As $q \rightarrow 0$, the second factor of the equation (3.10) vanishes whereas the other factors do not vanish. Hence, we arrive at the equation (3.7) for $q \in (0, 1)$. By analytic continuation the equation (3.7) is true for $|q| < 1$. ■

THEOREM 3.4. *If $P := \frac{\varphi(-q)\varphi(-q^9)}{\varphi^2(-q^3)}$ and $Q := \frac{\varphi(-q^2)\varphi(-q^{18})}{\varphi^2(-q^6)}$, then*

$$\frac{P^2}{Q^2} + \frac{Q^2}{P^2} = 4 - 3Q^2 + \frac{1}{Q^2}. \tag{3.11}$$

Proof. The proof of the equation (3.11) is similar to the proof of the equation (3.7). Hence, we omit the details. ■

THEOREM 3.5. *If $P := \frac{\varphi(-q)\varphi(-q^9)}{\varphi^2(-q^3)}$ and $Q := \frac{\varphi(-q^4)\varphi(-q^{36})}{\varphi^2(-q^{12})}$, then*

$$\begin{aligned} & \left(\frac{P^4}{Q^4} + \frac{Q^4}{P^4}\right) + 8\left(\frac{P^2}{Q^2} + \frac{Q^2}{P^2}\right) - 3\left(\frac{P^2}{Q^4} - \frac{3Q^4}{P^2}\right) - \left(\frac{1}{P^2Q^4} - 27P^2Q^4\right) \\ & - 8\left(\frac{1}{P^2Q^2} + 9P^2Q^2\right) - 12\left(\frac{1}{P^2} - 3P^2\right) + 3\left(9Q^4 + \frac{1}{Q^4}\right) = 24. \end{aligned} \tag{3.12}$$

Proof. Using the equation (2.9) along with the equations (2.1) and (2.2), we deduce

$$27P^4u^4Q^8v^2 + 12P^4u^4Q^4v + 12Q^8v^2P^2u^2 + P^6u^6 + Q^{12}v^3 = P^2u^2Q^4v, \tag{3.13}$$

where $u := \frac{q\psi^2(q)\psi^2(q^9)}{\psi^4(q^3)}$ and $v := u(q^4)$.

Invoking the equation (3.1), we find

$$u := \frac{1 - P^2}{3P^2 + 1} \quad \text{and} \quad v := \frac{1 - Q^2}{3Q^2 + 1}. \quad (3.14)$$

Using the equations (3.14) in the equation (3.13), we find

$$\begin{aligned} & (3Q^2P^4 + P^4 - 2Q^2P^2 - P^2 + 3P^2Q^4 + Q^4 - Q^2)(3Q^2P^2 + P^2 - 1 \\ & + 4QP + Q^2)(3Q^2P^2 + P^2 - 1 - 4QP + Q^2)(P^8 + 27P^6Q^8 - 72P^6Q^6 \\ & + 36P^6Q^4 + 8P^6Q^2 - 3P^6 + 27P^4Q^8 - 24P^4Q^4 + 3P^4 + 9P^2Q^8 \\ & + 8P^2Q^6 - 12P^2Q^4 - 8Q^2P^2 - P^2 + Q^8) = 0. \end{aligned} \quad (3.15)$$

As $q \rightarrow 0$, the last factor of the equation (3.15) vanishes whereas the other factors do not vanish. Hence, we arrive at the equation (3.12) for $q \in (0, 1)$. By analytic continuation the equation (3.12) is true for $|q| < 1$. ■

THEOREM 3.6. *If $P := \frac{\varphi(-q)\varphi(-q^9)}{\varphi^2(-q^3)}$ and $Q := \frac{\varphi(q)\varphi(q^9)}{\varphi^2(q^3)}$, then*

$$\left(\frac{P}{Q} + \frac{Q}{P}\right) + \left(3PQ - \frac{1}{PQ}\right) - 4 = 0. \quad (3.16)$$

Proof. Using the equation (2.9) along with the equations (2.1) and (2.2), we deduce

$$27P^8u^2Q^4v^4 + P^{12}u^3 + 12P^8u^2Q^2v^2 + 12P^4uQ^4v^4 + Q^6v^6 = P^4uQ^2v^2, \quad (3.17)$$

where $u := \frac{q\psi^2(q)\psi^2(q^9)}{\psi^4(q^3)}$ and $v := \frac{q\psi^2(-q)\psi^2(-q^9)}{\psi^4(-q^3)}$.

Invoking the equation (3.1), we find

$$u := \frac{1 - P^2}{3P^2 + 1} \quad \text{and} \quad v := \frac{Q^2 - 1}{3Q^2 + 1}. \quad (3.18)$$

Using the equations (3.18) in the equation (3.17), we find

$$\begin{aligned} & (3P^2Q^2 + Q^2 - 1 + 4PQ + P^2)(3P^2Q^2 + Q^2 - 1 - 4PQ + P^2)(3P^2Q^4 \\ & + Q^4 - Q^2 - 2P^2Q^2 - P^2 + 3P^4Q^2 + P^4)(27P^8Q^6 + 27P^8Q^4 + 9P^8Q^2 \\ & + P^8 - 72P^6Q^6 + 8P^6Q^2 + 36P^4Q^6 - 24P^4Q^4 - 12P^4Q^2 + 8P^2Q^6 \\ & - 8P^2Q^2 + Q^8 - 3Q^6 + 3Q^4 - Q^2) = 0. \end{aligned} \quad (3.19)$$

As $q \rightarrow 0$, the second factor of the equation (3.19) vanishes whereas the other factors do not vanish. Hence, we arrive at the equation (3.16) for $q \in (0, 1)$. By analytic continuation the equation (3.16) is true for $|q| < 1$. ■

THEOREM 3.7. *If $P := \frac{\sqrt{q}\psi(q)\psi(q^9)}{\psi^2(q^3)}$ and $Q := \frac{q^2\psi(q^4)\psi(q^{36})}{\psi^2(q^{12})}$, then*

$$\begin{aligned} & \left(\frac{P^4}{Q^4} + \frac{Q^4}{P^4}\right) + 8\left(\frac{P^2}{Q^2} + \frac{Q^2}{P^2}\right) - 8\left(3^2P^2Q^2 + \frac{1}{P^2Q^2}\right) + 3\left(\frac{3P^4}{Q^2} - \frac{Q^2}{P^4}\right) \\ & + \left(27P^4Q^2 - \frac{1}{P^4Q^2}\right) + 3\left(9P^4 + \frac{1}{P^4}\right) + 12\left(3Q^2 - \frac{1}{Q^2}\right) - 24 = 0. \end{aligned} \quad (3.20)$$

Proof. The proof of the equation (3.20) is similar to the proof of the equation (3.12). Hence, we omit the details. ■

THEOREM 3.8. *If $P := \frac{\sqrt{q}\psi(q)\psi(q^9)}{\psi^2(q^3)}$ and $Q := \frac{\sqrt{q}\psi(-q)\psi(-q^9)}{\psi^2(-q^3)}$, then*

$$\left(\frac{P^2}{Q^2} + \frac{Q^2}{P^2}\right) - \left(3PQ + \frac{1}{PQ}\right)\left(\frac{P}{Q} - \frac{Q}{P}\right) + 2 = 0. \quad (3.21)$$

Proof. The proof of the equation (3.21) is similar to the proof of the equation (3.16). Hence, we omit the details. ■

THEOREM 3.9. *If $L := \frac{\varphi(q^3)\varphi(q^5)}{\varphi(q)\varphi(q^{15})}$ and $M := \frac{\psi(-q^3)\psi(-q^5)}{q\psi(-q)\psi(-q^{15})}$, then*

$$M = \frac{1+L}{1-L}. \quad (3.22)$$

Proof. The equations (2.7) and (2.8) can be rewritten as

$$1 + \sqrt{\frac{m}{m'}} \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)} \right)^{1/8} = \sqrt{\frac{m}{m'}} - \frac{m}{m'} \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)} \right)^{1/8}. \quad (3.23)$$

Transforming the above equation (3.23) by using the equations (2.3) and (2.4), we arrive at equation (3.22). ■

THEOREM 3.10. *If $P := \frac{\psi(-q^3)\psi(-q^5)}{q\psi(-q)\psi(-q^{15})}$ and $Q := \frac{\psi(-q^6)\psi(-q^{10})}{q^2\psi(-q^2)\psi(-q^{30})}$, then*

$$\begin{aligned} &\left(\frac{P^2}{Q^2} + \frac{Q^2}{P^2}\right) + \left(\frac{P}{Q} + \frac{Q}{P}\right) - \left(PQ + \frac{1}{PQ}\right) \\ &+ \left(\sqrt{PQ} - \frac{1}{\sqrt{PQ}}\right) \left(\sqrt{\frac{P^3}{Q^3}} + \sqrt{\frac{Q^3}{P^3}} + \sqrt{\frac{P}{Q}} + \sqrt{\frac{Q}{P}}\right) = 0. \end{aligned} \quad (3.24)$$

Proof. Using the equation (2.2) in the equation (2.10), we find

$$\begin{aligned} &a^2P^4b^2Q^4 + a^4P^8b^4Q^8 - a^6P^{12} - 48a^4P^8b^2Q^4 - 12a^5P^{10}bQ^2 \\ &- 48a^2P^4b^4Q^8 - 76a^3P^6b^3Q^6 - b^6Q^{12} - 12b^5Q^{10}aP^2 = 0, \end{aligned} \quad (3.25)$$

where $a := a(q) := \frac{\varphi(q^3)\varphi(q^5)}{\varphi(q)\varphi(q^{15})}$ and $b := a(q^2)$.

Using the equation (3.22), we have

$$a := a(q) := \frac{P-1}{P+1} \quad \text{and} \quad b := b(q) := \frac{Q-1}{Q+1}. \quad (3.26)$$

Employing the equations (3.26) in the equation (3.25), we find

$$\begin{aligned}
& (PQ + P - 1 + Q)(P^2 - P - 2PQ + PQ^2 + Q^2)(P^2Q + P^2 - Q \\
& - 2PQ + Q^2)(12P^4Q^4 - P^2Q^2 - 5P^4Q^3 - 5P^4Q^2 - 5P^3Q^4 - P^3Q^3 \\
& - P^3Q^2 - 5P^2Q^4 - P^2Q^3 - P^6 - Q^6 - 3P^8 - 3Q^8 + 3P^6Q^2 + 7P^5Q^2 \\
& - 3P^6Q + 3P^2Q^6 + 7P^2Q^5 - 3PQ^6 - 7P^4Q^6 - 2P^4Q^5 + 5P^3Q^6 \\
& + 7P^3Q^5 - 12P^5Q^5 - 2P^5Q^4 + 7P^5Q^3 - 7P^6Q^4 + 5P^6Q^3 + P^6Q^6 \\
& - 5P^6Q^5 - 5P^5Q^6 + P^7Q^7 - P^7Q^6 + 5P^7Q^5 - P^6Q^7 + 5P^5Q^7 + 7P^7Q^4 \\
& + 7P^4Q^7 + 3Q^7 + Q^9 + 3P^7 + P^9 + 3P^9Q^2 + P^9Q^3 - 3P^3Q^8 - 3P^3Q^7 \\
& + P^3Q^9 + 3P^9Q - 9P^8Q^2 + 3P^7Q^2 - 9P^8Q + 9P^7Q - 3P^8Q^3 \\
& - 3P^7Q^3 + 3Q^9P^2 - 9Q^8P^2 + 3Q^7P^2 + 9Q^7P - 9Q^8P + 3PQ^9) \\
& (P^4 + Q^4 - P^2Q - PQ^2 - PQ + P^4Q - P^3Q^3 + P^3Q^2 + P^3Q \\
& + P^2Q^3 + PQ^4 + PQ^3 - P^3 - Q^3) = 0. \tag{3.27}
\end{aligned}$$

As $q \rightarrow 0$, the last factor of the equation (3.27) vanishes whereas the other factors do not vanish. Hence, we arrive at the equation (3.24) for $q \in (0, 1)$. By analytic continuation the equation (3.24) is true for $|q| < 1$. ■

THEOREM 3.11. *If $P := \frac{\psi(q^3)\psi(q^5)}{q\psi(q)\psi(q^{15})}$ and $Q := \frac{\psi(q^6)\psi(q^{10})}{q^2\psi(q^2)\psi(q^{30})}$, then*

$$\left(\frac{P}{Q} + \frac{Q}{P}\right) = \left(P - \frac{1}{P}\right) + 2. \tag{3.28}$$

Proof. Using the equation (2.1) in the equation (2.10), we deduce

$$\begin{aligned}
& a^2P^2b^2Q^2 + a^4P^4b^4Q^4 - a^6P^6 - 48a^4P^4b^2Q^2 - 12a^5P^5bQ \\
& - 48a^2P^2b^4Q^4 - 76a^3P^3b^3Q^3 - b^6Q^6 - 12b^5Q^5aP = 0, \tag{3.29}
\end{aligned}$$

where $a := a(q) := \frac{\varphi(-q^3)\varphi(-q^5)}{\varphi(-q)\varphi(-q^{15})}$ and $b := a(q^2)$.

Changing q to $-q$ in the equation (3.22), we find

$$a := \frac{1+P}{P-1} \quad \text{and} \quad b := \frac{1+Q}{Q-1}. \tag{3.30}$$

Using the equations (3.30) in the equation (3.29), we find

$$\begin{aligned}
& (-P + PQ - 1 - Q)(-P^2 + P^2Q + 2PQ - Q - Q^2)(P^2 + P - 2PQ \\
& + Q^2 - PQ^2)(-P^4 - P^3 - Q^3 - PQ^2 - PQ^3 + PQ^4 - P^2Q + P^2Q^3 \\
& - P^3Q + P^3Q^2 + P^3Q^3 + P^4Q - Q^4 + PQ)(-3P^8 - P^6 - 3P^7 - P^9 \\
& - P^2Q^2 + P^2Q^3 - 5P^2Q^4 + P^3Q^2 - P^3Q^3 + 5P^3Q^4 - 5P^4Q^2 + 5P^4Q^3 \\
& + 12P^4Q^4 + 3QP^6 + 9QP^7 + 9QP^8 + 7Q^3P^5 - 5Q^3P^6 - 3Q^3P^7 + 3Q^3P^8
\end{aligned}$$

$$\begin{aligned}
 & -7P^5Q^2 + 3P^6Q^2 - 3P^7Q^2 - 9P^8Q^2 - 3P^9Q^2 + P^9Q^3 + 2P^5Q^4 - 7P^6Q^4 \\
 & - 7P^7Q^4 + 3P^9Q + 2P^4Q^5 - 7P^4Q^6 - 7P^4Q^7 + 5P^6Q^5 + P^6Q^6 + P^6Q^7 \\
 & - 12P^5Q^5 + 5P^5Q^6 + 5P^5Q^7 + 5P^7Q^5 + P^7Q^6 + P^7Q^7 + 3PQ^6 + 9PQ^7 \\
 & + 9PQ^8 + 7P^3Q^5 - 5P^3Q^6 - 3P^3Q^7 + 3P^3Q^8 + 3PQ^9 - 7P^2Q^5 + 3P^2Q^6 \\
 & - 3P^2Q^7 - 9P^2Q^8 - 3P^2Q^9 + P^3Q^9 - 3Q^8 - Q^6 - 3Q^7 - Q^9 = 0.
 \end{aligned}
 \tag{3.31}$$

As $q \rightarrow 0$, the second factor of the equation (3.31) vanishes whereas the other factors do not vanish. Hence, we arrive at the equation (3.28) for $q \in (0, 1)$. By analytic continuation the equation (3.28) is true for $|q| < 1$. ■

THEOREM 3.12. *If $P := \frac{\varphi(-q^3)\varphi(-q^5)}{\varphi(-q)\varphi(-q^{15})}$ and $Q := \frac{\varphi(-q^6)\varphi(-q^{10})}{\varphi(-q^2)\varphi(-q^{30})}$, then*

$$\left(\frac{P}{Q} + \frac{Q}{P}\right) = \left(Q - \frac{1}{Q}\right) + 2. \tag{3.32}$$

Proof. The proof of the equation (3.32) is similar to the proof of the equation (3.28). Hence, we omit the details. ■

THEOREM 3.13. *If $P := \frac{\varphi(-q^3)\varphi(-q^5)}{\varphi(-q)\varphi(-q^{15})}$ and $Q := \frac{\varphi(-q^{12})\varphi(-q^{20})}{\varphi(-q^4)\varphi(-q^{60})}$, then*

$$\begin{aligned}
 & \left(\frac{P^2}{Q^2} + \frac{Q^2}{P^2}\right) + 3\left(Q^2 + \frac{1}{Q^2}\right) - 4\left(PQ + \frac{1}{PQ}\right) + 3\left(\frac{P}{Q^2} - \frac{Q^2}{P}\right) \\
 & - \left(PQ^2 + \frac{1}{PQ^2}\right) - 2\left(P - \frac{1}{P}\right) + 4\left(Q - \frac{1}{Q}\right) = 0.
 \end{aligned}
 \tag{3.33}$$

Proof. Using the equations (2.1) and (2.2) in the equation (2.10), we deduce $48Q^2b^2P^4a^4 - Q^4b^4P^4a^4 - Q^2b^2P^2a^2 + 48Q^4b^4P^2a^2 + P^6a^6 + 76Q^3b^3P^3a^3 + 12QbP^5a^5 + Q^6b^6 + 12Q^5b^5Pa = 0$,

where $a := a(q) := \frac{\psi(q^3)\psi(q^5)}{\psi(q)\psi(q^{15})}$ and $b := a(q^4)$.

Changing q to $-q$ in the equation (3.22), we find

$$a := \frac{1+P}{P-1} \quad \text{and} \quad b := \frac{1+Q}{Q-1}. \tag{3.35}$$

Employing the equations (3.35) in the equation (3.34), we deduce

$$\begin{aligned}
 & (Q^2P^2 - 2QP^2 + P^2 + 2P + 1 - 4QP + 2Q - 2Q^2P + Q^2)(QP^2 - P^2 \\
 & - P - Q + Q^2P - Q^2)(3Q^8 - 6P^4 - 15P^7 - 6P^8 + 45Q^8P^4 + 3QP^3 \\
 & + 18QP^4 + 45QP^5 + 60QP^6 + 45QP^7 + 18QP^8 - Q^2P + 8Q^2P^2 \\
 & + 10Q^2P^3 - 28Q^2P^4 - 68Q^2P^5 - 64Q^2P^6 - 34Q^2P^7 - 12Q^2P^8
 \end{aligned}$$

$$\begin{aligned}
& + Q^3P - 8Q^3P^2 - 12Q^3P^3 + 16Q^3P^4 + 38Q^3P^5 + 24Q^3P^6 + 4Q^3P^7 \\
& - 5Q^4P + 3Q^4P^2 - 9Q^4P^3 - 13Q^4P^4 + 33Q^4P^5 + 17Q^4P^6 - 19Q^4P^7 \\
& - 7Q^4P^8 - 7Q^5P + 19Q^5P^2 + 17Q^5P^3 - 33Q^5P^4 - 13Q^5P^5 + 9Q^5P^6 \\
& + 3Q^5P^7 + 5Q^5P^8 + 4Q^6P^2 - 24Q^6P^3 + 38Q^6P^4 - 16Q^6P^5 - 12Q^6P^6 \\
& + 8Q^6P^7 + Q^6P^8 + 3QP^9 - 3Q^2P^9 + Q^3P^9 + 3Q^8P^6 - 18Q^8P^5 \\
& + 15Q^9P^4 - 6Q^9P^5 + Q^9P^6 + Q^6 - P^3 - 20P^6 - 15P^5 - P^9 + 45Q^8P^2 \\
& + 34Q^7P^2 + 15Q^9P^2 - 64Q^7P^3 + 68Q^7P^4 - 28Q^7P^5 - 10Q^7P^6 + 8Q^7P^7 \\
& + Q^7P^8 - 60Q^8P^3 - 20Q^9P^3 + 3Q^7 + Q^9 - 6Q^9P - 18Q^8P - 12Q^7P) \\
& (P^4 + Q^4 + 3P^2 + P - 4QP - 4QP^2 + 2Q^2P - 2Q^2P^3 + 4Q^3P^2 \\
& - 4Q^3P^3 - 3Q^4P + 3Q^4P^2 - Q^4P^3 + 3P^3) = 0. \tag{3.36}
\end{aligned}$$

As $q \rightarrow 0$, the last factor of the equation (3.36) vanishes whereas the other factors do not vanish. Hence, we arrive at the equation (3.33) for $q \in (0, 1)$. By analytic continuation the equation (3.33) is true for $|q| < 1$. ■

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