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RECURRENT GENERALIZED (κ, μ) SPACE FORMS

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Abstract. In this paper we study generalized (k, μ) space forms by considering certain curvature conditions like generalized recurrent, ricci recurrent, generalized ϕ recurrent conditions. We found relations among associated functions f_1, f_2, f_3, f_4, f_5, f_6 in ϕ-concorcular recurrent, quasi-conformally ϕ-flat and quasi-conformally flat generalized (k, μ) space forms.

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1. Introduction

A generalized Sasakian space form was defined by Carriazo et al. in [1], as an almost contact metric manifold (M, ϕ, ξ, η, g) whose curvature tensor R is given by

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3, \tag{1}$$

where f_1, f_2, f_3 are some differentiable functions on M and

$$\begin{aligned} R_1(X, Y)Z &= g(Y, Z)X - g(X, Z)Y, \\ R_2(X, Y)Z &= g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z, \\ R_3(X, Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi, \end{aligned}$$

for any vector fields X, Y, Z on M.

In [7], the authors defined a generalized (k, μ) space form as an almost contact metric manifold (M, ϕ, ξ, η, g) whose curvature tensor can be written as

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3 + f_4 R_4 + f_5 R_5 + f_6 R_6, \tag{2}$$

where f_1, f_2, f_3, f_4, f_5, f_6 are differentiable functions on M and R_1, R_2, R_3 are tensors defined above and

$$R_4(X,Y)Z = g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y,$$
$$R_5(X,Y)Z = g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX,$$
$$R_6(X,Y)Z = \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi,$$

for any vector fields X, Y, Z, where $2h = L_\xi \phi$ and L is the usual Lie derivative.This manifold was denoted by $M(f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ f_6)$.
Natural examples of generalized (k, μ) space forms are (k, μ) space forms and generalized Sasakian space forms. The authors in [1] proved that contact metric generalized (k, μ) space forms are generalized (k, μ) spaces and if dimension is greater than or equal to 5, then they are (k, μ) spaces with constant $\phi-$sectional curvature $2f_6 - 1$. They gave a method of constructing examples of generalized (k, μ) space forms and proved that generalized (k, μ) space forms with trans-Sasakian structure reduces to generalized Sasakian space forms. Further in [3], it is proved that under D_a-homothetic deformation generalized (k, μ) space form structure is preserved for dimension 3, but not in general. In this paper, we study generalized (k, μ) space forms under the curvature conditions like generalized recurrent, ricci recurrent, generalized ϕ-recurrent, flat and ϕ-flat conditions. The paper is organised as follows. After preliminaries in section 2, we study generalized recurrent generalized (k, μ) space forms in section 3 and found the condition for $M(f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ f_6)$ to be co-symplectic. In section 4 we study generelized $\phi-$recurrent generalized (k, μ)-space forms and found relations among associated functions. In sections 5 and 6 we study concircular curvature tensor and quasi-conformal curvature of generalized (k, μ) space forms.

2. Preliminaries

In this section, some general definitions and basic formulas are presented which will be used later. A (2n+1)-dimensional Riemannian manifold (M, g) is said to be an almost contact metric manifold if it admits a tensor field ϕ of type (1,1), a vector field ξ, and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta \circ \phi = 0, \tag{3}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{4}$$

$$g(X, \phi Y) = -g(\phi X, Y),\ g(X, \phi X) = 0,\ g(X, \xi) = \eta(X). \tag{5}$$

Such a manifold is said to be a contact metric manifold if $d\eta = \Phi$, where $\Phi(X,Y) = g(X, \phi Y)$ is the fundamental 2-form of M.

It is well known that on a contact metric manifold (M, ϕ, ξ, η, g), the tensor h is defined by $2h = L_\xi \phi$ which is symmetric and satisfies the following relations.

$$h\xi = 0, \;\; h\phi = -\phi h, \;\; trh = 0, \;\; \eta \circ h = 0, \tag{6}$$

$$\nabla_X \xi = -\phi X - \phi hX, \;\; (\nabla_X \eta)Y = g(X + hX, \phi Y). \tag{7}$$

In a $(2n+1)$-dimensional (k, μ)-contact metric manifold, we have [6]

$$h^2 = (k-1)\phi^2, \;\; k \leq 1, \tag{8}$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{9}$$

$$\begin{aligned}(\nabla_X h)(Y) =& [(1-k)g(X, \phi Y) + g(X, h\phi Y)]\xi + \eta(Y)h(\phi X + \phi hX) \\ &- \mu\eta(X)\phi hY.\end{aligned} \tag{10}$$

Definition 1. *A contact metric manifold M is said to be*
(i) Einstein if $S(X,Y) = \lambda g(X,Y)$, where λ is a constant and S is the Ricci tensor,
(ii) η-Einstein if $S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y)$, where α and β are smooth functions on M.

In a $(2n+1)$-dimensional generalized (k, μ) space form, the following relations hold.

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y] + (f_4 - f_6)[\eta(Y)hX - \eta(X)hY], \tag{11}$$

$$\begin{aligned}QX =& [2nf_1 + 3f_2 - f_3]X + [(2n-1)f_4 - f_6]hX \\ &- [3f_2 + (2n-1)f_3]\eta(X)\xi,\end{aligned} \tag{12}$$

$$\begin{aligned}S(X,Y) =& [2nf_1 + 3f_2 - f_3]g(X,Y) + [(2n-1)f_4 - f_6]g(hX,Y) \\ &- [3f_2 + (2n-1)f_3]\eta(X)\eta(Y),\end{aligned} \tag{13}$$

$$S(X, \xi) = 2n(f_1 - f_3)\eta(X), \tag{14}$$

$$r = 2n[(2n+1)f_1 + 3f_2 - 2f_3], \tag{15}$$

where Q is the Ricci operator, S is the Ricci tensor and r is the scalar curvature of $M(f_1, ..., f_6)$.
The relation between the associated functions $f_i, i = 1, ..., 6$ of $M(f_1, ..., f_6)$ was recently discussed by Carriazo et al. [7].

3. Generalized recurrent generalized (k,μ) space forms

A generalized (k,μ) space form $M(f_1,...,f_6)$ is called generalized recurrent,[8] if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W + B(X)[g(Z,W)Y - g(Y,W)Z], \tag{16}$$

where A and B are two $1-forms$ and B is non-zero.

Theorem 1. *A generalized recurrent $M(f_1,...,f_6)$ is co-symplectic provided $f_1 \neq f_3$.*

Proof. Taking $Y = W = \xi$ in (1), we obtain

$$(\nabla_X R)(\xi,Z)\xi = A(X)R(\xi,Z)\xi + B(X)[\eta(Z)\xi - Z]. \tag{17}$$

By the definition of covariant derative, we have

$$(\nabla_X R)(\xi,Z)\xi = \nabla_X R(\xi,Z)\xi - R(\nabla_X\xi,Z)\xi - R(\xi,\nabla_X Z)\xi - R(\xi,Z)\nabla_X\xi. \tag{18}$$

Using (2), (7) and (10), we get

$$\begin{aligned}(\nabla_X R)(\xi,Z)\xi &= X(f_1-f_3)[\eta(Z)\xi - Z] - X(f_4-f_6)hZ - (f_4-f_6)[(1-k)g(X,\phi Z)\xi \\ &+ g(X,h\phi Z)\xi - \mu\eta(X)\phi hZ] + (f_1-f_3)\nabla_X Z - (f_4-f_6)g(-\phi X - \phi hX, hZ)\xi.\end{aligned} \tag{19}$$

Now comparing (17) and (19), we obtain

$$\begin{aligned}&[(X - A(X))(f_1-f_3) - B(X)][\eta(Z)\xi - Z] \\ &+ [(A(X) - X)(f_4-f_6)]hZ - (f_4-f_6)[(1-k)g(X,\phi Z)\xi \\ &+ g(X,h\phi Z)\xi - \mu\eta(X)\phi hZ] + (f_1-f_3)\nabla_X Z + (f_4-f_6)g(\phi X + \phi hX, hZ)\xi = 0.\end{aligned} \tag{20}$$

Taking $Z = \xi$ in (20), we obtain

$$(f_1-f_3)(\nabla_X\xi) = 0. \tag{21}$$

Thus M is co-symplectic if $f_1 \neq f_3$. Hence the proof.

Ricci-recurrent generalized (k,μ) space forms

A generalized (k,μ)- space form $M(f_1,...,f_6)$ is generalized Ricci-recurrent [9], if its Ricci tensor S satisfies the condition

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + 2nB(X)g(Y,Z), \tag{22}$$

where A and B are two non-zero $1-$forms.

Theorem 2. *In a generalized Ricci-recurrent $M(f_1, ..., f_6)$, $f_1 \neq f_3$ holds. Further the $1-$forms $A(X)$ and $B(X)$ are related by* (28).

Proof. By the definition of covariant derivative, we have

$$(\nabla_X S)(Y, \xi) = \nabla_X S(Y, \xi) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi). \tag{23}$$

Using (7) and (14) in (23), we get

$$\begin{aligned}(\nabla_X S)(Y, \xi) =& 2nd(f_1 - f_3)(X)\eta(Y) + 2n(f_1 - f_3)g(X + hX, \phi Y) \\ &+ (2nf_1 + 3f_2 - f_3)g(Y, \phi X + \phi hX) + [(2n-1)f_4 - f_6]g(hY, \phi X + \phi hX).\end{aligned} \tag{24}$$

Taking $Z = \xi$ in (22) and using (5) and (14), we obtain

$$(\nabla_X S)(Y, \xi) = 2n(f_1 - f_3)A(X)\eta(Y) + 2nB(X)\eta(Y). \tag{25}$$

From (24) and (25), we obtain

$$\begin{aligned}&2nd(f_1 - f_3)(X)\eta(Y) + 2n(f_1 - f_3)g(X + hX, \phi Y) + (2nf_1 + 3f_2 - f_3)g(Y, \phi X + \phi hX) \\ &+ [(2n-1)f_4 - f_6]g(hY, \phi X + \phi hX) - 2n(f_1 - f_3)A(X)\eta(Y) - 2nB(X)\eta(Y) = 0.\end{aligned} \tag{26}$$

Taking $Y = \xi$ in (26), we obtain

$$2nd(f_1 - f_3)(X) = 2n(f_1 - f_3)A(X) - 2nB(X). \tag{27}$$

If $(f_1 - f_3) = c$, a constant, then (27) reduces to

$$B(X) = cA(X). \tag{28}$$

Since $B(X)$ is not zero, we have $f_1 \neq f_3$.

It is easy to see that a generalized recurrent $M(f_1, ..., f_6)$ is always generalized Ricci-recurrent. It follows from theorem 1 and theorem 2 that

Corollary 3. *A generalized recurrent $M(f_1, ..., f_6)$ is always co-symplectic.*

4. Generalized $\phi-$recurrent $M(f_1, ..., f_6)$

A generalized (k, μ) space form $M(f_1, ..., f_6)$ is called

Definition 2. *A generalized $\phi-$Ricci recurrent [4],[9], if*

$$\phi^2((\nabla_X Q)(Y)) = A(X)QY + 2nB(X)Y \tag{29}$$

Definition 3. *$\phi-$Ricci symmetric, if*

$$\phi^2((\nabla_X Q)(Y)) = 0, \tag{30}$$

where Q is the Ricci operator, $A(X)$ and $B(X)$ are non-zero $1-$forms.

Theorem 4. *In a generalized (k, μ) space form which is $\phi-$Ricci recurrent the relation $3f_2 + (2n-1)f_3 = 0$ holds.*

Proof. Using (4) and (3), we have

$$-\nabla_X QY + Q(\nabla_X Y) + \eta((\nabla_X Q)Y)\xi = A(X)QY + 2nB(X)Y. \tag{31}$$

Taking $Y = \xi$ in (31) and contracting with respect to Z, we obtain

$$\begin{aligned} &- g(\nabla_X Q\xi, Z) + g(Q(\nabla_X \xi), Z) + \eta((\nabla_X Q)\xi)\eta(Z) \\ &= A(X)g(Q\xi, Z) + 2nB(X)\eta(Z) \end{aligned} \tag{32}$$

Using (7) and (12) in (32), we obtain

$$\begin{aligned} &2n(f_1 - f_3)[g(\phi X, Z) + g(\phi hX, Z)] - S(\phi X, Z) - S(\phi hX, Z) \\ &= 2n[(f_1 - f_3)A(X) + B(X)]\eta(Z). \end{aligned} \tag{33}$$

Replacing X by ϕX in (33), we get

$$\begin{aligned} &2n(f_1 - f_3)[g(\phi^2 X, Z) + g(\phi h\phi X, Z)] - S(\phi^2 X, Z) - S(\phi h\phi X, Z) \\ &= 2n[(f_1 - f_3)A(\phi X) + B(\phi X)]\eta(Z). \end{aligned} \tag{34}$$

Using (3), (13) and (14) in (34), we get

$$\begin{aligned} S(X, Z) =& [2n(f_1 - f_3) - ((2n-1)f_4 - f_6)(k-1)]g(X, Z) + [3f_2 + (2n-1)f_3]g(hX, Z) \\ &+ (k-1)[(2n-1)f_4 - f_6]\eta(X)\eta(Z) + 2n[(f_1 - f_3)A(\phi X) + B(\phi X)]\eta(Z). \end{aligned} \tag{35}$$

Replacing Z by ϕZ in (35), we get

$$S(X, \phi Z) = [2n(f_1 - f_3) - ((2n-1)f_4 - f_6)(k-1)]g(X, \phi Z) + [3f_2 + (2n-1)f_3]g(hX, \phi Z). \tag{36}$$

Again from (13), we have

$$S(X,\phi Z) = [2nf_1 + 3f_2 - f_3)]g(X,\phi Z) + [(2n-1)f_4 - f_6]g(hX,\phi Z). \tag{37}$$

From (37) and (36), we get

$$3f_2 + (2n-1)f_3 = 0. \tag{38}$$

If $A(X)$ and $B(X)$ are zero in (35), then $M(f_1,...,f_6)$ is called $\phi-$Ricci symmetric [9].
It is easy to see that relation (38) holds in $\phi-$Ricci symmetric generalized (k,μ) space form .
Conversely suppose $3f_2 + (2n-1)f_3 = 0$ holds in $\phi-$Ricci symmetric generalized (k,μ) space form, then from (12)

$$QY = (2nf_1 + 3f_2 - f_3)Y + [(2n-1)f_4 - f_6]hY.$$

Differentiating covariantly with respect to X, we obtain

$$(\nabla_X Q)Y = \nabla_X((2nf_1 + 3f_2 - f_3)Y) + \nabla_X(((2n-1)f_4 - f_6)hY).$$

Applying ϕ^2 on both sides, we obtain

$$\phi^2((\nabla_X Q)Y) = d(2nf_1 + 3f_2 - f_3)(X)\phi^2 Y + d((2n-1)f_4 - f_6)(X)\phi h^2 Y.$$

Therefore $M(f_1,...,f_6)$ is $\phi-$Ricci symmetric if and only if $2nf_1 + 3f_2 - f_3$ and $(2n-1)f_4 - f_6$ are constants.

5. Concircular curvature tensor of generalized (k,μ) space forms

The Concircular curvature tensor of $M(f_1,...,f_6)$ is given by [11]

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y]. \tag{39}$$

$M(f_1,...,f_6)$ is said to be

Definition 4. *$\phi-$concircular recurrent[12], if*

$$\phi^2((\nabla_W \tilde{C})(X,Y)Z) = A(W)\tilde{C}(X,Y)Z, \tag{40}$$

where $A(W)$ is a non-zero $1-$form.

Definition 5. *:$\phi-$concircular symmetric, if*

$$\phi^2((\nabla_W \tilde{C})(X, Y)Z) = 0. \tag{41}$$

Theorem 5. *In a $\phi-$concircular recurrent generalized (k, μ) space form, the relation $(2n-1)f_3 + 3f_2 = 0$ holds.*

Proof. Taking the covariant differentiation of (5), we get

$$(\nabla_W \tilde{C})(X, Y)Z = (\nabla_W R)(X, Y)Z - \frac{dr(W)}{2n(2n+1)}[g(Y, Z)X - g(X, Z)Y]. \tag{42}$$

Applying ϕ^2 on both sides, we get

$$\phi^2((\nabla_W \tilde{C})(X, Y)Z) = \phi^2((\nabla_W R)(X, Y)Z) - \frac{dr(W)}{2n(2n+1)}[g(Y, Z)\phi^2 X - g(X, Z)\phi^2 Y]. \tag{43}$$

Suppose $M(f_1, ..., f_6)$ is $\phi-$concircular recurrent. Then from (3) and (40) in (43) and taking $X = \xi$, we obtain

$$A(W)\tilde{C}(\xi, Y)Z = -(\nabla_W R)(\xi, Y)Z + \eta((\nabla_W R)(\xi, Y)Z)\xi + \frac{dr(W)}{2n(2n+1)}\eta(Z)\phi^2 Y. \tag{44}$$

Suppose the vector fields X, Y and Z are orthogonal to ξ. Then taking $X = \xi$ in (5) and using (2), and (3), we get

$$\tilde{C}(\xi, Y)Z = [(f_1 - f_3) - \frac{r}{2n(2n+1)}]g(Y, Z)\xi + (f_4 - f_6)g(hZ, Y)\xi. \tag{45}$$

Now using (2), (3) and (45) in (44) and contracting with respect to ξ, we obtain

$$A(W)\left[\left((f_1 - f_3) - \frac{r}{2n(2n+1)}\right)g(Y, Z) + (f_4 - f_6)g(hZ, Y)\right] = 0. \tag{46}$$

Taking $Z = \xi$ in (46) and using (15), we obtain

$$(2n-1)f_3 + 3f_2 = 0. \tag{47}$$

5.1. Concircular curvature tensor of (k, μ) space forms

In a (k, μ) space form M, curvature tensor R is given by

$$\begin{aligned} R(X,Y)Z &= (\frac{c+3}{4})[g(Y,Z)X - g(X,Z)Y] \\ &+ (\frac{c-1}{4})[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z] \\ &+ (\frac{c+3}{4} - k)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi] \\ &+ [g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y] \\ &+ \frac{1}{2}[g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX] \\ &+ (1-\mu)[\eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi], \end{aligned} \tag{48}$$

where c is the constant $\phi-$sectional curvature of M.
From (48), we have

$$R(\xi,Y)Z = k[g(Y,Z)\xi - \eta(Z)Y] + \mu[g(Y,hZ)\xi - \eta(Z)hY], \tag{49}$$

$$r = n[c(n+1) + 3(n-1) + 4k]. \tag{50}$$

Theorem 6. *In a $\phi-$concircular recurrent (k, μ) space form, the constant $\phi-$sectional curvature of M is given by $c = \frac{k(4n-2)-3(n-1)}{n+1}$.*

Proof. Suppose M is $\phi-$concircular recurrent. Then from (3) and (40) in (43) and taking $X = \xi$, we obtain

$$A(W)\tilde{C}(\xi,Y)Z = -(\nabla_W R)(\xi,Y)Z + \eta((\nabla_W R)(\xi,Y)Z)\xi + \frac{dr(W)}{2n(2n+1)}\eta(Z)\phi^2 Y. \tag{51}$$

Suppose the vector fields X, Y and Z are orthogonal to ξ. Then taking $X = \xi$ in (5) and using (49), (50) and (3), we get

$$\tilde{C}(\xi,Y)Z = \left(k - \frac{c(n+1) + 3(n-1) + 4k}{2(2n+1)}\right) g(Y,Z)\xi + \mu g(hZ,Y)\xi. \tag{52}$$

Using (52), (49) and (3) in (51) and contracting with respect to ξ, we obtain

$$A(W)\left[\left(k - \frac{c(n+1) + 3(n-1) + 4k}{2(2n+1)}\right) g(Y,Z) + \mu g(hZ,Y)\right] = 0. \tag{53}$$

Taking $Z = \xi$ in (53), we get

$$c = \frac{k(4n-2) - 3(n-1)}{n+1}. \tag{54}$$

6. Quasi-conformal curvature tensor on generalized (k, μ) space forms

In a $(2n+1)$-dimensional generalized (k, μ) space form $M(f_1, ..., f_6)$, the quasi-conformal curvature tensor [11] is given by

$$W(X,Y)Z = aR(X,Y)Z + b[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QX] - \frac{a + 2b(2n)}{2n(2n+1)} r[g(Y,Z)X - g(X,Z)Y], \tag{55}$$

where a and b are arbitrary constants such that $ab \neq 0$.

Definition 6. *A generalized (k, μ) space form $M(f_1, ..., f_6)$ is said to be quasi-conformally $\phi-$flat if*

$$g(W(X,Y)Z, \phi W) = 0. \tag{56}$$

Definition 7. *A generalized (k, μ) space form $M(f_1, ..., f_6)$ is said to be quasi-conformally flat if*

$$W(X,Y)Z = 0. \tag{57}$$

6.1. Quasi-conformally $\phi-$flat generalized (k, μ) space forms

In a (2n+1)-dimensional almost contact metric manifold M, [10], if $\{e_1, ...e_{2n}, \xi\}$ is a local orthonormal basis of vector fields in M, then $\{\phi e_1, ...\phi e_{2n}, \xi\}$ is also a local orthonormal basis and

$$\sum_{i=1}^{2n} g(e_i, e_i) = \sum_{i=1}^{2n} g(\phi e_i, \phi e_i) = 2n, \tag{58}$$

$$\sum_{i=1}^{2n} g(e_i, Z)S(Y, e_i) = \sum_{i=1}^{2n} g(\phi e_i, Z)S(Y, \phi e_i) = S(Y,Z) - S(Y,\xi)\eta(Z), \tag{59}$$

$$\sum_{i=1}^{2n} g(e_i, \phi Z)S(Y, e_i) = \sum_{i=1}^{2n} g(\phi e_i, \phi Z)S(Y, \phi e_i) = S(Y, \phi Z), \tag{60}$$

for all Y, $Z \in TM$. In a generalized (k, μ) space form, we have

$$\sum_{i=1}^{2n} S(e_i, e_i) = \sum_{i=1}^{2n} S(\phi e_i, \phi e_i) = r - 2n(f_1 - f_3), \tag{61}$$

$$\sum_{i=1}^{2n} R(e_i, \acute{Y}, Z, e_i) = \sum_{i=1}^{2n} R(\phi e_i, \acute{Y}, Z, \phi e_i) = S(Y,Z) - ((f_1 - f_3)[g(Y,Z) - \eta(Z)\eta(Y)] + (f_4 - f_6)g(hZ,Y)) . \tag{62}$$

Theorem 7. *A quasi-conformally ϕ-flat generalized (k, μ) space form is an $\eta-$Einstein manifold.*

Proof. From (55), we have

$$\begin{aligned} &g(W(X,Y)Z, \phi W) = \\ &ag(R(X,Y)Z, \phi W) + b[S(X,Z)g(Y,\phi W) - S(Y,Z)g(X,\phi W) + g(X,Z)S(Y,\phi W) \\ &- g(Y,Z)S(X,\phi W)] - \frac{a + 2b(2n)}{2n(2n+1)} r[g(Y,Z)g(X,\phi W) - g(X,Z)g(Y,\phi W)], \end{aligned} \tag{63}$$

for X, Y, Z, $W \in TM$.
For a local orthonormal basis $\{e_1, ...e_{2n}, \xi\}$ of vector fields in $M(f_1, ..., f_6)$, putting $X = \phi e_i$ and $W = e_i$ in (63) and using (58), (59), (60), (61) and (62), we obtain

$$\begin{aligned} \sum_{i=1}^{2n} g(W(\phi e_i, Y)Z, \phi e_i) &= a[S(Y,Z) - (f_1 - f_3)(g(Y,Z) - \eta(Z)\eta(Y)) \\ &- (f_4 - f_6)g(hZ,Y)] + b[(2 - 2n)S(Y,Z) - S(\xi,Z)\eta(Y) - S(Y,\xi)\eta(Z) \\ &- g(Y,Z)(r - 2n(f_1 - f_3))] - \frac{a + 2b(2n)}{2n(2n+1)} r[g(Y,Z)2n - g(\phi Y, \phi Z)]. \end{aligned} \tag{64}$$

If $M(f_1, ..., f_6)$ is quasiconformally ϕ-flat, then (64) reduces to

$$[b(2n-2) - a]S(Y,Z) = pg(Y,Z) + q\eta(Y)\eta(Z) - a(f_4 - f_6)g(hZ,Y), \tag{65}$$

where

$$\begin{aligned} p &= -a(f_1 - f_3) - b(r - 2n(f_1 - f_3)) - m(2n-1), \\ q &= a(f_1 - f_3) - 4nb(f_1 - f_3) - m, \\ m &= \frac{a + 2b(2n)}{2n(2n+1)} r. \end{aligned}$$

Replacing Z by hZ in (65) and using (13) and (8), we obtain

$$g(hZ,Y) = \frac{[-a(f_4 - f_6) - (b(2n-2) - a)e]}{[b(2n-2) - a]t - p}(k-1)[\eta(Z)\eta(Y) - g(Y,Z)], \tag{66}$$

with

$$t = 2nf_1 + 3f_2 - f_3,$$
$$e = (2n-1)f_4 - f_6.$$

Now substituting for $g(hZ, Y)$ in (65), we obtain

$$S(Y,Z) = \alpha g(Y,Z) + \beta\eta(Y)\eta(Z), \tag{67}$$

where

$$\alpha = \frac{p+l}{b(2n-2)-a},$$
$$\beta = \frac{q-l}{b(2n-2)-a},$$
$$l = \frac{a(f_4-f_6)[-a(f_4-f_6)-(b(2n-2)-a)e]}{(b(2n-2)-a)t-p}(k-1).$$

Therefore $M(f_1, ..., f_6)$ is an $\eta-$Einstein.

Putting $Z = \xi$ in (65) and using (14), we obtain

$$2nb(f_1-f_3)(2n-1) - 2na(f_1-f_3) = -br - \left(\frac{a+4nb}{2n+1}r\right). \tag{68}$$

If $f_1 = f_3$, then from (68), we have

$$r = 0 \ or \ a + b + 6nb = 0.$$

Thus we have

Proposition 8. *In a quasi-conformally ϕ-flat $M(f_1, ..., f_6)$, either $r = 0$ or $a + b + 6nb = 0$ provided $f_1 = f_3$.*

6.2. Quasi-conformally flat generalized (k, μ) space forms

Theorem 9. *In a quasi-conformally flat generalized (k, μ) space form which is $\phi-$ricci recurrent the scalar curvature is given by $\frac{-(2bt+m)}{a}$.*

Proof. Suppose $M(f_1, ..., f_6)$ is Quasi-conformally flat, then from (55) and (57), we obtain

$$\begin{aligned} aR(X,Y)Z = &- b[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QX] \\ &+ \frac{a+2b(2n)}{2n(2n+1)}r[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{69}$$

Using (12) and (13) in the above equation, it reduces to

$$\begin{aligned} R(X,Y)Z =& \frac{1}{a}\,(-(2bt+m)[g(X,Z)Y-g(Y,Z)X]-be[g(hX,Z)Y-g(hY,Z)X] \\ &+bs[g(X,Z)\eta(Y)-g(Y,Z)\eta(X)]\xi \\ &-be[g(X,Z)hY-g(Y,Z)hX]+bs[\eta(X)Y-\eta(Y)X]\eta(Z))\,, \end{aligned} \tag{70}$$

where

$$\begin{aligned} t &= 2nf_1+3f_2-f_3, \\ s &= 3f_2+(2n-1)f_3, \\ e &= (2n-1)f_4-f_6, \\ m &= \frac{a+2b(2n)}{2n(2n+1)}r. \end{aligned}$$

If $M(f_1,...,f_6)$ is $\phi-$Ricci recurrent, then $s=0$ and $e=0$.
Then from (70), we obtain

$$R(X,Y)Z = (\frac{-(2bt+m)}{a})[g(X,Z)Y-g(Y,Z)X]. \tag{71}$$

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