Electronic Journal of Mathematical Analysis and Applications
Vol. 4(2) Jul. 2016, pp. 25- 37.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

UNIQUENESS OF A MEROMORPHIC FUNCTIONS THAT SHARE ONE SMALL FUNCTION AND ITS DERIVATIVE.

HARINA P. WAGHAMORE , HUSNA V.

Abstract. In this paper we consider the problem of uniqueness of meromorphic functions that share one small function and its derivatives, and obtain two theorems which improve the result of Qingcai Zhang [11].

1. Introduction

Let f be a non-constant meromorphic function defined in the whole complex plane $\mathbb{C}$. It is assumed that the reader is familiar with the following notations of Nevanlinna theory such as $T(r,f), m(r,f), N(r,f), S(r,f)$ and so on, that can be found, for instance in [1,2].

Let f and g be two non-constant meromorphic functions, $a \in \mathbb{C}\cup\{\infty\}$, we say that f and g share the value a CM (counting multiplicity) if $f-a$ and $g-a$ have the same zeroes with the same multiplicities and they share the value a IM (ignoring multiplicities) if we do not consider the multiplicities. When $a=\infty$ the zeroes of $f-a$ means the poles of f(see [7]).

Let k be a non-negative integer or infinity. For any $a \in \mathbb{C}\cup\{\infty\}$, we denote by $E_k(a,f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \le k$ and $k+1$ times if $m > k$. If $E_k(a,f) = E_k(a,g)$, we say that f, g share the value a with weight k.(see[3],[5]).

We write f and g share (a,k) to mean that f and g share the value a with weight k. Clearly, if f and g share (a,k), then f and g share (a,p) for all integers p with $0 \le p \le k$. Also, we note that f, g share a value a IM or CM if and only if they share $(a,0)$ or (a,∞) respectively.

A function $a(z)$ is said to be a small function of f if $a(z)$ is a meromorphic function satisfying $T(r,a) = S(r,f)$, i.e,$T(r,a) = o(T(r,f))$ as $r \to +\infty$ possibly outside of set of finite linear measure. Similarly, we define that f and g share a small function a IM or CM or with weight k by $f-a$ and $g-a$ sharing the value 0 IM or CM or with weight k respectively.

For any constant a, we denote by $N_{k)}(r, \frac{1}{f-a})$ the counting function for zeros of $f-a$ with multiplicity no more than k, and by $\overline{N}_{k)}(r, \frac{1}{f-a})$ the corresponding one

2010 *Mathematics Subject Classification.* Primary 30D35 .
Key words and phrases. Uniqueness, Meromorphic function, Shared value, Small function.
Submitted Oct. 17, 2015.

for which multiplicity is not counted. Let $N_{(k}(r,\frac{1}{f-a})$ be the counting function for zeros of $f-a$ with multiplicity at least k and $\overline{N}_{(k}(r,\frac{1}{f-a})$ be the corresponding one for which multiplicity is not counted. Set
$N_k(r,\frac{1}{f-a})=\overline{N}(r,\frac{1}{f-a})+\overline{N}_{(2}(r,\frac{1}{f-a})+...+\overline{N}_{(k}(r,\frac{1}{f-a})$.
We define

$$\Theta(a,f)=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)}\quad,\quad\delta(a,f)=1-\limsup_{r\longrightarrow\infty}\frac{N(r,\frac{1}{f-a})}{T(r,f)},$$

We further define

$$\delta_k(a,f)=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}_k(r,\frac{1}{f-a})}{T(r,f)}$$

Clearly,

$$0\leq\delta(a,f)\leq\delta_k(a,f)\leq\delta_{k-1}(a,f)...\leq\delta_2(a,f)\leq\delta_1(a,f)=\Theta(a,f).$$

In additional, we shall also use the following notations:

Let f and g be two non-constant meromorphic functions such that f and g share 1 IM. We denote by $\overline{N}_L(r,\frac{1}{f-1})$ the counting function for 1-point of both f and g about which f has larger multiplicity than g, with multiplicity being not counted, and denote by $N_{11}(r,\frac{1}{f-1})$ the counting function for common simple 1-point of both f and g, and denote by $N_{(22}(r,\frac{1}{f-1})$ the counting function of those same multiplicity 1-point of both f and g and the multiplicity is ≥ 2. In the same way, we can define $\overline{N}_L(r,\frac{1}{g-1}), N_{11}(r,\frac{1}{g-1})$, and $N_{(22}(r,\frac{1}{g-1})$. Especially, if f and g share 1 CM, then $\overline{N}_L(r,\frac{1}{g-1})=0$.

R.Bruck [4] first considered the uniqueness problems of an entire function sharing one value with its derivative and proved the following result.

Theorem A. Let f be a entire function which is not constant. If f and f' share the value 1 CM and if $N(r,\frac{1}{f'})=S(r,f)$, then $\frac{f'-1}{f-1}\equiv c$ for some nonzero constant $c\in\mathbb{C}\setminus\{0\}$.

Bruck [4] further posed the following conjecture.

Conjecture 1.1. Let f be an entire function, which is not constant, $\rho_1(f)$ be the first iterated order of f. If $\rho_1(f)<+\infty$ and $\rho_1(f)$ is not a positive integer, and if f and f' share one value a CM, then $\frac{f'-a}{f-a}\equiv c$ for some nonzero constant $c\in\mathbb{C}\setminus\{0\}$.

Yang [8] proved that the conjecture is true if f is an entire function of finite order. Zhang[10] extended Theorem A to meromorphic functions. Yu[9] recently considered the problem of an entire or meromorphic function sharing one small function with its derivative and proved the following two theorems.

Theorem B([9]). Let f be a non-constant entire function and $a\equiv a(z)$ be a meromorphic function such that $a\not\equiv 0,\infty$ and $T(r,a)=o(T(r,f))$ as $r\to+\infty$. If $f-a$ and $f^{(k)}-a$ share the value 0 CM and $\delta(0,f)>\frac{3}{4}$, then $f\equiv f^{(k)}$.

Theorem C([9]). Let f be a non-constant, non-entire meromorphic function and $a\equiv a(z)$ be a meromorphic function such that $a\not\equiv 0,\infty$ and $T(r,a)=o(T(r,f))$ as $r\to+\infty$. If

(i) f and a have no common poles,

(ii) $f-a$ and $f^{(k)}-a$ share the value 0 CM,

(iii) $4\delta(0,f)+2\Theta(\infty,f)>19+2k$,

then $f\equiv f^{(k)}$ where k is a positive integer.

In the same paper, Yu[9] further posed the following open questions.

(i) Can a CM shared be replaced by an IM shared value ?
(ii) Can the condition $\delta(0, f) > \frac{3}{4}$ of Theorem B be further relaxed ?
(iii) Can the condition (iii) of Theorem C be further relaxed ?
(iv) Can in general the condition (i) of Theorem C be dropped ?

Lahiri[5] improved the results of Zhang[10] with weighted shared value obtained the following two theorems.

Theorem D([5]). Let f be a non-constant meromorphic function and k be a positive integer. If f and $f^{(k)}$ share (1,2) and

$$2\overline{N}(r, f) + N_2(r, \frac{1}{f^{(k)}}) + N_2(r, \frac{1}{f}) < (\lambda + o(1))T(r, f^{(k)})$$

for $r \in I$, where $0 < \lambda < 1$ and I is a set of infinite linear measure, then $\frac{f^{(k)}-1}{f-1} \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

Theorem E([5]). Let f be a non-constant meromorphic function and k be a positive integer. If f and $f^{(k)}$ share (1,1) and

$$2\overline{N}(r, f) + N_2(r, \frac{1}{f^{(k)}}) + 2\overline{N}(r, \frac{1}{f}) < (\lambda + o(1))T(r, f^{(k)})$$

for $r \in I$, where $0 < \lambda < 1$ and I is a set of infinite linear measure, then $\frac{f^{(k)}-1}{f-1} \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

In the same paper Lahiri[5] also obtained the following result which is an improvement of Theorem C.

Theorem F([5]). Let f be a non-constant meromorphic function and k be a positive integer. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic function such that $T(r, a) = S(r, f)$. If

(i) a has no zero(pole) which is also a zero(pole) of f or $f^{(k)}$ with the same multiplicity.
(ii) $f - a$ and $f^{(k)} - a$ share (0,2) CM,
(iii) $2\delta_{2+k}(0, f) + (4 + k)\Theta(\infty, f) > 5 + k$,

then $f \equiv f^{(k)}$.

In 2005, Zhang[11] improved the above results and proved the following theorems.

Theorem G([11]). Let f be a non-constant meromorphic function and $k(\geq 1), l(\geq 0)$ be integers. Also, let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic function such that $T(r, a) = S(r, f)$. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. If $l \geq 2$ and

$$2\overline{N}(r, f) + N_2(r, \frac{1}{f^{(k)}}) + N_2(r, \frac{1}{(f/a)'}) < (\lambda + o(1))T(r, f^{(k)}), \tag{1}$$

or $l = 1$ and

$$2\overline{N}(r, f) + N_2(r, \frac{1}{f^{(k)}}) + 2\overline{N}(r, \frac{1}{(f/a)'}) < (\lambda + o(1))T(r, f^{(k)}), \tag{2}$$

or $l = 0$, i.e, $f - a$ and $f^{(k)} - a$ share the value 0 IM and

$$4\overline{N}(r, f) + 3N_2(r, \frac{1}{f^{(k)}}) + 2\overline{N}(r, \frac{1}{(f/a)'}) < (\lambda + o(1))T(r, f^{(k)}) \tag{3}$$

for $r \in I$, where $0 < \lambda < 1$ and I is a set of infinite linear measure, then $\frac{f^{(k)}-a}{f-a} \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

Theorem H([11]). Let f be a non-constant meromorphic function and $k(\geq 1), l(\geq$

0) be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic function such that $T(r, a) = S(r, f)$. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. If $l \geq 2$ and

$$(3 + k)\Theta(\infty, f) + 2\delta_{2+k}(0, f) > k + 4, \tag{4}$$

or $l = 1$ and

$$(4 + k)\Theta(\infty, f) + 3\delta_{2+k}(0, f) > k + 6, \tag{5}$$

or $l = 0$ ie $f - a$ and $f^{(k)} - a$ share the value 0 IM and

$$(6 + 2k)\Theta(\infty, f) + 5\delta_{2+k}(0, f) > 2k + 10, \tag{6}$$

then $f \equiv f^{(k)}$.
In this paper we pay our attention to the uniqueness of more generalised form of a function namely f^n and $(f^{(k)})^m$sharing a small function for two arbitrary positive integer n and m.

Theorem 1.1. Let f be a non-constant meromorphic function and $k(\geq 1), n(\geq 1), m(\geq 2), l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic function such that $T(r, a) = S(r, f)$. Suppose that $f^n - a$ and $(f^{(k)})^m - a$ share $(0, l)$.
If $l \geq 2$ and

$$\frac{2}{m}\overline{N}(r, f) + \frac{2}{m}\overline{N}(r, \frac{1}{f^{(k)}}) + N_2(r, \frac{1}{(f/a)'}) < (\lambda + o(1))T(r, f^{(k)}) \tag{7}$$

or $l = 1$ and

$$\frac{2}{m}\overline{N}(r, f) + \frac{2}{m}\overline{N}(r, \frac{1}{f^{(k)}}) + 2N(r, \frac{1}{(f/a)'}) < (\lambda + o(1))T(r, f^{(k)}) \tag{8}$$

or $l = 0$ ie $f - a$ and $(f^{(k)})^m - a$ share the value 0 IM and

$$\frac{4}{m}\overline{N}(r, f) + \frac{6}{m}\overline{N}(r, \frac{1}{f^{(k)}}) + 2\overline{N}(r, \frac{1}{(f/a)'}) < (\lambda + o(1))T(r, f^{(k)}) \tag{9}$$

for $r \in I$, where $0 < \lambda < 1$ and I is a set of infinite linear measure, then $\frac{(f^{(k)})^m - a}{f^n - a} \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

Theorem 1.2. Let f be a non-constant meromorphic function and $k(\geq 1), n(\geq 1), m(\geq 2), l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic function such that $T(r, a) = S(r, f)$. Suppose that $f^n - a$ and $(f^{(k)})^m - a$ share $(0, l)$.
If $l \geq 2$ and

$$(3 + 2k)\Theta(\infty, f) + 2\Theta(0, f) + 2\delta_{1+k}(0, f) > 2k + 7 - n \tag{10}$$

or $l = 1$ and

$$(4 + 2k)\Theta(\infty, f) + 4\Theta(0, f) + 2\delta_{1+k}(0, f) > 2k + 10 - n \tag{11}$$

or $l = 0$ ie $f - a$ and $(f^{(k)})^m - a$ share the value 0 IM and

$$(6 + 4k)\Theta(\infty, f) + 6\Theta(0, f) + \delta_{1+k}(0, f) > 16 + 4k - n, \tag{12}$$

then $f^n \equiv (f^{(k)})^m$.
From Theorem 1.2 we have the following corollary.

Corollary 1.3. Let f be a non-constant entire function and $a \equiv a(z)(\neq 0, \infty)$ be a meromorphic function such that $T(r, a) = S(r, f)$. If $f^n - a$ and $(f^{(k)})^m - a$ share the value 0 CM and $\delta(0, f) > 1 - \frac{n}{2}$, or if $f^n - a$ and $(f^{(k)})^m - a$ share the value 0 IM and $\delta(0, f) > 1 - \frac{n}{4}$, then $f^n \equiv (f^{(k)})^m$.

2. Main Lemmas

Lemma 2.1[5]. Let f be a non-constant meromorphic function, k be a positive integer,then

$$N_p(r, \frac{1}{f^{(k)}}) \leq N_{p+k}(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f)$$

Lemma 2.2[7]. Let f be a non-constant meromorphic function, n be a positive integer. $P(f) = a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f$ where a_i is a meromorphic function such that $T(r, a_i) = S(r, f)(i = 1, 2, ...n)$ Then $T(r, P(f)) = nT(r, f) + S(r, f)$.

3. Proof of Theorem 1.1

Let $F = \frac{f^n}{a}$, $G = \frac{(f^{(k)})^m}{a}$, then $F - 1 = \frac{f^n - a}{a}$, $G - 1 = \frac{(f^{(k)})^m - a}{a}$. Since $f^n - a$ and $(f^{(k)})^m - a$ share $(0, l)$, F and G share $(1, l)$ except the zeros and poles of $a(z)$. Define

$$H = (\frac{F''}{F'} - \frac{2F'}{F-1}) - (\frac{G''}{G'} - \frac{2G'}{G-1}), \tag{13}$$

we have the following two cases to investigate
Case 1. $H \equiv 0$. Integration yields

$$\frac{1}{F-1} \equiv C\frac{1}{G-1} + D, \tag{14}$$

where C and D are constants and $C \neq 0$. If there exists a pole z_0 of f with multiplicity p which is not the pole and zero of $a(z)$, then z_0 is the pole of F with multiplicity p and the pole of G with multiplicity $p+k$. This contradicts with (14). So

$$\overline{N}(r, f) \leq \overline{N}(r, a) + \overline{N}(r, \frac{1}{a}) = S(r, f), \tag{15}$$

$$\overline{N}(r, F) = S(r, f) \quad \overline{N}(r, G) = S(r, f)$$

(14) also shows F and G share the value 1 CM. Next we prove $D = 0$. We first assume that $D \neq 0$, then

$$\frac{1}{F-1} \equiv \frac{D(G - 1 + \frac{C}{D})}{G-1} \tag{16}$$

So,

$$\overline{N}(r, \frac{1}{G - 1 + \frac{C}{D}}) = \overline{N}(r, F) = S(r, f) \tag{17}$$

If $\frac{C}{D} \neq 1$, by the second fundamental theorem and (15),(17) and $S(r, G) = S(r, f)$, we have

$$T(r, G) \leq \overline{N}(r, G) + \overline{N}(r, \frac{1}{G}) + \overline{N}(r, \frac{1}{G - 1 + \frac{C}{D}}) + S(r, G)$$

$$\leq \overline{N}(r, \frac{1}{G}) + S(r, f) \leq T(r, G) + S(r, f)$$

$$So, \qquad T(r, G) = \overline{N}(r, \frac{1}{G}) + S(r, f), \tag{18}$$

i.e, $$T(r, (f^{(k)})^m) = \overline{N}(r, \frac{1}{(f^{(k)})^m}) + S(r, f)$$

$mT(r,(f^{(k)})) = \overline{N}(r,\frac{1}{f^{(k)}}) + S(r,f).$
this contradicts with conditions (1),(2) and (3) of this theorem.
If $\frac{C}{D}=1$, from (16) we know

$$\frac{1}{F-1} \equiv C\frac{G}{G-1}$$

then

$$(F-1-\frac{1}{C})G = -\frac{1}{C}.$$

Noticing that $F=\frac{f^n}{a}, \quad G=\frac{(f^{(k)})^m}{a}, \quad wehave$

$$\frac{1}{f^n(f^n-(1+\frac{1}{C})a)} \equiv \frac{-C}{a^2}.\frac{(f^{(k)})^m}{f^n} \tag{19}$$

By Lemma 2.2 and (15) and (19), then

$$2T(r,f^n) = T(r,f^n(f^n-(1+\frac{1}{C})a)) + S(r,f) \tag{20}$$

$$\begin{aligned}
2nT(r,f) &= T(r,\frac{1}{f^n(f^n-(1+\frac{1}{C})a)}) + S(r,f)\\
&= T(r,\frac{(f^{(k)})^m}{f^n}) + S(r,f)\\
&\leq N(r,\frac{1}{f^n}) + m\overline{N}(r,f^{(k)}) + S(r,f)\\
&\leq nN(r,\frac{1}{f}) + S(r,f)\\
&\leq nT(r,f) + S(r,f)
\end{aligned}$$

So, $nT(r,f) = S(r,f)$, which is impossible. Hence D=0, and $\frac{G-1}{F-1} \equiv C$, ie, $\frac{(f^{(k)})^m-a}{f^n-a} \equiv C$. This is just the conclusion of this theorem.
Case 2.$H \not\equiv 0$, From (13) it is easy to see that $m(r,H) = S(r,f)$.
Subcase 2.1. $l \geq 1$. From (13) we have

$$\begin{aligned}
N(r,H) \leq\ & \overline{N}(r,F) + \overline{N}_{(l+1}(r,\frac{1}{F-1}) + \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G})\\
&+ \overline{N}_0(r,\frac{1}{G'}) + \overline{N}(r,a) + \overline{N}(r,\frac{1}{a}).
\end{aligned} \tag{21}$$

where $N_0(r,\frac{1}{F'})$ denotes the counting function of the zeros of F' which are not the zeros of F and $F-1$, and $\overline{N}_0(r,\frac{1}{F'})$ denotes its reduced form. In the same way, we can define $N_0(r,\frac{1}{G'})$ and $\overline{N}_0(r,\frac{1}{G'})$, Let z_0 be a simple zero of $F-1$ but $a(z_0) \neq 0,\infty$, then z_0 is also the simple zero of $G-1$. By calculating z_0 is the zero of H, So

$$N_{1)}(r,\frac{1}{F-1}) \leq N(r,\frac{1}{H}) + N(r,a) + N(r,\frac{1}{a}) \leq N(r,H) + S(r,f) \tag{22}$$

Noticing that $N_{1)}(r,\frac{1}{G}) = N_{1)}(r,\frac{1}{F}) + S(r,f)$
we have

$$\begin{aligned}\overline{N}(r,\frac{1}{G-1}) &= N_{1)}(r,\frac{1}{F-1}) + \overline{N}_{(2}(r,\frac{1}{F-1})\\ &\leq \overline{N}(r,F) + \overline{N}_{(l+1}(r,\frac{1}{F-1}) + \overline{N}_{(2}(r,\frac{1}{F-1})\\ &\quad + \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + \overline{N}_0(r,\frac{1}{F'}) + \overline{N}_0(r,\frac{1}{G}) + S(r,f)\end{aligned} \tag{23}$$

By the second fundamental theorem and (23) and noticing

$$\overline{N}(r,F) = \overline{N}(r,G) + S(r,f),$$

then

$$\begin{aligned}T(r,G) &\leq \overline{N}(r,G) + \overline{N}(r,\frac{1}{G}) + \overline{N}(r,\frac{1}{G-1}) - N_0(r,\frac{1}{G'}) + S(r,G)\\ &\leq 2\overline{N}(r,F) + \overline{N}(r,\frac{1}{G}) + \overline{N}_{(2}(r,\frac{1}{G}) + \overline{N}_{(2}(r,\frac{1}{F})\\ &\quad + \overline{N}_{(l+1}(r,\frac{1}{F-1}) + \overline{N}_{(2}(r,\frac{1}{F-1}) + \overline{N}_0(r,\frac{1}{F'}) + S(r,f).\end{aligned} \tag{24}$$

While $l \geq 2$,

$$\overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(l+1}(r,\frac{1}{F-1}) + \overline{N}_{(2}(r,\frac{1}{F-1}) + \overline{N}_0(r,\frac{1}{F'}) \leq N_2(r,\frac{1}{F'}), \tag{25}$$

So

$$T(r,G) \leq 2\overline{N}(r,F) + N_2(r,\frac{1}{G}) + N_2(r,\frac{1}{F'}) + S(r,f)$$

i.e,

$$mT(r,f^{(k)}) \leq 2\overline{N}(r,f) + N_2(r,\frac{1}{(f^{(k)})^m}) + N_2(r,\frac{1}{(\frac{f^n}{a})'}) + S(r,f)$$

$$T(r,f^{(k)}) \leq \frac{2}{m}\overline{N}(r,f) + \frac{2}{m}\overline{N}(r,\frac{1}{f^{(k)}}) + N_2(r,\frac{1}{(\frac{f^n}{a})'}) + S(r,f)$$

this contradicts with (1).
While $l = 1$, (25) turns into

$$\overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(l+1}(r,\frac{1}{F-1}) + \overline{N}_{(2}(r,\frac{1}{F-1}) + \overline{N}_0(r,\frac{1}{F'}) \leq 2\overline{N}(r,\frac{1}{F})$$

Similarly as above , we have

$$T(r,f^{(k)}) \leq \frac{2}{m}\overline{N}(r,f) + \frac{2}{m}\overline{N}(r,\frac{1}{f^{(k)}}) + 2N(r,\frac{1}{(\frac{f^n}{a})'}) + S(r,f)$$

This contradicts with (2).
Subcase 2.2. $l = 0$. In this case, F and G share 1 IM except the zeros and poles of $a(z)$. Let z_0 be the zero of $F-1$ with multiplicity p and the zero of $G-1$ with multiplicity q.
We denote by $N_E^{1)}(r,\frac{1}{F})$ the counting function of the zeros of $F-1$ where $p-q=1$; by $N_E^{2)}(r,\frac{1}{F})$ the counting function of the zeros of $F-1$ where $p=q\geq 2$; by $\overline{N}_L(r,\frac{1}{F})$ the counting function of the zeros of $F-1$ where $p>q\geq 1$, each point

in these counting functions is counted only once. In the same way, we can define $N_E^{1)}(r,\frac{1}{G})$,$N_E^{(2}(r,\frac{1}{G})$ and $\overline{N}_L(r,\frac{1}{G})$. It is easy to see that

$$N_E^{1)}(r,\frac{1}{F-1}) = N_E^{1)}(r,\frac{1}{G-1}) + S(r,f),$$

$$\overline{N}_E^{2)}(r,\frac{1}{F-1}) = \overline{N}_E^{(2}(r,\frac{1}{G-1}) + S(r,f),$$

$$\begin{aligned}\overline{N}(r,\frac{1}{F-1}) &= \overline{N}(r,\frac{1}{G-1}) + S(r,f)\\ &= N_E^{1)}(r,\frac{1}{F-1}) + N_E^{(2}(r,\frac{1}{F-1}) + \overline{N}_L(r,\frac{1}{F-1})\\ &+ \overline{N}_L(r,\frac{1}{G-1}) + S(r,f)\end{aligned} \tag{26}$$

From (13) we have now

$$\begin{aligned}N(r,H) \leq{}& \overline{N}(r,F) + \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + \overline{N}_L(r,\frac{1}{F-1})\\ &+ \overline{N}_L(r,\frac{1}{G-1}) + \overline{N}_0(r,\frac{1}{F'}) + \overline{N}_0(r,\frac{1}{G'}) + S(r,f).\end{aligned} \tag{27}$$

In this case, (22) is replaced by

$$N_E^{1)}(r,\frac{1}{F-1}) \leq N(r,H) + S(r,f). \tag{28}$$

From (26),(27) and (28), we have

$$\begin{aligned}\overline{N}(r,\frac{1}{G-1}) &\leq \overline{N}(r,F) + \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + \overline{N}_E^{(2}(r,\frac{1}{F-1})\\ &+ 2\overline{N}_L(r,\frac{1}{F-1}) + 2\overline{N}_L(r,\frac{1}{G-1}) + \overline{N}_0(r,\frac{1}{F'})\\ &+ \overline{N}_0(r,\frac{1}{G'}) + S(r,f)\\ &\leq \overline{N}(r,F) + 2\overline{N}(r,\frac{1}{F'}) + 2\overline{N}_L(r,\frac{1}{G-1})\\ &+ \overline{N}_{(2}(r,\frac{1}{G}) + \overline{N}_0(r,\frac{1}{G'}) + S(r,f)\end{aligned}$$

By the second fundamental theorem, then

$$\begin{aligned}T(r,G) &\leq \overline{N}(r,G) + \overline{N}(r,\frac{1}{G}) + \overline{N}(r,\frac{1}{G-1}) - N_0(r,\frac{1}{G'}) + S(r,G)\\ &\leq 2\overline{N}(r,G) + 2\overline{N}(r,\frac{1}{F'}) + \overline{N}(r,\frac{1}{G}) + 2\overline{N}(r,\frac{1}{G'}) + S(r,f)\end{aligned}$$

From Lemma 2.1 for $p=1, k=1$ we know

$$\overline{N}(r,\frac{1}{G'}) \leq N_2(r,\frac{1}{G}) + \overline{N}(r,G) + S(r,G),$$

So,

$$T(r,G) \leq 4\overline{N}(r,F) + 3N_2(r,\frac{1}{G}) + 2\overline{N}(r,\frac{1}{F'}) + S(r,f)$$

i.e,

$$mT(r,f^{(k)}) \le 4\overline{N}(r,f) + 3N_2(r,\frac{1}{(f^{(k)})^m}) + 2\overline{N}(r,\frac{1}{(\frac{f^n}{a})'}) + S(r,f).$$

$$T(r,f^{(k)}) \le \frac{4}{m}\overline{N}(r,f) + \frac{6}{m}\overline{N}(r,\frac{1}{f^{(k)}}) + 2\overline{N}(r,\frac{1}{(\frac{f^n}{a})'}) + S(r,f)$$

This contradicts with (3). The proof is complete.

4. Proof of Theorem 1.2

The proof is similar to that of Theorem 1.1. We define F and G and (13) as above, and we also distinguish two cases to discuss.

Case 3. $H \equiv 0$. We also have (14). From (15) we know that $\Theta(\infty,f)=1$, and from (4),(5) and (6), We further know $\delta_{1+k}(0,f) > 1-\frac{n}{2}$. Assume that $D \neq 0$, then

$$\frac{-D(F-1-\frac{1}{D})}{F-1} \equiv C\frac{1}{G-1},$$

so

$$\overline{N}(r,\frac{1}{F-1-\frac{1}{D}}) = \overline{N}(r,G) = S(r,f).$$

If $D \neq -1$, using the second fundamental theorem for F, similarly as (18) we have $T(r,F) = \overline{N}(r,\frac{1}{F}) + S(r,f)$,

i.e., $T(r,f^n) = \overline{N}(r,\frac{1}{f^n}) + S(r,f)$,

$nT(r,f) = \overline{N}(r,\frac{1}{f}) + S(r,f)$

Hence $\Theta(0,f)=0$, this contradicts with $\Theta(0,f) \ge \delta_{1+k}(0,f) > 1-\frac{n}{2}$.

If $D=-1$, then $\overline{N}(r,\frac{1}{F}) = S(r,f)$, i.e., $\overline{N}(r,\frac{1}{f}) = S(r,f)$, and

$$\frac{F}{F-1} \equiv C\frac{1}{G-1}.$$

Then $F(G-1-C) \equiv -C$

and thus,

$$(f^{(k)})^m((f^{(k)})^m-(1+C)a) \equiv -C\frac{a^2(f^{(k)})^m}{f^n}. \tag{29}$$

As same as (20), by Lemma 2.2 and (15) and $\overline{N}(r,\frac{1}{f}) = S(r,f)$. from (29) we have

$$\begin{aligned}
2T(r,(f^{(k)})^m) &= T(r,\frac{(f^{(k)})^m}{f}) + S(r,f)\\
&= N(r,\frac{(f^{(k)})^m}{f}) + S(r,f)\\
&\le mk\overline{N}(r,f) + m\overline{N}(r,\frac{1}{f}) + S(r,f)\\
&= S(r,f)
\end{aligned}$$

So, $T(r,(f^{(k)})^m) = S(r,f)$ and $T(r,\frac{(f^{(k)})^m}{f}) = S(r,f)$.
Hence

$$\begin{aligned} T(r,f^n) &\leq T(r,\frac{f^n}{(f^{(k)})^m}) + T(r,(f^{(k)})^m) + O(1) \\ &= T(r,\frac{(f^{(k)})^m}{f^n}) + mT(r,f^{(k)}) + O(1) \\ &= S(r,f), \end{aligned}$$

this is impossible. Therefore $D = 0$, and from (14) then

$$G - 1 \equiv \frac{1}{C}(F-1)$$

If $C \neq 1$, then $G = \frac{1}{C}(F-1+C)$,
and $N(r,\frac{1}{G}) = N(r,\frac{1}{F-1+C})$
By the second fundamental theorem and (15) we have

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{F-1+C}) + S(r,G) \\ &\leq \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G}) + S(r,f) \end{aligned}$$

By Lemma 2.1 for $p = 1$ and (15), we have

$$\begin{aligned} nT(r,f) &\leq \overline{N}(r,\frac{1}{f^n}) + \overline{N}(r,\frac{1}{(f^{(k)})^m}) + S(r,G) \\ &\leq \overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{f^{(k)}}) + S(r,f) \\ &\leq 2N_{1+k}(r,\frac{1}{f}) + S(r,f) \end{aligned}$$

Hence $\delta_{1+k}(0,f) \leq 1-\frac{n}{2}$. This is a contradiction with $\delta_{1+k}(0,f) \leq 1-\frac{n}{2}$. So $C = 1$ and $F \equiv G$, i.e., $f^n = (f^{(k)})^m$. This is just the conclusion of this theorem.
Case 4. $H \not\equiv 0$
Subcase 4.1 $l \geq 1$ As similar as Subcase 2.1, From (21) and (22) we have

$$\begin{aligned} \overline{N}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{G-1}) &= N_{1)}(r,\frac{1}{F-1}) + \overline{N}_{(2}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{G-1}) \\ &\leq \overline{N}(r,F) + \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + \overline{N}_{(l+1}(r,\frac{1}{G-1}) \\ &+ \overline{N}_{(2}(r,\frac{1}{G-1}) + \overline{N}(r,\frac{1}{G-1}) + \overline{N}_0(r,\frac{1}{F'}) \\ &+ \overline{N}_0(r,\frac{1}{G'}) + S(r,f) \end{aligned}$$

While $l \geq 2$,

$$\overline{N}_{(l+1}(r,\frac{1}{G-1}) + \overline{N}_{(2}(r,\frac{1}{G-1}) + \overline{N}(r,\frac{1}{G-1}) \leq N(r,\frac{1}{G-1}) \leq T(r,G) + O(1),$$

So,

$$\begin{aligned} \overline{N}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{G-1}) &\leq \overline{N}(r,F) + \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) \\ &+ \overline{N}_0(r,\frac{1}{F'}) + \overline{N}_0(r,\frac{1}{G'}) + T(r,G) + S(r,f). \end{aligned}$$

By the second fundamental theorem, we have

$$\begin{aligned}T(r,F)+T(r,G) &\le \overline{N}(r,F)+\overline{N}(r,G)+\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{G})+\overline{N}(r,\frac{1}{F-1})\\ &+\overline{N}(r,\frac{1}{G-1})-N_0(r,\frac{1}{F'})-N_0(r,\frac{1}{G'})+S(r,F)+S(r,G)\\ &\le 3\overline{N}(r,F)+N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+T(r,G)+S(r,f),\end{aligned}$$

$$\begin{aligned}So,\ \ T(r,F) &\le 3\overline{N}(r,F)+N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+S(r,f),\\ i.e,\ \ nT(r,f) &\le 3\overline{N}(r,f)+N_2(r,\frac{1}{f})+N_2(r,\frac{1}{(f^{(k)})^m})+S(r,f)\\ nT(r,f) &\le 3\overline{N}(r,f)+N_2(r,\frac{1}{f})+2N(r,\frac{1}{f^{(k)}})+S(r,f)\end{aligned}$$

By Lemma 2.1 for $p=2$ we have

$$nT(r,f)\le(3+2k)\overline{N}(r,f)+2\overline{N}(r,\frac{1}{f})+2N_{1+k}(r,\frac{1}{f})+S(r,f)$$

So, $\qquad (3+2k)\Theta(\infty,f)+2\Theta(0,f)+2\delta_{1+k}(0,f)\le 7+2k-n.$
This contradicts with (4).
While $l=1$,

$$\overline{N}_{(l+1}(r,\frac{1}{G-1})+\overline{N}(r,\frac{1}{G-1})\le N(r,\frac{1}{G-1})\le T(r,G)+O(1),$$

so by Lemma 2.1 for $p=1,k=1$, we have

$$\begin{aligned}\overline{N}(r,\frac{1}{F-1})+\overline{N}(r,\frac{1}{G-1}) &\le \overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G})+\overline{N}_{(2}(r,\frac{1}{F-1})+\overline{N}_0(r,\frac{1}{F'})\\ &+\overline{N}_0(r,\frac{1}{G'})+T(r,G)+S(r,f).\\ &\le \overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{G})+\overline{N}(r,\frac{1}{F'})+\overline{N}_0(r,\frac{1}{G'})+T(r,G)+S(r,f)\\ &\le 2\overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{G})+N_2(r,\frac{1}{F})+\overline{N}_0(r,\frac{1}{G'})+T(r,G)+S(r,f)\end{aligned}$$

As same as above, by the second fundamental theorem we have

$$T(r,F)+T(r,G)\le 4\overline{N}(r,F)+2N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+T(r,G)+S(r,f),$$

so

$$T(r,F)\le 4\overline{N}(r,F)+2N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+S(r,f),$$

i.e.,

$$nT(r,f)\le 4\overline{N}(r,f)+2N_2(r,\frac{1}{f^n})+N_2(r,\frac{1}{(f^{(k)})^m})+S(r,f),$$

$$\begin{aligned}nT(r,f) &\le 4\overline{N}(r,f)+4\overline{N}(r,\frac{1}{f})+2\overline{N}(r,\frac{1}{f^{(k)}})+S(r,f)\\ &\le 4\overline{N}(r,f)+4\overline{N}(r,\frac{1}{f})+2\{N_{1+k}(r,\frac{1}{f})+k\overline{N}(r,f)\}+S(r,f)\end{aligned}$$

By Lemma 2.1 for p=2 we have

$$nT(r,f) \leq (4+2k)\overline{N}(r,f) + 2N_{1+k}(r,\frac{1}{f}) + 4\overline{N}(r,\frac{1}{f}) + S(r,f)$$

So,

$$(4+2k)\Theta(\infty,f) + 4\Theta(0,f) + 2\delta_{1+k}(0,f) \leq 10 + 2k - n$$

This contradicts with (5).

Subcase 4.2. $l = 0$. From (26),(27) and (28) and Lemma 2.1 for $p = 1, k = 1$, noticing

$$N_E^{(2}(r,\frac{1}{G-1}) + \overline{N}_L(r,\frac{1}{G-1}) + \overline{N}(r,\frac{1}{G-1}) \leq N(r,\frac{1}{G-1}) \leq T(r,G) + S(r,f)$$

then

$$\begin{aligned}
\overline{N}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{G-1}) &= N_E^{1)}(r,\frac{1}{F-1}) + N_E^{(2}(r,\frac{1}{F-1}) + \overline{N}_L(r,\frac{1}{F-1}) + \overline{N}_L(r,\frac{1}{G-1}) \\
&\quad + \overline{N}(r,\frac{1}{G-1}) \\
&\leq \overline{N}(r,F) + \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + 2\overline{N}_L(r,\frac{1}{F-1}) + \overline{N}_L(r,\frac{1}{G-1}) \\
&\quad + \overline{N}_E^{(2}(r,\frac{1}{G-1}) + \overline{N}_L(r,\frac{1}{G-1}) + \overline{N}(r,\frac{1}{G-1}) + \overline{N}_0(r,\frac{1}{F'}) + \overline{N}_0(r,\frac{1}{G'}) \\
&\quad + S(r,f) \\
&\leq \overline{N}(r,F) + 2\overline{N}(r,\frac{1}{F'}) + \overline{N}(r,\frac{1}{G'}) + T(r,G) + S(r,f) \\
&\leq 4\overline{N}(r,F) + 2N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + T(r,G) + S(r,f)
\end{aligned}$$

As same as above, by the second fundamental theorem,we can obtain

$$T(r,F) + T(r,G) \leq 6\overline{N}(r,F) + 3N_2(r,\frac{1}{F}) + 2N_2(r,\frac{1}{G}) + T(r,G) + S(r,f)$$

So

$$\begin{aligned}
T(r,F) &\leq 6\overline{N}(r,F) + 3N_2(r,\frac{1}{F}) + 2N_2(r,\frac{1}{G}) + S(r,f), \\
nT(r,f) &\leq 6\overline{N}(r,f) + 6\overline{N}(r,\frac{1}{f^n}) + 2N_2(r,\frac{1}{(f^{(k)})^m}) + S(r,f) \\
nT(r,f) &\leq 6\overline{N}(r,f) + 6\overline{N}(r,\frac{1}{f}) + 4\overline{N}(r,\frac{1}{f^{(k)}}) + S(r,f)
\end{aligned}$$

By Lemma 2.1 for $p = 2$ we have

$$nT(r,f) \leq (6+4k)\overline{N}(r,f) + 6\overline{N}(r,\frac{1}{f}) + 4N_{1+k}(r,\frac{1}{f}) + S(r,f)$$

$$(6+4k)\Theta(\infty,f) + 6\Theta(0,f) + 4\delta_{1+k}(0,f) \leq 16 + 4k - n$$

this contradicts with (6). Now the proof has been completed.

References

[1] W. K. Hayman, "Meromorphic functions," Oxford, Clarendon Press, 1964.

[2] L.Yang, "Distribution Theory," Springer-Verlag, Berlin, 1993.

[3] I.Lahiri, "Weighted sharing and uniqueness of meromorphic function," Nagoya Math. J., 161, 193-206, 2001.

[4] R.Bruck, "On entire functions which share one value CM with their first derivative," Results in Math, 30,21-24, 1996.

[5] I.Lahiri, "Uniqueness of meromorphic function and its derivative," J. Inequal. Pure Appl. Math. 5(1) Art.20, 2004. (Online:http://jipam.vu.edu.au/).

[6] C.C.Yang and H.Y.Yi "Uniqueness theory of a meromorphic functions," Beijing/New York,Science Press/Kluwer Academic Publishers, 2003.

[7] C.C.Yang, "On deficiencies of differential polynomials" II,Math.Z., 125, 107-112, 1972.

[8] L.Z.Yang, "Solution of a differential equation and its applications," Kodai. Math. J., 22, 458-464, 1999.

[9] K.W.Yu, "On entire and meromorphic functions that share small functions with their derivatives," J. Inequal. Pure Appl. Math. 4(1), 2003 Art.21(Online:http://jipam.vu.edu.au/).

[10] Q.C. Zhang, "The uniqueness of meromorphic functions with their derivatives," Kodai Math.J.,179-184, 1998.

[11] Q.C. Zhang, "Meromorphic function that shares one small function with their derivatives," J. Inequal. Pure. Appl. Math. 6(4), 2005 Art.116 (Online:http://jipam.vu.edu.au/).

HARINA P. WAGHAMORE

Department of Mathematics, Jnanabharathi Campus,Bangalore University, Bengaluru-560056, INDIA

E-mail address: harinapw@gmail.com

Husna V.

Department of Mathematics, Jnanabharathi Campus,Bangalore University, Bengaluru-560056, INDIA

E-mail address: husnav43@gmail.com