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Schläfli-type Mixed Modular Equations of Degrees 1, 3, n and $3n$

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Abstract

In this paper, we establish several new Schläfli-type mixed modular equations of composite degrees. These equations are analogous to those recorded by Ramanujan in his second notebook. As an application, we establish several new explicit values for the Ramanujan-Weber class invariant G_n for $n = 12, 48, 51, 57, 3/4, 3/16, 3/17$ and $3/19$.

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1 Introduction

For any complex number a and $|q| < 1$, we employ the standard notation

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

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Recall that Ramanujan's theta-function $f(a,b)$ is defined as:

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \ |ab| < 1.$$

By, Jacobi's fundamental factorization formula the above identity $f(a,b)$ takes the form

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

The special cases of theta-functions $f(a,b)$ are defined as:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty, \tag{1.1}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{1.2}$$

We define modular equation in brief. The ordinary or Gaussian hypergeometric function is defined by

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} z^n, \quad 0 \le |z| < 1,$$

where a, b, c are complex numbers, $c \neq 0, -1, -2, \ldots$, and

$$(a)_0 = 1, \ (a)_n = a(a+1)\cdots(a+n-1) \ \text{for any positive integer } n.$$

Now we recall the notion of modular equation. Let $K(k)$ be the complete elliptic integral of the first kind of modulus k. Recall that

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{\pi}{2}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2} k^{2n} = \frac{\pi}{2}\varphi^2(q), \ \ (0 < k < 1) \tag{1.3}$$

and set $K' = K(k')$, where $k' = \sqrt{1-k^2}$ is the so called complementary modulus of k. It is classical to set $q(k) = e^{-\pi K(k')/K(k)}$ so that q is one-to-one and increases from 0 to 1. In the same manner introduce $L_1 = K(\ell_1)$, $L_1' = K(\ell_1')$ and suppose that the following equality

$$n_1 \frac{K'}{K} = \frac{L_1'}{L_1} \tag{1.4}$$

holds for some positive integer n_1. Then a modular equation of degree n_1 is a relation between the moduli k and ℓ_1 which is induced by (1.4). Following Ramanujan, set $\alpha = k^2$ and $\beta = \ell_1^2$. We say that β is of degree n_1 over α. The multiplier m, corresponding to the degree n_1, is defined by

$$m = \frac{K}{L_1} = \frac{\varphi^2(q)}{\varphi^2(q^{n_1})}, \tag{1.5}$$

for $q = e^{-\pi K(k')/K(k)}$.

Let K, K', L_1, L_1', L_2, L_2', L_3 and L_3' denote complete elliptic integrals of the first kind corresponding, in pairs, to the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{\delta}$ and their complementary moduli, respectively. Let n_1, n_2 and n_3 be positive integers such that $n_3 = n_1 n_2$. Suppose that the equalities

$$n_1 \frac{K'}{K} = \frac{L_1'}{L_1}, \ \ n_2 \frac{K'}{K} = \frac{L_2'}{L_2} \ \text{ and } \ n_3 \frac{K'}{K} = \frac{L_3'}{L_3}, \tag{1.6}$$

hold. Then a "mixed" modular equation is a relation between the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{\delta}$ that is induced by (1.6). We say that β, γ and δ are of degrees n_1, n_2 and n_3 respectively over α. The multipliers $m = K/L_1$ and $m' = L_2/L_3$ are algebraic relation involving α, β, γ and δ.

In [12], L. Schläfli established relations between P and Q, where

$$P := 2^{1/6}\left[\alpha\beta(1-\alpha)(1-\beta)\right]^{1/24} \ \text{ and } \ Q := \left[\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right]^{1/24} \tag{1.7}$$

for β having degrees 3, 5, 7, 9, 11, 13, 17 and 19, respectively over α. Such modular equations are referred as Schläfli-type modular equations involving two moduli. On pages 86 and 88 of his first Notebook [13], Ramanujan recorded a total of 11 Schläfli-type modular equations for composite degrees involving four moduli. K. G. Ramanathan [11] observed that one of these equations follows from a modular equation recorded by Ramanujan. But these equations was proved by Berndt [5] using the theory of modular forms. N. D. Baruah [1] and [2], proved these equations by deriving some theta-function identities from Schröter's formulae. In the process, he also obtain three new Schläfli-type mixed modular equations. K. R. Vasuki and T. G. Sreeramurthy [14] have established certain new Ramanujan's Schläfli-type mixed modular equations, by employing Ramanujan's modular equations and Schläfli modular equations. Recently, M. S. Mahadeva Naika and K. S. Bairy [9] have established several new Schläfli-type mixed modular equations.

Motivated by these works in this paper, we establish several new Schläfli-type mixed modular equations for composite degrees involving four moduli. In Section 2, we collect the identities which are useful in deriving our main identities. In Section 3, we establish new Schläfli-type mixed modular equations for composite degrees $1, 3, n$ and $3n$ for $n = 3, 4, 9, 17$ and 19. In section 4, we evaluate some explicit values of the class invariant G_n. We conclude the introduction by recalling some definitions and notations, which will be used in the sequel.

Ramanujan-Weber class invariant is defined by

$$G_n := 2^{-1/4} q^{-1/24} \chi(q), \tag{1.8}$$

where $\chi(q) := (-q;q^2)_\infty$, $\quad q = e^{-\pi\sqrt{n}}$, n is any positive rational number.
In view of [3, Entry 12(vii), Ch. 17 p. 124]

$$\chi(q) = 2^{1/6}\{\alpha(1-\alpha)/q\}^{-1/24}, \tag{1.9}$$

the class invariant G_n becomes

$$G_n := \{4\alpha(1-\alpha)\}^{-1/24} \ \text{ and } \ G_{r^2 n} := \{4\beta(1-\beta)\}^{-1/24}, \tag{1.10}$$

where β is of degree r over α.
We set,

$$P := (256\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/24}, \tag{1.11}$$

$$Q := \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/24}, \tag{1.12}$$

$$R := \left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{1/24} \tag{1.13}$$

and

$$T := \left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/24}. \tag{1.14}$$

2 Preliminary results

In this section, we collect all the identities which play a vital role in proving our main results.

Lemma 2.1. *[3, Ch. 19, Entry 5(vii), p. 230] We have*

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2}, \tag{2.1}$$

$$\frac{9}{m^2} = \left(\frac{\alpha}{\beta}\right)^{1/2} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2}. \tag{2.2}$$

Lemma 2.2. *[3, Ch. 20, Entry 1(iii), p. 345] We have*

$$\frac{3\varphi\left(q^9\right)}{\varphi(q)} = 1 + \left(\frac{9\varphi^4\left(q^3\right)}{\varphi^4(q)} - 1\right)^{1/3}. \tag{2.3}$$

Lemma 2.3. *[3, Ch. 19, Entry 5(xii), p. 231]*
If $U := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$ and $V := \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}$, then

$$V + \frac{1}{V} + 2\sqrt{2}\left(U - \frac{1}{U}\right) = 0. \tag{2.4}$$

Lemma 2.4. *[6, Ch. 2, Theorem 2.3.5] If $X := \frac{\varphi(q)\varphi(q^4)}{\varphi(q^3)\varphi(q^{12})}$ and $Y := \frac{\varphi(q)\varphi(q^{12})}{\varphi(q^4)\varphi(q^3)}$, then*

$$Y^2 + \frac{1}{Y^2} - 16\left[Y + \frac{1}{Y}\right] + X^2 + \frac{3^2}{X^2} + 2\left[Y - \frac{1}{Y}\right]\left[X + \frac{3}{X}\right] + 20 = 0. \tag{2.5}$$

Lemma 2.5. *[8, Theorem 3.1] If $M := \frac{\varphi(q)}{\varphi(q^3)}$ and $N := \frac{\varphi(q^9)}{\varphi(q^{27})}$, then*

$$\begin{aligned}&\frac{N^5}{M^5} + 9\left(\frac{N^3}{M^3} - \frac{N^4}{M^4}\right) + 9\left(\frac{5N}{M} + \frac{9M}{N}\right) + 9\left(N^4 + \frac{9}{M^4}\right) - 9\left(M^2N^2 + \frac{3^2}{M^2N^2}\right)\\ &+ 3\left(N^2 + 3M^2\right)\left(MN + \frac{9}{M^3N^3}\right) = 135 + 3\left(\frac{N^2}{M^2} + \frac{9M^2}{N^2}\right) + \left(M^4N^4 + \frac{3^4}{M^4N^4}\right).\end{aligned} \tag{2.6}$$

Lemma 2.6. *[8, Theorem 3.2] If* $M := \dfrac{\varphi(q)\varphi(q^{17})}{\varphi(q^3)\varphi(q^{51})}$ *and* $N := \dfrac{\varphi(q)\varphi(q^{51})}{\varphi(q^3)\varphi(q^{17})}$, *then*

$$\begin{aligned}
&N^9 - \frac{1}{N^9} + 34\left(N^8 + \frac{1}{N^8}\right) + 272\left(N^7 - \frac{1}{N^7}\right) + 238\left(N^6 + \frac{1}{N^6}\right)\\
&-595\left(N^5 - \frac{1}{N^5}\right) - 510\left(N^4 + \frac{1}{N^4}\right) + 16303\left(N^3 - \frac{1}{N^3}\right)\\
&-5202\left(N^2 + \frac{1}{N^2}\right) - 26911\left(N - \frac{1}{N}\right) + \left(M^8 + \frac{3^8}{M^8}\right) + 20230\\
&= 17\left\{\left(M^2 + \frac{3^2}{M^2}\right)\left[7\left(N^6 + \frac{1}{N^6}\right) + 28\left(N^5 - \frac{1}{N^5}\right) + 34\left(N^4 + \frac{1}{N^4}\right)\right.\right.\\
&-168\left(N^3 - \frac{1}{N^3}\right) + 160\left(N^2 + \frac{1}{N^2}\right) + 378\left(N - \frac{1}{N}\right) - 210\Bigg]\\
&-\left(M^4 + \frac{3^4}{M^4}\right)\left[5\left(N^4 + \frac{1}{N^4}\right) + 21\left(N^3 - \frac{1}{N^3}\right) - 35\left(N^2 + \frac{1}{N^2}\right)\right.\\
&\left.\left.-14\left(N - \frac{1}{N}\right) - 30\right] - \left(M^6 + \frac{3^6}{M^6}\right)\left[\left(N^2 + \frac{1}{N^2}\right) - 2\left(N - \frac{1}{N}\right) - 2\right]\right\}.
\end{aligned} \tag{2.7}$$

Lemma 2.7. *[8, Theorem 3.3] If* $M := \dfrac{\varphi(q)\varphi(q^{19})}{\varphi(q^3)\varphi(q^{57})}$ *and* $N := \dfrac{\varphi(q)\varphi(q^{57})}{\varphi(q^3)\varphi(q^{19})}$, *then*

$$\begin{aligned}
&N^{10} - \frac{1}{N^{10}} + \left(M^9 + \frac{3^9}{M^9}\right) + 19\left\{26\left(N^8 - \frac{1}{N^8}\right) + 135\left(N^6 - \frac{1}{N^6}\right)\right.\\
&+1916\left(N^4 - \frac{1}{N^4}\right) + 1144\left(N^2 - \frac{1}{N^2}\right) - \left(M^7 + \frac{3^7}{M^7}\right)\left[\left(N^2 - \frac{1}{N^2}\right) - 4\right]\\
&-5\left(M^6 + \frac{3^6}{M^6}\right)\left[\left(N^2 - \frac{1}{N^2}\right)\right] + 3\left(M^5 + \frac{3^5}{M^5}\right)\left[2\left(N^4 + \frac{1}{N^4}\right)\right.\\
&\left.-14\left(N^2 + \frac{1}{N^2}\right) + 11\right] + 2\left(M^4 + \frac{3^4}{M^4}\right)\left[26\left(N^4 - \frac{1}{N^4}\right) - \left(N^2 - \frac{1}{N^2}\right)\right]\\
&-5\left(M^2 + \frac{3^2}{M^2}\right)\left[13\left(N^6 - \frac{1}{N^6}\right) - 4\left(N^4 - \frac{1}{N^4}\right) + 104\left(N^2 - \frac{1}{N^2}\right)\right]\\
&-\left(M^3 + \frac{3^3}{M^3}\right)\left[12\left(N^6 + \frac{1}{N^6}\right) - 65\left(N^4 + \frac{1}{N^4}\right) - 591 + 200\left(N^2 + \frac{1}{N^2}\right)\right]\\
&+3\left(M + \frac{3}{P}\right)\left[2\left(N^8 + \frac{1}{N^8}\right) - 29\left(N^6 + \frac{1}{N^6}\right) + 7\left(N^4 + \frac{1}{N^4}\right) + 697\right.\\
&\left.\left.-504\left(N^2 + \frac{1}{N^2}\right)\right]\right\} = 0.
\end{aligned} \tag{2.8}$$

3 Schläfli-type mixed modular equations

In this section, we establish several new Schläfli-type mixed modular equations.

Theorem 3.1. *If α, β, γ and δ have degrees 1, 3, 3 and 9 respectively, then*

$$\left[Q^6+\frac{1}{Q^6}\right]\left\{\left[T^3+\frac{1}{T^3}\right]+1\right\}-\left[T^6+\frac{1}{T^6}\right]-10\left[T^3+\frac{1}{T^3}\right]-20=0, \tag{3.1}$$

$$\left[R^9+\frac{1}{R^9}\right]+10\left[R^6+\frac{1}{R^6}\right]+19\left[R^3+\frac{1}{R^3}\right]=8\left[P^3+\frac{1}{P^3}\right]\left[R^3+\frac{1}{R^3}+1\right]-36, \tag{3.2}$$

$$2\left[\frac{P}{Q}+\frac{Q}{P}\right]-\left[Q^3+\frac{1}{Q^3}\right]=0, \tag{3.3}$$

$$\left[T^9+\frac{1}{T^9}\right]+10\left[T^6+\frac{1}{T^6}\right]+19\left[T^3+\frac{1}{T^3}\right]=8\left[P^3+\frac{1}{P^3}\right]\left[T^3+\frac{1}{T^3}+1\right]-36. \tag{3.4}$$

Proof of (3.1). Combining the equations (2.1) and (2.2), we get

$$m^2+a=\frac{9}{m^2}a+1, \tag{3.5}$$

where $a:=\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2}$.
Solving the above equation (3.5) for m^2, we get

$$m^2:=\frac{(1-a)+u}{2}, \tag{3.6}$$

where $u^2:=a^2+34a+1$.
Similarly,

$$(m')^2:=\frac{(1-b)+v}{2}, \tag{3.7}$$

where $v^2:=b^2+34b+1$ and $b:=\left(\frac{\delta(1-\delta)}{\gamma(1-\gamma)}\right)^{1/2}$.

The equation (2.3) can be written in the form

$$Mst+3M=3sN+Nt, \tag{3.8}$$

where $M:=\frac{\varphi(q)}{\varphi(q^3)}$, $N:=\frac{\varphi\left(q^3\right)}{\varphi(q^9)}$, $s:=\frac{\varphi^2(q)}{\varphi^2(q^3)}$ and $t:=\frac{\varphi^2\left(q^3\right)}{\varphi^2(q^9)}$.
Using the equations (3.6) and (3.7) in (3.8) after squaring both the sides, we deduce that

$$s(ab-av-bu+uv-a+11b+u-11v+25)=t(6u-6a-2b+2v+8). \tag{3.9}$$

Eliminating s and t in the equation (3.9), we get

$$\begin{aligned}&80-a^3b^2-16a^3b-a^3v-36a^2b^2-585a^2b+a^2u-27a^2v-198ab^2+19b^2u\\&-2b^2v-3420ab+10au+54av+403bu-33bv-80uv-a^2buv-19abuv\\&+355b-82v+82u-19buv+198abv+10auv+a^2uv+36a^2bv+313abu\\&+19ab^2u+a^3bv+a^2b^2u+16a^2bu-a^3-27a^2+2b^3+67b^2-108a=0.\end{aligned}\tag{3.10}$$

Now collecting u on one side of the above equation (3.10) and then squaring on both the sides, we get

$$\begin{aligned}&17746a^3b^2+13312a^3b+1024a^3v+43614a^2b^2-84654a^2b+3359754ab^2\\&+4096b^2v+2679390ab-118098av-1024bv+2a^3b^5+134a^3b^4+72a^2b^5\\&+2774a^3b^3-3366a^2b^4+396ab^5+2b^5v-153954a^2b^3+26604ab^4+100b^4v\\&+545886ab^3+1362b^3v+118098a-2774b^4-1024b-320850a^2bv-2a^3b^4v\\&-745452abv-4096a^3bv+65556a^2b^2v-396ab^4v+4590a^2b^3v-72a^2b^4v\\&-1362a^3b^2v-100a^3b^3v-19872ab^3v-265086ab^2v+1024a^3-17746b^3\\&-13312b^2-2b^6-134b^5=0.\end{aligned}\tag{3.11}$$

Similarly eliminating v from the above equation (3.11), we get

$$\begin{aligned}&a^5b^5+36a^5b^4+36a^4b^5-a^6b^2+198a^5b^3-2798a^4b^4+198a^3b^5-a^2b^6+ab\\&-36a^5b^2+7200a^4b^3+7200a^3b^4-36a^2b^5-a^5b+33608a^3b^3-ab^5-36a^4b\\&+7200a^2b^3-36ab^4-a^4+198a^3b-2798a^2b^2+198ab^3-b^4+36a^2b+36ab^2\\&+7200a^3b^2=0.\end{aligned}\tag{3.12}$$

Using the fact that $a := \dfrac{T^6}{Q^6}$ and $b := Q^6T^6$ in the equation (3.12), we get

$$\begin{aligned}&\big(Q^{12}T^9+Q^{12}T^6-Q^6T^{12}+Q^{12}T^3-10Q^6T^9-20Q^6T^6-10Q^6T^3+T^9-Q^6\\&+T^6+T^3\big)\big(Q^{12}T^9-Q^{12}T^6+Q^6T^{12}+Q^{12}T^3-10Q^6T^9+20Q^6T^6-10Q^6T^3\\&+T^9+Q^6-T^6+T^3\big)\big(Q^{24}T^{18}-Q^{24}T^{12}+18Q^{18}T^{18}+Q^{12}T^{24}+Q^{24}T^6-T^{12}\\&+62Q^{12}T^{18}+18Q^{18}T^6+200Q^{12}T^{12}+18Q^6T^{18}+62Q^{12}T^6+T^{18}+18Q^6T^6\\&+T^6+Q^{12}\big)=0.\end{aligned}\tag{3.13}$$

We see that the first factor of the above equation (3.13) vanishes for the specific value of $q := e^{-\pi}$ and the other factors do not vanish. Hence we arrive at (3.1). □

Proof of (3.2). The equation (2.4) is solved for V, we get

$$V^2 := a := \frac{(-x+A)^2}{4},\tag{3.14}$$

where $x := 2\sqrt{2}\left(U - \dfrac{1}{U}\right)$ and $A^2 := x^2 - 4$.
Similarly,

$$Y^2 := b := \frac{(-y+B)^2}{4}, \tag{3.15}$$

where $y := 2\sqrt{2}\left(X - \dfrac{1}{X}\right)$, $B^2 := y^2 - 4$, $X := \{16\gamma\delta(1-\gamma)(1-\delta)\}^{1/8}$ and $Y := \left(\dfrac{\delta(1-\delta)}{\gamma(1-\gamma)}\right)^{1/4}$.

Using the equations (3.14) and (3.15) in the equation (3.12), we get

$$\begin{aligned}
&Ax^9y^{10} - ABx^9y^9 + Bx^{10}y^9 - 28ABx^9y^7 - 28ABx^7y^9 + 26Ax^9y^8 + 28Ax^7y^{10}\\
&+ 28Bx^{10}y^7 + 26Bx^8y^9 - 26x^{10}y^8 + ABx^{11}y^3 - 3ABx^9y^5 + 3310ABx^7y^7\\
&- 3ABx^5y^9 + ABx^3y^{11} - Ax^{11}y^4 - 55Ax^9y^6 - 3366Ax^7y^8 + 3Ax^5y^{10} - Ax^3y^{12}\\
&- Bx^{12}y^3 + 3Bx^{10}y^5 - 3366Bx^8y^7 - 55Bx^6y^9 - Bx^4y^{11} + x^{12}y^4 + 55x^{10}y^6\\
&+ 3418x^8y^8 + 55x^6y^{10} + x^4y^{12} - 2ABx^{11}y + 478ABx^9y^3 - 24720ABx^7y^5\\
&- 24720ABx^5y^7 + 478ABx^3y^9 - 2ABxy^{11} - 544Ax^9y^4 + 31284Ax^7y^6\\
&+ 24714Ax^5y^8 - 476Ax^3y^{10} + 2Axy^{12} + 2Bx^{12}y - 476Bx^{10}y^3 + 24714Bx^8y^5\\
&+ 31284Bx^6y^7 - 544Bx^4y^9 + 4Bx^2y^{11} - 4x^{12}y^2 + 542x^{10}y^4 - 31394x^8y^6\\
&- 31394x^6y^8 + 542x^4y^{10} - 4x^2y^{12} - 506ABx^9y + 53632ABx^7y^3 - x^{10}y^{10}\\
&+ 53632ABx^3y^7 - 506ABxy^9 - 2Ax^{11} + 1334Ax^9y^2 - 96564Ax^7y^4\\
&- 52674Ax^3y^8 + 502Axy^{10} + 502Bx^{10}y - 52674Bx^8y^3 - 203281Bx^6y^5\\
&+ 1334Bx^2y^9 - 2By^{11} + 2x^{12} - 1326x^{10}y^2 + 95474x^8y^4 + 265739x^6y^6\\
&- 1326x^2y^{10} + 2y^{12} - 28800ABx^7y - 275616ABx^5y^3 - 275616ABx^3y^5\\
&- 28800ABxy^7 - 376Ax^9 + 99584Ax^7y^2 + 533834Ax^5y^4 + 383840Ax^3y^6\\
&+ 27784Axy^8 + 27784Bx^8y + 383840Bx^6y^3 + 533834Bx^4y^5 + 99584Bx^2y^7\\
&- 376By^9 + 372x^{10} - 96908x^8y^2 - 728054x^6y^4 - 728054x^4y^6 - 96908x^2y^8\\
&+ 372y^{10} + 127870ABx^5y + 359420ABx^3y^3 + 127870ABxy^5 - 16902Ax^7\\
&- 470342Ax^5y^2 - 801466Ax^3y^4 - 186490Axy^6 - 186490Bx^6y - 801466Bx^4y^3\\
&- 470342Bx^2y^5 - 16902By^7 + 16146x^8 + 672194x^6y^2 + 1673820x^4y^4\\
&+ 672194x^2y^6 + 16146y^8 - 110720ABx^3y - 110720ABxy^3 + 72576Ax^5\\
&+ 497664Ax^3y^2 + 306816Axy^4 + 306816Bx^4y + 497664Bx^2y^3 + 72576By^5\\
&- 107140x^6 - 1233804x^4y^2 - 1233804x^2y^4 - 107140y^6 + 6144ABxy\\
&- 92160Axy^2 - 92160Bx^2y - 55296By^3 + 165120x^4 + 558592x^2y^2\\
&- 36864x^2 - 36864y^2 - 55296Ax^3 - 96564Bx^4y^7 + 95474x^4y^8\\
&+ 153835ABx^5y^5 - 203281Ax^5y^6 - 26x^8y^{10} + 4Ax^{11}y^2 + 165120y^4 = 0.
\end{aligned} \tag{3.16}$$

Now collecting the terms containing A on one side of the above equation (3.16) and squaring on both the sides, we get

$$\begin{aligned}&x^{12}-x^{10}y^6-x^8y^8-x^6y^{10}-30x^{10}y^4-90x^8y^6-90x^6y^8-30x^4y^{10}-62x^{10}y^2\\&+2431x^8y^4+1404x^6y^6+2431x^4y^8-62x^2y^{10}+y^{12}+384x^{10}-13440x^8y^2\\&+74496x^6y^4+74496x^4y^6-13440x^2y^8+384y^{10}+55296x^8-851968x^6y^2\\&-962560x^4y^4-851968x^2y^6+3538944x^6-11403264x^4y^2+84934656y^4\\&-11403264x^2y^4+3538944y^6+84934656x^4+132120576x^2y^2+55296y^8=0.\end{aligned}\tag{3.17}$$

Further, substituting for x and y as in the equations (3.14) and (3.15) in the equation (3.17), we get

$$\begin{aligned}&\big(U^6-8U^5X^3-8U^4X^4-8U^3X^5+10U^5X+19U^4X^2+36U^3X^3+19U^2X^4\\&+X^6+10UX^5-8U^3X-8U^2X^2-8UX^3\big)\big(U^6X^6+10U^5X^5-8U^5X^3+1\\&+19U^4X^4-8U^3X^5-8U^4X^2+36U^3X^3-8U^2X^4-8U^3X+19U^2X^2+10UX\\&-8UX^3\big)\big(8U^5X^3-8U^4X^4+8U^3X^5+U^6-10U^5X+19U^4X^2-36U^3X^3\\&+19U^2X^4-10UX^5+X^6+8U^3X-8U^2X^2+8UX^3\big)\big(U^6X^6-10U^5X^5+1\\&+8U^5X^3+19U^4X^4+8U^3X^5-8U^4X^2-36U^3X^3-8U^2X^4+8U^3X+19U^2X^2\\&+8UX^3-10UX\big)=0.\end{aligned}\tag{3.18}$$

We see that the first factor of the above equation (3.18) vanishes for the specific value of $q := e^{-\pi}$ and the other factors do not vanish. And finally setting $UX := P^3$ and $\dfrac{X}{U} := R^3$, we arrive at (3.2). □

Proof of (3.3) *and* (3.4). Interchanging β and γ in the equation (3.1), we get

$$\left[Q^6+\frac{1}{Q^6}\right]\left\{\left[R^3+\frac{1}{R^3}\right]+1\right\}-\left[R^6+\frac{1}{R^6}\right]-10\left[R^3+\frac{1}{R^3}\right]-20=0.\tag{3.19}$$

Eliminating R from the equations (3.19) and (3.2), we arrive at (3.3).

Interchanging β and γ in the equation (3.2), we arrive at the equation (3.4). □

Theorem 3.2. *If α, β, γ and δ have degrees 1, 3, 4 and 12 respectively, then*

$$\begin{aligned}&\left[T^{18}+\frac{1}{T^{18}}\right]\left\{16\left[Q^{10}+\frac{1}{Q^{10}}\right]-\left[Q^{14}+\frac{1}{Q^{14}}\right]+39\left[Q^6+\frac{1}{Q^6}\right]\right.\\&\left.+64\left[Q^2+\frac{1}{Q^2}\right]\right\}+\left[T^{12}+\frac{1}{T^{12}}\right]\left\{1856\left[Q^{12}+\frac{1}{Q^{12}}\right]-112\left[Q^{16}+\frac{1}{Q^{16}}\right]\right.\\&\left.-6088\left[Q^8+\frac{1}{Q^8}\right]+14265\left[Q^4+\frac{1}{Q^4}\right]-11416\right\}-1125176\left[Q^{12}+\frac{1}{Q^{12}}\right]\\&+\left[T^6+\frac{1}{T^6}\right]\left\{38944\left[Q^{14}+\frac{1}{Q^{14}}\right]-32\left[Q^{22}+\frac{1}{Q^{22}}\right]-2119\left[Q^{18}+\frac{1}{Q^{18}}\right]\right.\\&\left.-172583\left[Q^{10}+\frac{1}{Q^{10}}\right]+159088\left[Q^6+\frac{1}{Q^6}\right]+20008\left[Q^2+\frac{1}{Q^2}\right]\right\}+15463590\end{aligned}$$

$$-\left[Q^{28}+\frac{1}{Q^{28}}\right]+176\left[Q^{24}+\frac{1}{Q^{24}}\right]-8313\left[Q^{20}+\frac{1}{Q^{20}}\right]+139168\left[Q^{16}+\frac{1}{Q^{16}}\right]$$
$$+4922543\left[Q^{8}+\frac{1}{Q^{8}}\right]+\left[Q^{4}+\frac{1}{Q^{4}}\right]\Bigg\}\left\{\left[T^{24}+\frac{1}{T^{24}}\right]-11519272\right\}=0, \tag{3.20}$$

$$2^{7}\left[R^{3}+\frac{1}{R^{3}}\right]\left\{5389P^{15}-288\left[2221P^{9}-\frac{2^{17}}{P^{9}}\right]+3072\left[5605P^{3}-\frac{6656}{P^{3}}\right]\right\}$$
$$+2^{11}\left[R^{6}+\frac{1}{R^{6}}\right]\left\{2\left[75631P^{6}-\frac{393216}{P^{6}}\right]-\left[2709P^{12}+\frac{2^{19}}{P^{12}}\right]+105984\right\}$$
$$+2^{11}\left[R^{9}+\frac{1}{R^{9}}\right]\left\{3\left[3117P^{9}+\frac{2^{19}}{P^{9}}\right]-2^{8}\left[1053P^{3}-\frac{1732}{P^{3}}\right]\right\}+11089152P^{12}$$
$$+2^{17}\left[R^{12}+\frac{1}{R^{12}}\right]\left\{360-\left[251P^{6}+\frac{19968}{P^{6}}\right]\right\}-786432\left[949P^{6}+\frac{19200}{P^{6}}\right] \tag{3.21}$$
$$+2^{18}\left[R^{15}+\frac{1}{R^{15}}\right]\left[111P^{3}+\frac{2176}{P^{3}}\right]-12582912\left[R^{18}+\frac{1}{R^{18}}\right]+P^{24}-23232P^{18}$$
$$+\frac{2^{21}}{P^{3}}\left[R^{21}+\frac{1}{R^{21}}\right]+10670309376=0,$$

$$\left[R^{18}+\frac{1}{R^{18}}\right]\Bigg\{16\left[Q^{10}+\frac{1}{Q^{10}}\right]-\left[Q^{14}+\frac{1}{Q^{14}}\right]+39\left[Q^{6}+\frac{1}{Q^{6}}\right]$$
$$+64\left[Q^{2}+\frac{1}{Q^{2}}\right]\Bigg\}+\left[R^{12}+\frac{1}{R^{12}}\right]\Bigg\{1856\left[Q^{12}+\frac{1}{Q^{12}}\right]-112\left[Q^{16}+\frac{1}{Q^{16}}\right]$$
$$-6088\left[Q^{8}+\frac{1}{Q^{8}}\right]+14265\left[Q^{4}+\frac{1}{Q^{4}}\right]-11416\Bigg\}-1125176\left[Q^{12}+\frac{1}{Q^{12}}\right]$$
$$+\left[R^{6}+\frac{1}{R^{6}}\right]\Bigg\{38944\left[Q^{14}+\frac{1}{Q^{14}}\right]-32\left[Q^{22}+\frac{1}{Q^{22}}\right]-2119\left[Q^{18}+\frac{1}{Q^{18}}\right] \tag{3.22}$$
$$-172583\left[Q^{10}+\frac{1}{Q^{10}}\right]+159088\left[Q^{6}+\frac{1}{Q^{6}}\right]+20008\left[Q^{2}+\frac{1}{Q^{2}}\right]\Bigg\}+15463590$$
$$-\left[Q^{28}+\frac{1}{Q^{28}}\right]+176\left[Q^{24}+\frac{1}{Q^{24}}\right]-8313\left[Q^{20}+\frac{1}{Q^{20}}\right]+139168\left[Q^{16}+\frac{1}{Q^{16}}\right]$$
$$+4922543\left[Q^{8}+\frac{1}{Q^{8}}\right]+\left[Q^{4}+\frac{1}{Q^{4}}\right]\Bigg\}\left\{\left[R^{24}+\frac{1}{R^{24}}\right]-11519272\right\}=0,$$

$$2^{7}\left[T^{3}+\frac{1}{T^{3}}\right]\left\{5389P^{15}-288\left[2221P^{9}-\frac{2^{17}}{P^{9}}\right]+3072\left[5605P^{3}-\frac{6656}{P^{3}}\right]\right\}$$
$$+2^{11}\left[T^{6}+\frac{1}{T^{6}}\right]\left\{2\left[75631P^{6}-\frac{393216}{P^{6}}\right]-\left[2709P^{12}+\frac{2^{19}}{P^{12}}\right]+105984\right\}$$
$$+2^{11}\left[T^{9}+\frac{1}{T^{9}}\right]\left\{3\left[3117P^{9}+\frac{2^{19}}{P^{9}}\right]-2^{8}\left[1053P^{3}-\frac{1732}{P^{3}}\right]\right\}+11089152P^{12}$$

$$+2^{17}\left[T^{12}+\frac{1}{T^{12}}\right]\left\{360-\left[251P^6+\frac{19968}{P^6}\right]\right\}-786432\left[949P^6+\frac{19200}{P^6}\right]$$
$$+2^{18}\left[T^{15}+\frac{1}{T^{15}}\right]\left[111P^3+\frac{2176}{P^3}\right]-12582912\left[T^{18}+\frac{1}{T^{18}}\right]-23232P^{18} \quad (3.23)$$
$$+\frac{2^{21}}{P^3}\left[T^{21}+\frac{1}{T^{21}}\right]+P^{24}+10670309376=0.$$

Since the proof of the Theorem (3.2) is similar to the proof of the Theorem (3.1), we omit the details.

Theorem 3.3. *If α, β, γ and δ have degrees 1, 3, 9 and 27 respectively, then*

$$\left[T^{24}+\frac{1}{T^{24}}\right]+\left[T^{18}+\frac{1}{T^{18}}\right]\left\{27\left[Q^6+\frac{1}{Q^6}\right]+496\left[Q^3+\frac{1}{Q^3}\right]+1170\right\}$$
$$+\left[T^{12}+\frac{1}{T^{12}}\right]\left\{27\left[Q^{12}+\frac{1}{Q^{12}}\right]-\left[Q^{15}+\frac{1}{Q^{15}}\right]-270\left[Q^9+\frac{1}{Q^9}\right]\right.$$
$$\left.+2692\left[Q^6+\frac{1}{Q^6}\right]-657\left[Q^3+\frac{1}{Q^3}\right]+9694\right\}+\left[T^6+\frac{1}{T^6}\right]\left\{18\left[Q^{15}+\frac{1}{Q^{15}}\right]\right.$$
$$-\left[Q^{18}+\frac{1}{Q^{18}}\right]-10\left[Q^{12}+\frac{1}{Q^{12}}\right]-954\left[Q^9+\frac{1}{Q^9}\right]+16254\left[Q^6+\frac{1}{Q^6}\right] \quad (3.24)$$
$$\left.-60488\left[Q^3+\frac{1}{Q^3}\right]+254610\right\}-\left[Q^{21}+\frac{1}{Q^{21}}\right]+36\left[Q^{18}+\frac{1}{Q^{18}}\right]$$
$$-613\left[Q^{15}+\frac{1}{Q^{15}}\right]+6642\left[Q^{12}+\frac{1}{Q^{12}}\right]-46513\left[Q^9+\frac{1}{Q^9}\right]$$
$$+225252\left[Q^6+\frac{1}{Q^6}\right]-662697\left[Q^3+\frac{1}{Q^3}\right]+1262162=0,$$

$$2^4\left[P^3+\frac{1}{P^3}\right]\left\{275\left[R^{21}+\frac{1}{R^{21}}\right]-6381\left[R^{18}+\frac{1}{R^{18}}\right]+79209\left[R^{15}+\frac{1}{R^{15}}\right]\right.$$
$$-621342\left[R^{12}+\frac{1}{R^{12}}\right]+3221555\left[R^9+\frac{1}{R^9}\right]-12146355\left[R^6+\frac{1}{R^6}\right]$$
$$\left.+27500625\left[R^3+\frac{1}{R^3}\right]-21172356\right\}+2^6\left[P^6+\frac{1}{P^6}\right]\left\{2881\left[R^{15}+\frac{1}{R^{15}}\right]\right.$$
$$-216\left[R^{18}+\frac{1}{R^{18}}\right]-39852\left[R^{12}+\frac{1}{R^{12}}\right]+235494\left[R^9+\frac{1}{R^9}\right]$$
$$\left.-883134\left[R^6+\frac{1}{R^6}\right]+1241577\left[R^3+\frac{1}{R^3}\right]-3772828\right\}+2^9\left[P^9+\frac{1}{P^9}\right]$$
$$\times\left\{3^3\left[R^{15}+\frac{1}{R^{15}}\right]-270\left[R^{12}+\frac{1}{R^{12}}\right]+5139\left[R^9+\frac{1}{R^9}\right]-7336\left[R^6+\frac{1}{R^6}\right]\right.$$
$$\left.+64818\left[R^3+\frac{1}{R^3}\right]+58860\right\}-2^{12}\left[P^{12}+\frac{1}{P^{12}}\right]\left\{\left[R^{12}+\frac{1}{R^{12}}\right]-6^2\left[R^9+\frac{1}{R^9}\right]\right.$$

$$+585\left[R^6+\frac{1}{R^6}\right]+2664\left[R^3+\frac{1}{R^3}\right]+1684\Bigg\}+2^{15}\left[P^{15}+\frac{1}{P^{15}}\right]\Bigg\{45\left[R^3+\frac{1}{R^3}\right]$$
$$-\left[R^6+\frac{1}{R^6}\right]+180\Bigg\}-2^{18}\left[P^{18}+\frac{1}{P^{18}}\right]+\left[R^{27}+\frac{1}{R^{27}}\right]-306\left[R^{24}+\frac{1}{R^{24}}\right]$$
$$+18081\left[R^{21}+\frac{1}{R^{21}}\right]-322272\left[R^{18}+\frac{1}{R^{18}}\right]+3645532\left[R^{15}+\frac{1}{R^{15}}\right] \quad (3.25)$$
$$-23695992\left[R^{12}+\frac{1}{R^{12}}\right]+105349788\left[R^9+\frac{1}{R^9}\right]-278500896\left[R^6+\frac{1}{R^6}\right]$$
$$+534589030\left[R^3+\frac{1}{R^3}\right]=956630700,$$

$$\left[R^{24}+\frac{1}{R^{24}}\right]+\left[R^{18}+\frac{1}{R^{18}}\right]\Bigg\{27\left[Q^6+\frac{1}{Q^6}\right]+496\left[Q^3+\frac{1}{Q^3}\right]+1170\Bigg\}$$
$$+\left[R^{12}+\frac{1}{R^{12}}\right]\Bigg\{27\left[Q^{12}+\frac{1}{Q^{12}}\right]-\left[Q^{15}+\frac{1}{Q^{15}}\right]-270\left[Q^9+\frac{1}{Q^9}\right]$$
$$+2692\left[Q^6+\frac{1}{Q^6}\right]-657\left[Q^3+\frac{1}{Q^3}\right]+9694\Bigg\}+\left[R^6+\frac{1}{R^6}\right]\Bigg\{18\left[Q^{15}+\frac{1}{Q^{15}}\right]$$
$$-\left[Q^{18}+\frac{1}{Q^{18}}\right]-10\left[Q^{12}+\frac{1}{Q^{12}}\right]-954\left[Q^9+\frac{1}{Q^9}\right]+16254\left[Q^6+\frac{1}{Q^6}\right] \quad (3.26)$$
$$-60488\left[Q^3+\frac{1}{Q^3}\right]+254610\Bigg\}-\left[Q^{21}+\frac{1}{Q^{21}}\right]+36\left[Q^{18}+\frac{1}{Q^{18}}\right]$$
$$-613\left[Q^{15}+\frac{1}{Q^{15}}\right]+6642\left[Q^{12}+\frac{1}{Q^{12}}\right]-46513\left[Q^9+\frac{1}{Q^9}\right]$$
$$+225252\left[Q^6+\frac{1}{Q^6}\right]-662697\left[Q^3+\frac{1}{Q^3}\right]+1262162=0,$$

$$2^4\left[P^3+\frac{1}{P^3}\right]\Bigg\{275\left[T^{21}+\frac{1}{T^{21}}\right]-6381\left[T^{18}+\frac{1}{T^{18}}\right]+79209\left[T^{15}+\frac{1}{T^{15}}\right]$$
$$-621342\left[T^{12}+\frac{1}{T^{12}}\right]+3221555\left[T^9+\frac{1}{T^9}\right]-12146355\left[T^6+\frac{1}{T^6}\right]$$
$$+27500625\left[T^3+\frac{1}{T^3}\right]-21172356\Bigg\}+2^6\left[P^6+\frac{1}{P^6}\right]\Bigg\{2881\left[T^{15}+\frac{1}{T^{15}}\right]$$
$$-216\left[T^{18}+\frac{1}{T^{18}}\right]-39852\left[T^{12}+\frac{1}{T^{12}}\right]+235494\left[T^9+\frac{1}{T^9}\right]$$
$$-883134\left[T^6+\frac{1}{T^6}\right]+1241577\left[T^3+\frac{1}{T^3}\right]-3772828\Bigg\}+2^9\left[P^9+\frac{1}{P^9}\right]$$
$$\times\Bigg\{3^3\left[T^{15}+\frac{1}{T^{15}}\right]-270\left[T^{12}+\frac{1}{T^{12}}\right]+5139\left[T^9+\frac{1}{T^9}\right]-7336\left[T^6+\frac{1}{T^6}\right]$$
$$+64818\left[T^3+\frac{1}{T^3}\right]+58860\Bigg\}-2^{12}\left[P^{12}+\frac{1}{P^{12}}\right]\Bigg\{\left[T^{12}+\frac{1}{T^{12}}\right]$$

$$-6^2\left[T^9+\frac{1}{T^9}\right]+585\left[T^6+\frac{1}{T^6}\right]+2664\left[T^3+\frac{1}{T^3}\right]+1684\Big\}+\left[T^{27}+\frac{1}{T^{27}}\right]$$
$$+2^{15}\left[P^{15}+\frac{1}{P^{15}}\right]\left\{45\left[T^3+\frac{1}{T^3}\right]-\left[T^6+\frac{1}{T^6}\right]+180\right\}-2^{18}\left[P^{18}+\frac{1}{P^{18}}\right]$$
$$-306\left[T^{24}+\frac{1}{T^{24}}\right]+18081\left[T^{21}+\frac{1}{T^{21}}\right]-322272\left[T^{18}+\frac{1}{T^{18}}\right] \tag{3.27}$$
$$+3645532\left[T^{15}+\frac{1}{T^{15}}\right]-23695992\left[T^{12}+\frac{1}{T^{12}}\right]+105349788\left[T^9+\frac{1}{T^9}\right]$$
$$-278500896\left[T^6+\frac{1}{T^6}\right]+534589030\left[T^3+\frac{1}{T^3}\right]=956630700.$$

Since the proof of the Theorem (3.3) is similar to the proof of the Theorem (3.1), we omit the details.

Theorem 3.4. *If α, β, γ and δ have degrees 1, 3, 17 and 51 respectively, then*

$$\left[Q^9+\frac{1}{Q^9}\right]-17\left[Q^6+\frac{1}{Q^6}\right]\left[T^2+\frac{1}{T^2}\right]+\left[Q^3+\frac{1}{Q^3}\right]\left\{68\left[T^4+\frac{1}{T^4}\right]\right.$$
$$\left.-136\left[T^2+\frac{1}{T^2}\right]-153\right\}-\left[T^8+\frac{1}{T^8}\right]-34\left[T^6+\frac{1}{T^6}\right]-153\left[T^4+\frac{1}{T^4}\right] \tag{3.28}$$
$$-272\left[T^2+\frac{1}{T^2}\right]-884=0,$$

$$\left[Q^9+\frac{1}{Q^9}\right]-17\left[Q^6+\frac{1}{Q^6}\right]\left[R^2+\frac{1}{R^2}\right]+\left[Q^3+\frac{1}{Q^3}\right]\left\{68\left[R^4+\frac{1}{R^4}\right]\right.$$
$$\left.-136\left[R^2+\frac{1}{R^2}\right]-153\right\}-\left[R^8+\frac{1}{R^8}\right]-34\left[R^6+\frac{1}{R^6}\right]-153\left[R^4+\frac{1}{R^4}\right] \tag{3.29}$$
$$-272\left[R^2+\frac{1}{R^2}\right]-884=0,$$

$$2^{24}\left[P^{24}+\frac{1}{P^{24}}\right]=2^{21}.17\left[P^{21}+\frac{1}{P^{21}}\right]\left\{3\left[R^3+\frac{1}{R^3}\right]+104\right\}+\left[R^{27}+\frac{1}{R^{27}}\right]$$
$$-2^{18}.17\left[P^{18}+\frac{1}{P^{18}}\right]\left\{60\left[R^6+\frac{1}{R^6}\right]-6480\left[R^3+\frac{1}{R^3}\right]+23009\right\}$$
$$+2^{15}.17\left[P^{15}+\frac{1}{P^{15}}\right]\left\{1100604\left[R^3+\frac{1}{R^3}\right]-45837\left[R^6+\frac{1}{R^6}\right]-1531364\right.$$
$$\left.+601\left[R^9+\frac{1}{R^9}\right]\right\}-2^{12}.17\left[P^{12}+\frac{1}{P^{12}}\right]\left\{3207\left[R^{12}+\frac{1}{R^{12}}\right]+8715\left[R^9+\frac{1}{R^9}\right]\right.$$
$$\left.-1426425\left[R^6+\frac{1}{R^6}\right]-30510807\left[R^3+\frac{1}{R^3}\right]+129757356\right\}+8704\left[P^9+\frac{1}{P^9}\right]$$

$$
\begin{aligned}
&\times\left\{8979\left[R^{15}+\frac{1}{R^{15}}\right]+759804\left[R^{12}+\frac{1}{R^{12}}\right]-20592271\left[R^{9}+\frac{1}{R^{9}}\right]\right.\\
&\left.+372890892\left[R^{6}+\frac{1}{R^{6}}\right]+2421488316\left[R^{3}+\frac{1}{R^{3}}\right]-1476754256\right\}\\
&-2^{6}.17\left[P^{6}+\frac{1}{P^{6}}\right]\left\{11970\left[R^{18}+\frac{1}{R^{18}}\right]+2530407\left[R^{15}+\frac{1}{R^{15}}\right]\right.\\
&+107090160\left[R^{12}+\frac{1}{R^{12}}\right]-838208230\left[R^{9}+\frac{1}{R^{9}}\right]+4624119972\left[R^{6}+\frac{1}{R^{6}}\right]\\
&\left.+14474993391\left[R^{3}+\frac{1}{R^{3}}\right]+126223725172\right\}-242188438622874\left[R^{3}+\frac{1}{R^{3}}\right]\\
&+2^{4}.17\left[P^{3}+\frac{1}{P^{3}}\right]\left\{2883\left[R^{21}+\frac{1}{R^{21}}\right]+875419539537\left[R^{3}+\frac{1}{R^{3}}\right]\right.\\
&+1580383\left[R^{18}+\frac{1}{R^{18}}\right]+177501345\left[R^{15}+\frac{1}{R^{15}}\right]+6404707098\left[R^{12}+\frac{1}{R^{12}}\right]\\
&\left.+67141173451\left[R^{9}+\frac{1}{R^{9}}\right]+337092950145\left[R^{6}+\frac{1}{R^{6}}\right]+469757334540\right\}\\
&-6018\left[R^{24}+\frac{1}{R^{24}}\right]+7296417\left[R^{21}+\frac{1}{R^{21}}\right]-261517664\left[R^{18}+\frac{1}{R^{18}}\right]\\
&-39535508083044\left[R^{9}+\frac{1}{R^{9}}\right]-133483203434400\left[R^{6}+\frac{1}{R^{6}}\right]\\
&-3605922603576\left[R^{12}+\frac{1}{R^{12}}\right]-89812064676\left[R^{15}+\frac{1}{R^{15}}\right]\\
&-524832474901132,
\end{aligned}
\tag{3.30}
$$

$$
\begin{aligned}
&2^{24}\left[P^{24}+\frac{1}{P^{24}}\right]=2^{21}.17\left[P^{21}+\frac{1}{P^{21}}\right]\left\{3\left[T^{3}+\frac{1}{T^{3}}\right]+104\right\}+\left[T^{27}+\frac{1}{T^{27}}\right]\\
&-2^{18}.17\left[P^{18}+\frac{1}{P^{18}}\right]\left\{60\left[T^{6}+\frac{1}{T^{6}}\right]-6480\left[T^{3}+\frac{1}{T^{3}}\right]+23009\right\}\\
&+2^{15}.17\left[P^{15}+\frac{1}{P^{15}}\right]\left\{1100604\left[T^{3}+\frac{1}{T^{3}}\right]-45837\left[T^{6}+\frac{1}{T^{6}}\right]-1531364\right.\\
&\left.+601\left[T^{9}+\frac{1}{T^{9}}\right]\right\}-2^{12}.17\left[P^{12}+\frac{1}{P^{12}}\right]\left\{3207\left[T^{12}+\frac{1}{T^{12}}\right]+8715\left[T^{9}+\frac{1}{T^{9}}\right]\right.\\
&\left.-1426425\left[T^{6}+\frac{1}{T^{6}}\right]-30510807\left[T^{3}+\frac{1}{T^{3}}\right]+129757356\right\}+8704\left[P^{9}+\frac{1}{P^{9}}\right]\\
&\times\left\{8979\left[T^{15}+\frac{1}{T^{15}}\right]+759804\left[T^{12}+\frac{1}{T^{12}}\right]-20592271\left[T^{9}+\frac{1}{T^{9}}\right]\right.\\
&\left.+372890892\left[T^{6}+\frac{1}{T^{6}}\right]+2421488316\left[T^{3}+\frac{1}{T^{3}}\right]-1476754256\right\}\\
&-2^{6}.17\left[P^{6}+\frac{1}{P^{6}}\right]\left\{11970\left[T^{18}+\frac{1}{T^{18}}\right]+2530407\left[T^{15}+\frac{1}{T^{15}}\right]\right.
\end{aligned}
$$

$$+107090160\left[T^{12}+\frac{1}{T^{12}}\right]-838208230\left[T^{9}+\frac{1}{T^{9}}\right]+4624119972\left[T^{6}+\frac{1}{T^{6}}\right]$$
$$+14474993391\left[T^{3}+\frac{1}{T^{3}}\right]+126223725172\Big\}-242188438622874\left[T^{3}+\frac{1}{T^{3}}\right]$$
$$+2^{4}.17\left[P^{3}+\frac{1}{P^{3}}\right]\Big\{2883\left[T^{21}+\frac{1}{T^{21}}\right]+875419539537\left[T^{3}+\frac{1}{T^{3}}\right]$$
$$+1580383\left[T^{18}+\frac{1}{T^{18}}\right]+177501345\left[T^{15}+\frac{1}{T^{15}}\right]+6404707098\left[T^{12}+\frac{1}{T^{12}}\right]$$
$$+67141173451\left[T^{9}+\frac{1}{T^{9}}\right]+337092950145\left[T^{6}+\frac{1}{T^{6}}\right]+469757334540\Big\} \quad (3.31)$$
$$-6018\left[T^{24}+\frac{1}{T^{24}}\right]+7296417\left[T^{21}+\frac{1}{T^{21}}\right]-261517664\left[T^{18}+\frac{1}{T^{18}}\right]$$
$$-39535508083044\left[T^{9}+\frac{1}{T^{9}}\right]-133483203434400\left[T^{6}+\frac{1}{T^{6}}\right]$$
$$-3605922603576\left[T^{12}+\frac{1}{T^{12}}\right]-89812064676\left[T^{15}+\frac{1}{T^{15}}\right]$$
$$-524832474901132.$$

Since the proof of the Theorem (3.4) is similar to the proof of the Theorem (3.1), we omit the details.

Theorem 3.5. *If α, β, γ and δ have degrees 1, 3, 19 and 57 respectively, then*

$$\left[Q^{10}+\frac{1}{Q^{10}}\right]-\left[T^{9}+\frac{1}{T^{9}}\right]+\left[T^{6}+\frac{1}{T^{6}}\right]\left\{57\left[Q^{2}+\frac{1}{Q^{2}}\right]-152\right\}$$
$$-\left[Q^{6}+\frac{1}{Q^{6}}\right]\left\{19\left[T^{3}+\frac{1}{T^{3}}\right]+76\right\}-\left[Q^{4}+\frac{1}{Q^{4}}\right]\left\{266\left[T^{3}+\frac{1}{T^{3}}\right]+570\right\} \quad (3.32)$$
$$-\left[Q^{2}+\frac{1}{Q^{2}}\right]\left\{190\left[T^{3}+\frac{1}{T^{3}}\right]+133\right\}+627\left[T^{3}+\frac{1}{T^{3}}\right]+2280=0,$$

$$\left[Q^{10}+\frac{1}{Q^{10}}\right]-\left[R^{9}+\frac{1}{R^{9}}\right]+\left[R^{6}+\frac{1}{R^{6}}\right]\left\{57\left[Q^{2}+\frac{1}{Q^{2}}\right]-152\right\}$$
$$-\left[Q^{6}+\frac{1}{Q^{6}}\right]\left\{19\left[R^{3}+\frac{1}{R^{3}}\right]+76\right\}-\left[Q^{4}+\frac{1}{Q^{4}}\right]\left\{266\left[R^{3}+\frac{1}{R^{3}}\right]+570\right\} \quad (3.33)$$
$$-\left[Q^{2}+\frac{1}{Q^{2}}\right]\left\{190\left[R^{3}+\frac{1}{R^{3}}\right]+133\right\}+627\left[R^{3}+\frac{1}{R^{3}}\right]+2280=0,$$

$$2^{27}\left[P^{27}+\frac{1}{P^{27}}\right]=2^{24}.19\left[P^{24}+\frac{1}{P^{24}}\right]\left\{3\left[R^{3}+\frac{1}{R^{3}}\right]+156\right\}-2^{21}.19\left[P^{21}+\frac{1}{P^{21}}\right]$$
$$\times\left\{69\left[R^{6}+\frac{1}{R^{6}}\right]-13818\left[R^{3}+\frac{1}{R^{3}}\right]+80985\right\}+2^{18}.19\left[P^{18}+\frac{1}{P^{18}}\right]\{4251856$$
$$+826\left[R^{9}+\frac{1}{R^{9}}\right]-64063\left[R^{6}+\frac{1}{R^{6}}\right]+1012639\left[R^{3}+\frac{1}{R^{3}}\right]\Big\}-2^{15}.19\left[P^{15}+\frac{1}{P^{15}}\right]$$

$$
\begin{aligned}
&\times\left\{5553\left[R^{12}+\frac{1}{R^{12}}\right]+418098\left[R^{9}+\frac{1}{R^{9}}\right]-430665\left[R^{6}+\frac{1}{R^{6}}\right]-303115482\right.\\
&\left.+252627537\left[R^{3}+\frac{1}{R^{3}}\right]\right\}+2^{12}.19\left[P^{12}+\frac{1}{P^{12}}\right]\left\{21150\left[R^{15}+\frac{1}{R^{15}}\right]\right.\\
&+3121665\left[R^{12}+\frac{1}{R^{12}}\right]+193551996\left[R^{9}+\frac{1}{R^{9}}\right]+2009104488\left[R^{6}+\frac{1}{R^{6}}\right]\\
&\left.-3581220420\left[R^{3}+\frac{1}{R^{3}}\right]-5960839518\right\}-2^{9}.19\left[P^{9}+\frac{1}{P^{9}}\right]\left\{43708\left[R^{18}+\frac{1}{R^{18}}\right]\right.\\
&+5651612\left[R^{15}+\frac{1}{R^{15}}\right]+516915495\left[R^{12}+\frac{1}{R^{12}}\right]-21266027137\left[R^{9}+\frac{1}{R^{9}}\right]\\
&\left.+105734011183\left[R^{6}+\frac{1}{R^{6}}\right]-551428273038\left[R^{3}+\frac{1}{R^{3}}\right]+354589485186\right\}\\
&+2^{6}.19\left[P^{6}+\frac{1}{P^{6}}\right]\left\{43374\left[R^{21}+\frac{1}{R^{21}}\right]+1369884706560\left[R^{9}+\frac{1}{R^{9}}\right]\right.\\
&-1242910572\left[R^{15}+\frac{1}{R^{15}}\right]+79389654171\left[R^{12}+\frac{1}{R^{12}}\right]-3163011\left[R^{18}+\frac{1}{R^{18}}\right]\\
&\left.-6969407218269\left[R^{6}+\frac{1}{R^{6}}\right]+13667855540046\left[R^{3}+\frac{1}{R^{3}}\right]-17815633008150\right\} \qquad (3.34)\\
&-2^{3}.19\left[P^{3}+\frac{1}{P^{3}}\right]\left\{15153\left[R^{24}+\frac{1}{R^{24}}\right]-424749912984\left[R^{12}+\frac{1}{R^{12}}\right]\right.\\
&-16772709\left[R^{21}+\frac{1}{R^{21}}\right]-671353242\left[R^{18}+\frac{1}{R^{18}}\right]-29188644591\left[R^{15}+\frac{1}{R^{15}}\right]\\
&-18507945894885\left[R^{9}+\frac{1}{R^{9}}\right]-171460814709255\left[R^{3}+\frac{1}{R^{3}}\right]\\
&\left.+150274131850458\left[R^{6}+\frac{1}{R^{6}}\right]+368182659125454\right\}-33132700514985924\\
&+11628\left[R^{27}+\frac{1}{R^{27}}\right]+30310282\left[R^{24}+\frac{1}{R^{24}}\right]+11527045932\left[R^{21}+\frac{1}{R^{21}}\right]\\
&+845540398845\left[R^{18}+\frac{1}{R^{18}}\right]+45780429297808\left[R^{15}+\frac{1}{R^{15}}\right]+\left[R^{30}+\frac{1}{R^{30}}\right]\\
&+744284748373080\left[R^{12}+\frac{1}{R^{12}}\right]+8159758099239184\left[R^{9}+\frac{1}{R^{9}}\right]\\
&-13401561663284990\left[R^{6}+\frac{1}{R^{6}}\right]+63405689474887112\left[R^{3}+\frac{1}{R^{3}}\right],
\end{aligned}
$$

$$
\begin{aligned}
2^{27}\left[P^{27}+\frac{1}{P^{27}}\right]&=2^{24}.19\left[P^{24}+\frac{1}{P^{24}}\right]\left\{3\left[T^{3}+\frac{1}{T^{3}}\right]+156\right\}-2^{21}.19\left[P^{21}+\frac{1}{P^{21}}\right]\\
&\times\left\{69\left[T^{6}+\frac{1}{T^{6}}\right]-13818\left[T^{3}+\frac{1}{T^{3}}\right]+80985\right\}+2^{18}.19\left[P^{18}+\frac{1}{P^{18}}\right]\{4251856
\end{aligned}
$$

$$
\begin{aligned}
&+826\left[T^9+\frac{1}{T^9}\right]-64063\left[T^6+\frac{1}{T^6}\right]+1012639\left[T^3+\frac{1}{T^3}\right]\bigg\}-2^{15}.19\left[P^{15}+\frac{1}{P^{15}}\right]\\
&\times\bigg\{5553\left[T^{12}+\frac{1}{T^{12}}\right]+418098\left[T^9+\frac{1}{T^9}\right]-430665\left[T^6+\frac{1}{T^6}\right]-303115482\\
&+252627537\left[T^3+\frac{1}{T^3}\right]\bigg\}+2^{12}.19\left[P^{12}+\frac{1}{P^{12}}\right]\bigg\{21150\left[T^{15}+\frac{1}{T^{15}}\right]\\
&+3121665\left[T^{12}+\frac{1}{T^{12}}\right]+193551996\left[T^9+\frac{1}{T^9}\right]+2009104488\left[T^6+\frac{1}{T^6}\right]\\
&-3581220420\left[T^3+\frac{1}{T^3}\right]-5960839518\bigg\}-2^9.19\left[P^9+\frac{1}{P^9}\right]\bigg\{43708\left[T^{18}+\frac{1}{T^{18}}\right]\\
&+5651612\left[T^{15}+\frac{1}{T^{15}}\right]+516915495\left[T^{12}+\frac{1}{T^{12}}\right]-21266027137\left[T^9+\frac{1}{T^9}\right]\\
&+105734011183\left[T^6+\frac{1}{T^6}\right]-551428273038\left[T^3+\frac{1}{T^3}\right]+354589485186\bigg\}\\
&+2^6.19\left[P^6+\frac{1}{P^6}\right]\bigg\{43374\left[T^{21}+\frac{1}{T^{21}}\right]+1369884706560\left[T^9+\frac{1}{T^9}\right]\\
&-1242910572\left[T^{15}+\frac{1}{T^{15}}\right]+79389654171\left[T^{12}+\frac{1}{T^{12}}\right]-3163011\left[T^{18}+\frac{1}{T^{18}}\right]\\
&-6969407218269\left[T^6+\frac{1}{T^6}\right]+13667855540046\left[T^3+\frac{1}{T^3}\right]-17815633008150\bigg\}\\
&-2^3.19\left[P^3+\frac{1}{P^3}\right]\bigg\{15153\left[T^{24}+\frac{1}{T^{24}}\right]-424749912984\left[T^{12}+\frac{1}{T^{12}}\right]\\
&-16772709\left[T^{21}+\frac{1}{T^{21}}\right]-671353242\left[T^{18}+\frac{1}{T^{18}}\right]-29188644591\left[T^{15}+\frac{1}{T^{15}}\right]\\
&-18507945894885\left[T^9+\frac{1}{T^9}\right]-171460814709255\left[T^3+\frac{1}{T^3}\right]\\
&+150274131850458\left[T^6+\frac{1}{T^6}\right]+368182659125454\bigg\}-33132700514985924\\
&+11628\left[T^{27}+\frac{1}{T^{27}}\right]+30310282\left[T^{24}+\frac{1}{T^{24}}\right]+11527045932\left[T^{21}+\frac{1}{T^{21}}\right]\\
&+845540398845\left[T^{18}+\frac{1}{T^{18}}\right]+45780429297808\left[T^{15}+\frac{1}{T^{15}}\right]+\left[T^{30}+\frac{1}{T^{30}}\right]\\
&+744284748373080\left[T^{12}+\frac{1}{T^{12}}\right]+8159758099239184\left[T^9+\frac{1}{T^9}\right]\\
&-13401561663284990\left[T^6+\frac{1}{T^6}\right]+63405689474887112\left[T^3+\frac{1}{T^3}\right].
\end{aligned}
\tag{3.35}
$$

Since the proof of the Theorem (3.5) is similar to the proof of the Theorem (3.1), we omit the details.

4 Explicit Evaluations of G_n

In this section, we evaluate some new values for the class invariant G_n for $n = 12, 48, 51, 57, 3/4, 3/16, 3/17$ and $3/19$. For details on this notation and the link to class invariants and the work of Ramanujan one can see [5].

Theorem 4.1. *We have*

$$G_{12}^4 = \frac{\left(9\sqrt{6}+22\right)^{1/3}}{\left(1+\sqrt{3}\right)^{1/2}\left(6\sqrt{2}-8\right)^{1/2}}, \tag{4.1}$$

$$G_{3/4}^4 = \left(1+\sqrt{3}\right)^{1/2}\left(6\sqrt{2}-8\right)^{1/2}\left(9\sqrt{6}+22\right)^{1/3}, \tag{4.2}$$

$$G_{48}^2 = \frac{\left(358+144\sqrt{6}+9\sqrt{3118+1273\sqrt{6}}\right)^{1/6}}{\sqrt{2}\left(-14-6\sqrt{6}+\frac{3}{2}\sqrt{262+107\sqrt{6}}-\sqrt{\left(14+6\sqrt{6}-\frac{3}{2}\sqrt{262+107\sqrt{6}}\right)^2-1}\right)^{1/4}}, \tag{4.3}$$

$$\begin{aligned} G_{3/16}^2 = &\left\{\frac{3}{2}\sqrt{262+107\sqrt{6}}-14-6\sqrt{6}-\sqrt{\left(14+6\sqrt{6}-\frac{3}{2}\sqrt{262+107\sqrt{6}}\right)^2-1}\right\}^{1/4} \\ &\times\frac{\left(358+144\sqrt{6}+9\sqrt{3118+1273\sqrt{6}}\right)^{1/6}}{\sqrt{2}}. \end{aligned} \tag{4.4}$$

Proofs of the equations (4.1) *and* (4.2). In the equation (3.21), using the definition of class invariant G_n, we note that $P = \dfrac{1}{G_nG_{9n}G_{16n}G_{144n}}$ and $R = \dfrac{G_nG_{9n}}{G_{16n}G_{144n}}$. Now by setting $n = 1/12$ and using the fact that $G_n = G_{1/n}$, the equation (3.21) reduces to

$$\begin{aligned} &(32M^6-176M^3-1)(2M-1)^2(4M^2+2M+1)^2(64M^6-88M^3+1)^2 \\ &\times(4M^3+1)^4 = 0, \end{aligned} \tag{4.5}$$

where $M = G_{3/4}^2G_{12}^2$.
We see that the first factor of the above equation (4.5) vanishes for the particular value of $q = e^{-\pi\sqrt{1/12}}$, hence we deduce that

$$32M^6-176M^3-1 = 0. \tag{4.6}$$

Solving the above equation (4.6) for M, we get

$$M = \frac{\left(9\sqrt{6}+22\right)^{1/3}}{2}. \tag{4.7}$$

Now in the equation (3.20), using the definition of class invariant G_n, we note that $Q = \dfrac{G_{9n}G_{16n}}{G_nG_{144n}}$ and $T = \dfrac{G_nG_{16n}}{G_{9n}G_{144n}}$. By setting $n = 1/12$ and using the fact that $G_n = G_{1/n}$, the equation (3.20) reduces to

$$\begin{aligned}&(Q^8+8Q^6-36Q^4+8Q^2+1)(Q^4+4Q^2+1)^2(Q^4-3Q^3+Q^2-3Q+1)^2\\&\times(Q^6+2Q^4-3Q^2+1)^2(Q^4+3Q^3+Q^2+3Q+1)^2(Q^6-3Q^4+2Q^2+1)^2=0,\end{aligned}\tag{4.8}$$

where $Q=\dfrac{G_{3/4}^2}{G_{12}^2}$.

We see that the first factor of the above equation (4.8) vanishes for the particular value of $q=e^{-\pi\sqrt{1/12}}$, hence we deduce that

$$x^2+8x-38=0,\tag{4.9}$$

where $x:=Q^2+\dfrac{1}{Q^2}$.

Solving (4.9) for x, we find that

$$x:=-4+3\sqrt{6},\ -4-3\sqrt{6}.\tag{4.10}$$

Since $x>0$, we deduce that

$$Q^2+\frac{1}{Q^2}=-4+3\sqrt{6}.\tag{4.11}$$

Solving the above equation (4.11), we get

$$Q=\frac{\sqrt{-8-8\sqrt{3}+6\sqrt{2}+6\sqrt{6}}}{2}.\tag{4.12}$$

Now combining the equations (4.7) and (4.12), we arrive at (4.1) and (4.2). □

Proofs of the equations (4.3) *and* (4.4). By setting $n=1/3$ and using the fact that $G_{1/3}=G_3=2^{1/12}$[5, Ch.34, p.189] in the equation (3.21), we deduce that

$$2048M^{24}-366592M^{18}+481152M^{12}-250432M^6-1=0,\tag{4.13}$$

where $M=G_{3/16}G_{48}$.

Solving the above equation (4.13) for M, we get

$$M=\frac{\left(358+144\sqrt{6}+9\sqrt{3118+1273\sqrt{6}}\right)^{1/6}}{\sqrt{2}}.\tag{4.14}$$

Now in the equation (3.20), using the definition of class invariant G_n with $n=1/3$ and using the fact that $G_n=G_{1/n}$, the equation (3.20) reduces to

$$\begin{aligned}&(Q^{32}+112Q^{28}-1736Q^{24}+4624Q^{20}-7460Q^{16}+4624Q^{12}-1736Q^8\\&+112Q^4+1)(Q^{16}+8Q^{12}-36Q^8+8Q^4+1)^2=0,\end{aligned}\tag{4.15}$$

where $Q = \dfrac{G_{3/16}}{G_{48}}$.
We see that the first factor of the above equation (4.15) vanishes for the particular value of $q = e^{-\pi\sqrt{1/3}}$, hence we deduce that

$$Q^{32} + 112Q^{28} - 1736Q^{24} + 4624Q^{20} - 7460Q^{16} + 4624Q^{12} - 1736Q^{8} + 112Q^{4} + 1 = 0. \tag{4.16}$$

Solving the above equation (4.16), we get

$$Q = \left(-14 - 6\sqrt{6} + \frac{3}{2}\sqrt{262 + 107\sqrt{6}} - \sqrt{\left(14 + 3\sqrt{6} - \frac{3}{2}\sqrt{262 + 107\sqrt{6}}\right)^2 - 1}\right)^{1/4}. \tag{4.17}$$

Now combining the equations (4.14) and (4.17), we arrive at (4.3) and (4.4). □

Theorem 4.2. *We have*

$$G_{51}^4 = \frac{1}{3^{1/3}}\left(2 + (9 - \sqrt{17})^{1/3} + (9 + \sqrt{17})^{1/3}\right)\left(t + s + 17 + \sqrt{(t + s + 17)^2 - 9}\right)^{1/3}, \tag{4.18}$$

$$G_{3/17}^4 = \frac{3^{1/3}\left(2 + (9 - \sqrt{17})^{1/3} + (9 + \sqrt{17})^{1/3}\right)}{\left(t + s + 17 + \sqrt{(t + s + 17)^2 - 9}\right)^{1/3}}, \tag{4.19}$$

where $s = \left(3427 + 1701\sqrt{17}\right)^{1/3}$ *and* $t = \left(3427 - 1701\sqrt{17}\right)^{1/3}$.

Proofs of the equations (4.18) *and* (4.19). Using G_n and setting $n = 1/51$ in the equation (3.30), we deduce that

$$(M^3 - 6M^2 - 2)(2M^3 + 6M - 1)(M^4 - 4M^3 + 7M^2 - 4M + 1)(M - 1)^2(M - 2)^2 \times (2M - 1)^2 = 0, \tag{4.20}$$

where $M = G_{3/17}G_{51}$.
We see that the first factor of the above equation (4.20) vanishes for the particular value of $q = e^{-\pi\sqrt{1/51}}$, hence we deduce that

$$M^3 - 6M^2 - 2 = 0, \tag{4.21}$$

Solving the above equation (4.21) for M, we get

$$M = 2 + (9 - \sqrt{17})^{1/3} + (9 + \sqrt{17})^{1/3}. \tag{4.22}$$

Now in the equation (3.28), using the definition of class invariant G_n with $n = 1/51$ and using the fact that $G_n = G_{1/n}$, the equation (3.28) reduces to

$$Q^{18} - 34Q^{15} - 289Q^{12} - 1804Q^{9} - 289Q^{6} - 34Q^{3} + 1 = 0, \tag{4.23}$$

where $Q = \dfrac{G_{3/17}}{G_{51}}$.
Solving the above equation (4.23), we get

$$Q = \frac{3^{1/3}}{\left(t+s+17+\sqrt{(t+s+17)^2-9}\right)^{1/3}}, \tag{4.24}$$

where s and t are as in (4.19).
Now combining the equations (4.22) and (4.24), we arrive at (4.18) and (4.19). □

Theorem 4.3. *We have*

$$G_{57}^4 = \left(2+\sqrt{3}\right)\left(\frac{3\sqrt{19}+13}{\sqrt{2}}\right)^{2/3}, \tag{4.25}$$

$$G_{3/19}^4 = \left(2-\sqrt{3}\right)\left(\frac{3\sqrt{19}+13}{\sqrt{2}}\right)^{2/3}. \tag{4.26}$$

Proofs of the equations (4.25) *and* (4.26). Using G_n and setting $n = 1/57$ in the equation (3.34), we deduce that

$$(M^6-340M^3+1)(M-2)^2(2M-1)^2(M^2+2M+4)^2(4M^2+2M+1)^2 = 0, \tag{4.27}$$

where $M = G_{3/19}G_{57}$.
We see that the first factor of the above equation (4.27) vanishes for the particular value of $q = e^{-\pi\sqrt{1/57}}$, hence we deduce that

$$M^6 - 340M^3 + 1 = 0. \tag{4.28}$$

Solving the above equation (4.28) for M, we get

$$M = (170+39\sqrt{19})^{1/3}. \tag{4.29}$$

Now in the equation (3.32), using the definition of class invariant G_n with $n = 1/57$ and using the fact that $G_n = G_{1/n}$, the equation (3.32) reduces to

$$\begin{aligned}&(Q^2-4Q+1)(Q^{12}+16Q^{10}+112Q^8+226Q^6+112Q^4+16Q^2+1)\\&\times(Q^2+4Q+1)(Q-1)^2(Q+1)^2 = 0,\end{aligned} \tag{4.30}$$

where $Q = \dfrac{G_{3/19}}{G_{57}}$.
We see that the first factor of the above equation (4.30) vanishes for the particular value of $q = e^{-\pi\sqrt{1/57}}$, hence we deduce that

$$Q^2 - 4Q + 1 = 0, \tag{4.31}$$

Solving the above equation (4.31), we get

$$Q = 2-\sqrt{3}. \tag{4.32}$$

Now combining the equations (4.29) and (4.32), we arrive at (4.25) and (4.26).

Remark 4.4. For an alternate proof of (4.25), one can see [5].

□

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