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Embedding in distance degree regular and distance degree injective graphs

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Abstract

The eccentricity $e(u)$ of a vertex u is the maximum distance of u to any other vertex of G.The distance degree sequence (dds) of a vertex u in a graph $G = (V, E)$ is a list of the number of vertices at distance 1, 2,. . . , $e(u)$ in that order, where $e(u)$ denotes the eccentricity of u in G. Thus the sequence $(d_{i_0}, d_{i_1}, d_{i_2}, \ldots, d_{i_j}, \ldots)$ is the dds of the vertex v_i in G where d_{i_j} denotes number of vertices at distance j from v_i. A graph is distance degree regular (DDR) graph if all vertices have the same dds. A graph is distance degree injective (DDI) graph if no two vertices have the same dds.

In this paper, we consider the construction of a DDR graph having any given graph G as its induced subgraph. Also we consider construction of some special class of DDI graphs.

Keywords: Distance degree sequence, Distance degree regular (DDR) graphs, Almost DDR graphs, Distance degree injective(DDI) graphs.

2010 MSC: 05C12.

1 Introduction

Unless mentioned otherwise for terminology and notation the reader may refer Buckley and Harary [6], new ones will be introduced as and when found necessary.

In this paper we consider simple undirected graphs without self-loops.

The distance $d(u, v)$ from a vertex u of G to a vertex v is the length of a shortest u to v path. The eccentricity $e(v)$ of v is the distance to a farthest vertex from v. If $dist(u, v) = e(u), (v \neq u)$, we say that v is an eccentric vertex of u.

The distance degree sequence (dds) of a vertex v in a graph $G = (V, E)$ is a list of the number of vertices at distance 1, 2,. . . ,$e(v)$ in that order, where $e(v)$ denotes the eccentricity of v in G. Thus, the sequence $(d_{i_0}, d_{i_1}, d_{i_2}, \ldots, d_{i_j}, \ldots)$ is the dds of the vertex v_i in G where, d_{i_j} denotes number of vertices at distance j from v_i. The concept of distance degree regular (DDR) graphs was introduced by G. S. Bloom et.al.[3], as the graphs for which all vertices have the same dds. For example, the three dimensional cube $Q_3 = K_2 \times K_2 \times K_2$,cycles,complete graphs are all DDR graphs. By definition it is clear that the DDR graphs must be regular but not conversely. The DDR graphs are studied by Bloom et.al [3], [4]. In [9] Halberstam et.al. have dealt the problem of path degree sequence and distance degree sequence using algorithms. All properties of cubic graphs up to a specified order are listed by Bussemaker et,al [7].The cubic graph generation is looked into by Brinkmann [5]. In [3], Bloom et.al have proved a result which states that "every regular graph with diameter at most two is DDR".This result shows that getting a DDR graph of higher diameter is challenging. In [12] Itagi Huilgol et.al. have listed all DDR graphs of diameter three with extremal degree regularity. But, the question of characterizing DDR graphs of diameter greater than two still remains open. In [12] Itagi Huilgol et.al. have shown the existence of a diameter three DDR graph of arbitrary regularity. In [13] Itagi Huilgol et.

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al. have constructed some more DDR graphs of higher diameter and considered the behavior of DDR graphs under other graph binary operations. In [14], Itagi Huilgol et.al.have constructed higher order DDR graphs by considering the simplest of the products viz.the cartesian and normal product. Another famous product is the lexicographic product of graphs. The lexicographic product is defined as follows. Given graphs G and H, the lexicographic product $G[H]$ has vertex set $\{(g, h) : g \in V(G), h \in V(H)\}$ and two vertices (g, h), (g', h') are adjacent if and only if either gg' is an edge of G or $g = g'$ and hh' is an edge of H.

The other extreme of DDR graphs is DDI graphs. The concept of DDI graphs was introduced in [4]. A graph G is said to be DDI graph if no two of its vertices have same distance degree sequence. In literature, in comparison to DDR graphs the number of DDI graphs is very less. So construction of new DDI graphs is also a challenging one. In [14], Itagi Huilgol et. al. have constructed higher order DDI graphs by using the products. A question was posed on the existence of r-regular DDI graphs by Bloom et. al. in [4]. In [9], Halberstam and Quintas showed the existence of a cubic DDI graph of diameter 10 and order 24. It was reduced to order 22 and diameter 8 by Martinez and Quintas in [15]. They also constructed a general cubic DDI graph with 22+2k points and diameter 8+k. It was further reduced to order 18 and diameter 7 by J. Volf in [19].

Characterization of graphs with a given property in terms of other properties is very common. A trend has been developed in characterizing the graphs with given property in terms of certain class of graphs which are not induced subgraphs of graphs with the property considered, that is, in terms of the " forbidden subgraphs". The first and foremost such characterization was given by Kuratowski [6] in case of planar graphs. From the definition it is clear that the study of planar graphs necessarily involves the topology of the plane. In general, the notion of embedding is extended to other surfaces too, viz, mobius band, torous. The problem gets interesting as we know that not all graphs can be embedded in the plane, or any other surface. In recent years, this type of characterization is considered as a "good characterization". Such a characterization has been used by many researchers. To quote a few Bieneke [6] in case of line graphs, Cook [8] for the graphs corresponding to $(0, 1)$- matrices, Berge [2] for perfect graph conjecture.

In this paper, we consider the embedding of a graph in a DDR graph and/or DDI graphs. As mentioned above, the DDR and DDI graphs are quite different. We relax a condition to introduce the concepts of almost DDR or ADDR, in short and almost DDI or ADDI in short. Here, we have also considered the embedding into ADDR and ADDI graphs.

2 Embeddings

DDR graphs exhibit high regularity in terms of the vertices and their distance distribution. If we relax for only one vertex to have different dds, then we can call the graph to be almost DDR, or in short ADDR. Similarly, we can define almost DDI graphs or ADDI in short.

Definition 2.1. *A graph G of order p is said to be almost DDR if $p-1$ vertices have same dds and one vertex with different dds.*

Definition 2.2. *A graph G of order p is said to be almost DDI if $p-2$ vertices have different dds and two vertices with same dds.*

In [17], Nandakumar et. al have proved that " For each vertex u with $e(u) > r(G)$, one of its neighbors v satisfies $e(v) = e(u) - 1$", which we are using to prove the following result.

Theorem 2.1. *If G is almost DDR, then $r(G) \leq diam(G) \leq r(G) + 1$.*

Proof. Let G be almost DDR. The left hand inequality follows from the definition of radius and diameter. Suppose on contrary, if $diam(G) \geq r(G) + 2$. Let u be a vertex with $e(u) = r(G) + 2$. From Nandakumar [17], there exists a vertex v adjacent to u with $e(v) = e(u) - 1$. Hence there exist three vertices having distinct eccentricities, a contradiction. Hence $diam(G) \leq r(G) + 1$. □

Remark 2.1. *Let G be a DDI graph having a vertex v such that $|d_2(v) - d_2(v_i)| \neq 1$, for all $v_i \in V(G)$ then adding a vertex u and making it adjacent with all the neighbors of v we get an almost DDI graph.*

Proposition 2.1. *Any path can be embedded in an almost DDI graph.*

Proof. Consider a DDI graph G as in [4] on $p+1$ vertices having path on p vertices as its induced subgraph as shown in Figure(1) below. Now adding an edge $(v,3)$ in G, we obtain an almost DDI graph in which $dds(u) = dds(v)$ as shown in the Figure(2). □

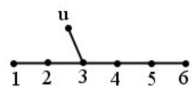

Figure 1: DDI graph on p+1 vertices

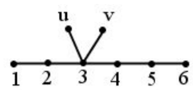

Figure 2: Embedding of P_p in an almost DDI graph

Theorem 2.2. *Any graph can be embedded in a DDR graph.*

Proof. First we prove that any regular graph can be embedded in a DDR graph. Let G be any regular graph of order k, the generalized lexicographic product $C_p[G,G,G,\ldots,G]$, is a DDR graph with diameter $\lfloor \frac{p}{2} \rfloor$ and having the dds of each vertex as $dds(v) = (1, 2k+r, 3k-r-1, 2k, 2k, \ldots, 2k)$ if p is odd and $dds(v) = (1, 2k+r, 3k-r-1, 2k, 2k, \ldots, k)$ if p is even. It is clear that G is an induced subgraph of $C_p[G,G,G,\ldots,G]$. Hence, any regular graph can be embedded in a DDR graph.
Next, we prove that any non-regular graph G can be embedded in a regular graph of regularity $\Delta(G)$, where $\Delta(G)$ is the maximum degree of G. Then by embedding it in a DDR graph we achieve the result.
Let G be any non regular graph. Let $t = \sum_{i=1}^{l}(\Delta - deg(v_i))$, where Δ and l are the maximum degree and order of G, respectively. Here two cases arise,
Case(i): $t = n\Delta$, for some $n \geq 1$,
Case(ii): $t = n\Delta + s$, $s < \Delta$, $n \geq 1$
Case(i): If $t = n\Delta$, consider $\overline{K}_n$ and add all $n\Delta$ edges such that (i) every edge has one end in $\overline{K}_n$ and the other end in G, (ii) every vertex of $\overline{K}_n$ receives exactly Δ edges, (iii) degree of every vertex in G becomes Δ. The resulting graph G' is a regular graph having G as its induced subgraph.
Case(ii): $t = n\Delta + s$, $s < \Delta$. Here we need to consider four subcases,
Case(a): s even and Δ even. Consider $\overline{K}_{n+s}$. Let S_1 and S_2 be the partition of $n+s$ vertices, such that $|S_1| = n$ and $|S_2| = s$. Add $n\Delta$ edges such that (i) every edge has one end in S_1 and the other end in G, (ii) every vertex of S_1 receives exactly Δ edges and add the remaining s edges such that (i) every edge has one end in S_2 and the other end in G, (ii) every vertex of S_2 receives exactly 1 edge. To make the vertices of S_2, Δ - regular we need exactly $\Delta - 1$ edges incident to each vertex of S_2. For this, take a complete graph $K_{\Delta+1}$ on $\Delta + 1$ vertices. Now we add the edges between S_2 and $K_{\Delta+1}$ preserving the regularity of $K_{\Delta+1}$. Suppose v and w are any two vertices in S_2, remove an edge (u_i, u_j) from $K_{\Delta+1}$ and add two edges (v, u_i) and (v, u_j) continuing the process of removing and adding the edges, we can make the degree of v equal to $\Delta - 1$ as Δ is odd. We have to add one more edge e_v(say) to v so that degree of v becomes Δ. Adding the edges to w as above, we can make the degree of w equal to $\Delta - 1$. We have to add one more edge e_w(say) to w so that degree of w becomes Δ, for that remove an edge (u_i', u_j') and add the edges (v, u_i') and (v, u_j'). In this way we can make degree of every vertex in S_2 equal to Δ as $|S_2|$ is even.

Similarly we can do it for the other cases given as below
Case(b): s odd and Δ even. This case is similar to Case(a) when we replace s by $s-1$. Since $\Delta - 1$ is odd, there exists a vertex u_1 in S_2 with degree $\Delta - 1$ after removing and adding the edges as in above case. To make the degree of u_1 equal to Δ, take an isomorphic copy(G_2') of the above resulting graph(G_1') and make u_1 adjacent with its mirror image u_1' in G_2'.

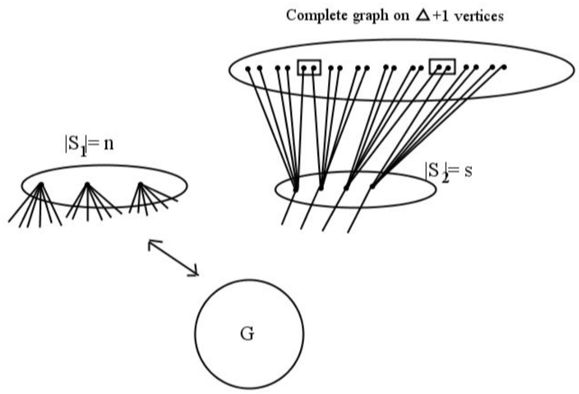

Figure 3: Case(a)

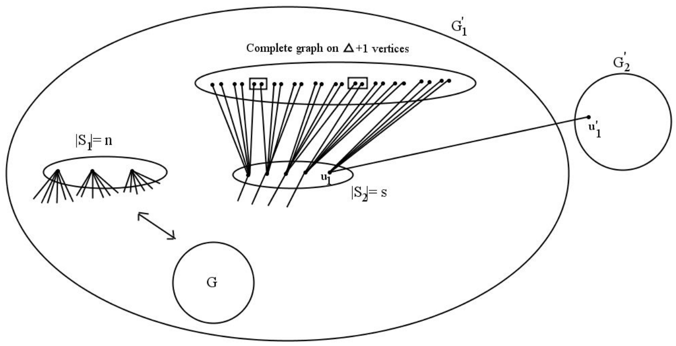

Figure 4: Case(b)

Case(c): s odd and Δ odd. As in case(a), we can make degree of every vertex of S_1 equal to Δ and since $\Delta-1$ is even, it is possible to add $\Delta-1$ edges to each vertex of S_2 by removing $\frac{\Delta-1}{2}$ number of edges from $K_{\Delta+1}$.

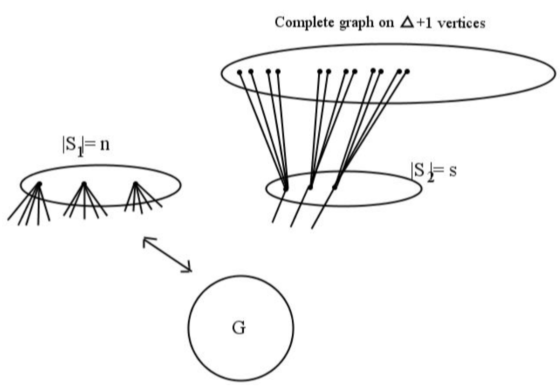

Figure 5: Case(c)

Case(d): s even and Δ odd. Proof of this case is similar to the proof of Case(c). Hence the proof. $\square$

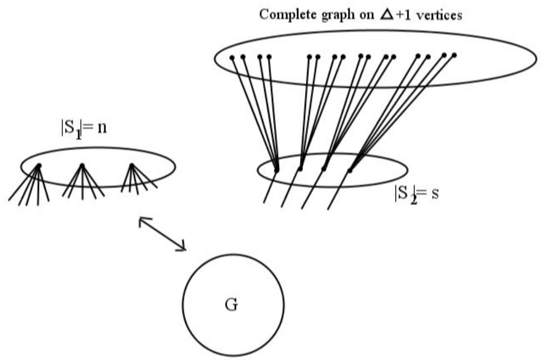

Figure 6: Case(d)

Theorem 2.3. *Any connected/disconnected graph G can be embedded in an almost DDR graph.*

Proof. First we prove any regular graph can be embedded in an almost DDR graph. Let G be any regular graph of regularity r. Add a vertex v to G and make it adjacent to all the vertices of G. The resulting graph is an almost DDR graph with the dds of all p vertices of G as $(1, r+1, p-r-1)$ and $dds(v) = (1, p)$ and having G as its induced subgraph. We know from the above theorem that any graph can be embedded in a regular graph. Hence any graph G can be embedded in an almost DDR graph. □

Ex:

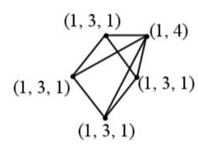

Figure 7: Embedding of C_4 in an almost DDR graph

Theorem 2.4. *Every cycle can be embedded in a DDI graph.*

Proof. Let the vertices of a cycle C_p be labeled as $1, 2, 3, \ldots, p$ and $P_1, P_2, P_3, \ldots, P_p$ be paths of lengths $1, 2, 3, \ldots, p$, respectively. Concatinating a pendent vertex of each P_i with a vertex i on C_p, the resulting graph G is shown to be a DDI graph. Now we prove this by showing no two vertices have same dds. Here three cases arise, Case(i): No two vertices on each path will have same dds.
Case(ii): No two vertices on the cycle C_p will have same dds.
Case(iii): No two vertices from two different paths will have same dds.
Case(i): Eccentricity of every vertex on a path is different as eccentricity increases by one as we move one step towards the pendent vertex of that path. Hence no two vertices on each path will have same dds.
Case(ii): Number of vertices at distance i from a vertex i, where $2 \leq i \leq p$ is always greater than the number of vertices at distance $i, 2 \leq i \leq p$ from a vertex $j, 1 \leq j \leq i-1$. Hence no two vertices on the cycle C_p have same dds.
Case(iii): No two vertices from two different paths have same dds as these vertices lie on the paths having different lengths. □

Note: Adding a pendent vertex at p^{p-1} in above DDI graph, we get an almost DDI graph.

Lemma 2.1. *If a graph G containing two vertices u and v which are the only eccentric vertices of each other with eccentricities equal to three and $deg(u) = deg(v)$ then G is non DDI.*

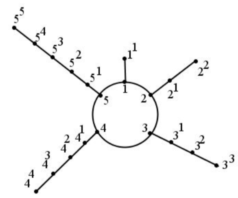

Figure 8: Embedding of C_p in a DDI graph

Proof. Let G be a graph containing two vertices u and v with eccentricities equal to three such that u is the only eccentric vertex of v and v is the only eccentric vertex of u and $deg(u) = deg(v) = k$. Let S_1, S_2 be vertex sets at distance one and two respectively from u. Every vertex in S_1 is adjacent to at least one vertex of S_2 otherwise, the eccentricity of v will be more than three, a contradiction. Now for S_2 two cases arise.
Case(i): $|S_2| = k$ and Case(ii): $|S_2| > k$.
Case(i): $|S_2| = k$, clearly $dds(u) = (1, k, k, 1) = dds(v)$.
Case(ii): $|S_2| > k$. Let $|S_2| - k = t$ be the number of vertices non adjacent to v. Here consider two cases,
Case(a): Suppose there is a vertex w in $S_2 - N(v)$ non adjacent with any of the vertices in $N(v)$, then w is at distance three from v, a contradiction. Hence this case is excluded.
Case(b): Suppose every vertex in $S_2 - N(v)$ is adjacent to at least one vertex in $N(v)$, then every vertex in $S_2 - N(v)$ is at distance two from both u and v and hence $dds(u) = (1, k, k + |S_2 - N(v)|, 1) = dds(v)$. Hence G is non DDI. □

Lemma 2.2. *There exists no regular self centered DDI graph of diameter three.*

Proof. Let G be a regular self centered graph of radius three. The dds of any vertex is given by $dds(v) = (1, k, d_2, d_3)$. Since the graph G is self centered, d_2 and d_3 will satisfy the following $d_2 + d_3 = p - k - 1$, $1 \leq d_2 \leq p - k - 2$ and $1 \leq d_3 \leq p - k - 2$. Therefore the number of vertices having distinct dds is atmost $p - k - 2$. Any set containing at least $p - k - 1$ contains at least two vertices having same dds. Hence G is non DDI. □

The u.e.n graphs are defined by Nandakumar et. al. as follows:
Definition[17]:A graph G is said to be unique eccentric node (u.e.n) graph if every vertex has a unique eccentric vertex.

Corollary 2.1. *There exist no regular u.e.n. DDI graph with diameter three.*

Proof. Let G be a regular u.e.n. graph with diameter three. There exist two vertices u and v at distance three from each other. Let $dds(u) = (1, k, d_2, 1)$ and $dds(v) = (1, k, d_2', 1)$. Comparing the dds of u and v, we get $d_2 = d_2'$. Hence $dds(u) = dds(v)$, implying G is not DDI. □

Corollary 2.2. *There exists no regular self centered u.e.n. DDI graph with diameter four.*

Proof. Let G be a regular self centered u.e.n. graph of diameter four. The dds of any vertex is given by $dds(v) = (1, k, d_2, d_3, 1)$. It is clear from lemma[2.2] that there exist at least two vertices having same dds. Hence G is non DDI. □

Note: Combining the above two results we can say "There exists no regular u.e.n. DDI graph with diameter atmost four".

Lemma 2.3. *If G is a graph with radius two containing at least two central vertices having same degree, then G is non DDI.*

Proof. Let G be a graph with radius two containing at least two central vertices u and v having same degree, then their dds are given by $dds(u) = (1, k, p - k - 1)$ and $dds(v) = (1, k, p - k - 1)$, i.e., $dds(u) = dds(v)$. Hence G is non DDI. □

Remark 2.2. *If a regular self centered graph G with radius three has at least two vertices having same number of vertices at distance two or three then G is non DDI.*

Lemma 2.4. *There exists no regular self centered DDI graph whose complement is also DDI.*

Proof. In [11], If $d(G) \geq 3$, then $d(\overline{G}) \leq 3$, but in lemma[2.2] we have proved that there exists no regular self centered DDI graph of radius three and also it is clear that there exists no self centered DDI graph of radius two. Hence there exists no regular self centered DDI graph whose complement is also DDI. □

Combining the results in [4], "If both G and $\overline{G}$ are DDI then both G and $\overline{G}$ are of diameter three" and in [6], "If G is regular with diameter 3, then $d(\overline{G}) = 2$", we have the following remark.

Remark 2.3. *There exists no regular DDI graph whose complement is also DDI.*

Remark 2.4. *A graph G having two vertices u and v such that $N(u) \cap N(v) = N(u) = N(v)$ is non DDI, where $N(u)$ and $N(v)$ are sets of vertices adjacent to u and v respectively.*

Lemma 2.5. *There exists a u.e.n. DDI graph having diameter $d = 2n + 1$,where $n \geq 3$*

Proof. Let $P_p : 1, 2, 3, \ldots, p-2, p-1, p(\geq 6)$ be a path on p vertices and $P_{p-2}, P_{p-3}, P_{p-4}, \ldots, P_2$ be paths on $p-2, p-3, p-4, \ldots, 2$ respectively. The graph obtained by concatination of a pendant vertex of each path P_{p-i} with a vertex i, where $2 \leq i \leq p-2$ on the above said path P_p is a u.e.n. DDI graph having diameter $d = 2n + 1$,where $n \geq 3$. □

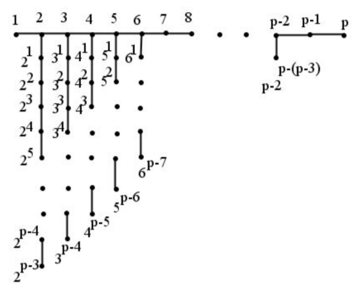

Figure 9: A u.e.n. DDI graph having diameter $d = 2n + 1$,where $n \geq 3$

Lemma 2.6. *There are at least $p - 5$ non-isomorphic DDI graphs of order p,where $p \geq 7$.*

Proof. Let $v_1, v_2, v_3, \ldots, v_{p-1}$ be a path on $p-1$ vertices and v_p be a vertex to be made adjacent to a vertex of above said path on $p-1$ vertices such that resulting graph is DDI. Amongst $p-1$ points on the path(of length $p-2$) we can not join the vertex to the end vertices v_1 and v_{p-1}, otherwise, the induced graph would be a path and hence is not DDI. We also can not join the vertex to either v_2 or v_{p-2}, since v_1 and v_p or v_{p-2} and v_p would have the same dds, contradicting to the fact that G is DDI. Now, if the path induced by $p-1$ vertices is of odd length, then we can join a vertex at any of the vertices $v_3, v_4, \ldots, v_{p-3}$, without any repetition of dds. Hence we can join a vertex at $p-5$ vertices to get different DDI graphs. If the path induced by $p-1$ vertices is of even length then, we can join v_p at the vertices $v_3, v_4, \ldots, v_{\frac{p-1}{2}-1}, v_{\frac{p-1}{2}+1}, \ldots, v_{p-3}$; i.e., except at the central vertex of the path. So we can join at $p-6$ vertices to get different DDI graphs. □

We conclude this paper with a couple of open problems.
<u>**Problem 1 :**</u> Characterize DDR graphs of diameter higher than 3.
<u>**Problem 2 :**</u> Can any graph be embedded in a DDI graph?
<u>**Problem 3 :**</u> Does there exist DDI r-regular graph for $r \geq 4$?

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