Int. J. of Mathematical Sciences and Applications,
Vol. 1, No. 3, September 2011

www.journalshub.com

RADIUS-VITAL EDGES IN A GRAPH

Medha Itagi Huilgol
Department of Mathematics, Bangalore University,
Central College Campus, Bangalore-560001
medha@bub.ernet.in

Abstract:

The graph resulting from contracting edge "e" is denoted as G/e and the graph resulting from deleting edge "e" is denoted as G-e. An edge "e" is radius-essential if rad(G/e) < rad(G), radius-increasing if rad(G-e)>rad(G), and radius-vital if it is both radius-essential and radius-increasing. We partition the edges that are not radius-vital into three categories. In this paper, we study realizability questions relating to the number of edges that are not radius-vital in the three defined categories. A graph is radius-vital if all its edges are radius-vital. We give a structural characterization of radius-vital graphs.

Keywords: Radius-vital edges, radius-increasing edges, radius-essential edges

AMS Classification 2010: 05C12

1.Introduction:

The terminology used throughout this paper is based on Buckley and Harary [1], Harary[3].

Let G be a connected graph with *vertex set V(G)* and *edge set E(G)* with *p* and *q* representing *order* and *size* of G. The *distance d(u,v)* between vertices *u* and *v* is the *length of a shortest path* joining u and v. The *eccentricity e(v)* of *v* is the *distance to a farthest vertex from v*. The radius *rad(G) and diameter diam(G)* are *minimum and maximum eccentricities*, respectively. The *center C(G) and P(G), periphery* of a graph G consists of the sets of vertices of *minimum and maximum eccentricity*, respectively. Vertices in *C(G)* are called *central vertices* and those in *P(G)* are called the *peripheral vertices*. An *elementary contraction* of an *edge e=uv* in G is obtained by *removing u and v, inserting a new vertex w and inserting an edge between w and any vertex to which either u or v (or both) were adjacent* and *G/e* denotes the resulting graph. The graph resulting from deleting edge e is denoted by *G-e*.

As in [1], the *sequential join* $G_1+G_2+G_3+\dots\dots+G_k$ of graphs $G_1,G_2,\dots.G_k$ is the graph formed by taking one copy of each of the graphs $G_1,G_2,\dots..G_k$ and adding in additional edges from each vertex of G_i to each vertex in G_{i+1}, for $1\leq i \leq k-1$.

An edge *e* is *radius-essential* if *rad(G/e) < rad(G)* and *radius-increasing* if

rad(G-e) > rad(G). We studied *radius-essential edges* in [9]. If every edge in a graph *G* is *radius-increasing*, then G is a *radius-minimal graph*. Gliviak [2] established various existence resutls for radius-minimal graphs.
Definition 1.1: An *edge e* is *radius-vital* if it is both *radius-essential* and *radius-increasing*; otherwise, it is *radius-non-vital*.

Thus, a radius-vital edge *e* has the property that contracting *e* decreases the radius and deleting *e* increases the radius.

An edge *e* is *deletable* if its deletion does not alter the radius, that is, *rad(G-e)= rad(G).* (Gliviak [2] refers such edges *superflous*). An edge *e* is *contractible* if its contraction does not alter the radius, that is, *rad(G/e)=rad(G)*. In view of these definitions, we can partition *E(G)* into four sets:

***radius-vital edges**: $E_v(G)=\{e: rad(G/e) < rad(G) \text{ and } rad(G\text{-}e) > rad(G)\}$,*

***contracible, radius-increasing edges:** $E_c(G)=\{e: rad(G/e)=rad(G) \text{ and } rad(G\text{-}e)>rad(G)\}$,*

***deletable, radius-essential edges:** $E_d(G)=\{e: rad(G\text{-}e)=rad(G) \text{ and } rad(G/e)<rad(G)\}$,*

and

***contractible and deletable edges:** $E_{cd}(G)=\{e: rad(G\text{-}e)= rad(G) \text{ and } rad(G/e)= rad(G)\}$.*

An edge *e is radius-non-vital(non-vital)*, if $e \in E_c(G) \cup E_d(G) \cup E_{cd}(G)$.

In this paper, we shall study the vital and non-vital edges of graphs. After characterizing graphs for which every edge is vital, we examine realizability questions relating to the sizes of the sets $E_c(G)$, $E_d(G)$ and $E_{cd}(G)$ and study which triples (x, y ,z) of integers are realizable for $(|E_c(G)|,|E_d(G)|,|E_{cd}(G)|)$.

We mention that a similar study was done for 3-connectedness in graphs. Reid and Wu[6] studied edges *"e"* in 3-connected graphs for which either deletion of *"e"* or the contraction of *"e"*, but not both, alters the 3-connectedness of the graph.

Definition 1.2: *A graph G is radius-vital if all its edges are radius-vital.*

We recall some results from Walikar, Buckley and Itagi[9]. Let $\sigma_r(G)$ be the number of essential edges in G. That is,

$\sigma_r(G)=|\{e \in E(G): rad(G/e) < rad(G)\}|$.

Since an essential edge is not contractible, $\sigma_r(G) =|E_v(G)|+|E_d(G)|$.

Let *p* and *q* denote the number of vertices and edges, respectively, in G. We shall need the following.

Proposition 1.3[2]: *A non-trivial graph is radius minimal if and only if G is a tree.*

Proposition 1.4[9] :*For a tree T , $\sigma_r(G)=q$, if and only if T is a path on even number of vertices.*

2.Results:

The following result characterizes *radius-vital graphs*.

Proposition 2.1: *Let G be a graph with rad(G)=r. Then G is radius-vital if and only if G is a path on even number of vertices.*

Proof: A non-trivial graph G is radius-minimal if and only if G is a tree, by Proposition 1.1[2]. By Proposition 1.2[9], $\sigma_r(T)=q$ if and only if T is a path on even number of vertices. Combining the two results the proof follows.■

We now focus on the non-vital edges of a graph. We begin with a definition and several preliminary observations.

Definition 2.2: *For any three non-negative integers x ,y, z, a graph G is said to be an (x, y, z)-graph, if $|E_c(G)|= x$, $|E_d(G)|= y$ and $|E_{cd}(G)|= z$, and the triple (x, y, z) is realizable if there exists an (x, y, z)-graph G.*

By *Proposition 2.1*, it is clear that only (0, 0, 0) graphs are paths on even number of vertices.

Remark 2.3: *If $\sigma_r(G)=0$,then all edges are contractible no matter whether they are deletable or not. Hence G contains no vital edges. Therefore, if G is an (x, y, z)- graph with $\sigma_r(G)=0$, we have y = 0 and x + z= q.*

A graph G is a radius-edge-invariant graph (r.e.i. graph) if for each $e\in E(G)$, rad(G-e) = rad(G), that is, every edge of G is deletable. Refer Walikar, Buckley and Itagi [8] for detailed study of these graphs.

Remark 2.4: *If $\sigma_r(G)= q$, then no edge of G is contractible. Hence x = z = 0 and $y \le q$. If y = q, then G is radius-edge-invariant, otherwise, there exists at least one vital edge in G.*

Remark 2.5: *If G is radius-edge-invariant graph then every edge is deletable, so there are no vital edges in G. Thus, for a r.e.i. graph , x = 0 and y + z = q.*

Remark 2.6: *If G is radius-minimal then no edge is deletable. Hence y = z = 0 and $x \le q$. Thus for a diameter minimal graph G, if $\sigma_r(G) = 0$,then x = q and if $\sigma_r(G) > 0$, there exists at least one vital edge.*

Next we consider realizability of triple of integers.

Lemma 2.7: *The triple (0,1,0) is not realizable*.

Proof: On the contrary assume that *(0,1 ,0)* is realizabe. Then there exists a graph *G*, containing only one edge, say $e\in E_d(G)$. Then (i) *rad(G-e) = rad(G)*and (ii) *rad(G/e) < rad(G)* hold. And all other edges *e′* in *G* are vital, hence (i′) *rad(G-e′) > rad(G)* and (ii′) *rad(G/e′) < rad(G)*. From (ii) and (ii′) it follows that $\sigma_r(G)= q$. Hence, for *e*, there exists a radius-preserving spanning tree which avoids *e*. But this edge can be contracted too without altering the radius of *G*, a contradiction to the fact that $\sigma_r(G) = q$.■

Lemma 2.8: *The triple (x, 0, 0) is realizable, for all $x \ge 0$.*

Proof: If *x = 0*, then by *Proposition 2.1*, *G* is a path on even number of vertices. For $x \ge 1$, consider a graph *G*, obtained by joining *x* pendent edges to any one of the central vertices of path $P_{2n,}\ n \ge 3$.Clearly, this graph has *x* edges belonging to $E_c(G)$ and rest all vital. Hence the result.■

Lemma 2.9: *The triple (x,1,0) is not realizable for all values of $x \ge 0$.*

Proof: On the contrary assume that the triple is realizable. Hence there exists a graph *G* containing one edge, say *e*, $e \in E_d(G)$ and *"x"* edges belonging to $E_c(G)$, that is, $\sigma_r(G) = q-x$. For *e*, there exists a radius-preserving spanning tree, which does not contain *e*, as *rad(G-e) = rad(G)*. By contraction of this edge radius remains unaltered contradicting the fact that the edge $e \in E_d(G)$, proving the result.■

Lemma 2.10: *The triple (0,y,0) is realizable for* $y = 2m, m \geq 2$, *or* $y = 2k + mn, k \geq 2, m \geq 2, n \geq 1$ *or* $y = mn, m \geq 2, n \geq 1$.

Proof: To show that $(0,y,0)$ is realizable, it is sufficient to show the existence of a graph for values given in the hypothesis. For $y = 0$, the realizability follows from *Proposition 2.1*. It is clear that an edge $e \in E_d(G)$ edge lies on a block of G. Since $x = z = 0$, all other edges of G must be vital. For different values of *"y"*, we have different structure of blocks containing *"y"* edges. If y is even i.e. $y \geq 2m, m \geq 2$, consider a graph $G_1 = K_1 + \overline{K_m} + K_1$, $m \geq 2$, as in Figure 1.

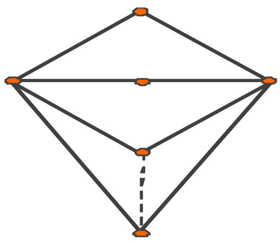

Figure 1

For the graph of Figure 2, all $2m$ edges belong to $E_d(G)$. Hence G_1 is a $(0, y, 0)$ graph.

For $y = mn, m \geq 2, n \geq 1$, consider a graph $G_2 = K_1 + \overline{K_m} \Box F \Box \overline{K_m} + K_1$, where F denotes the one factor between $\overline{K_m}$ *and* $\overline{K_m}$ as in Figure 2.

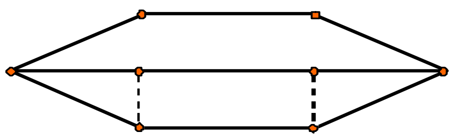

Figure 2

G_2 is an $(0, y, 0)$ graph as all $3m$ edges are deletable, radius-increasing.

Next consider a graph $G_3 = K_1 + \overline{K_m} \Box F \Box \overline{K_m} \Box F \Box\Box\Box F \Box \overline{K}_m + K_1$, where F denotes one factor between two consecutive $\overline{K_m}$'s, as in Figure 3.

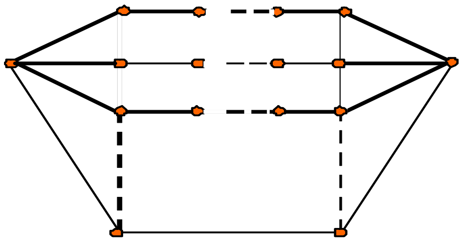

Figure 3

Clearly G_3 is an (0, y, 0)-graph for $y = mn$, $m \geq 2$, $n \geq 2$, as all *mn* edges of G_3 belong to $E_d(G)$. Hence any combination of the above discussed values of *"y"* can be realized for *(0, y, 0)*. So the realizing graph will be as shown in Figure 4.

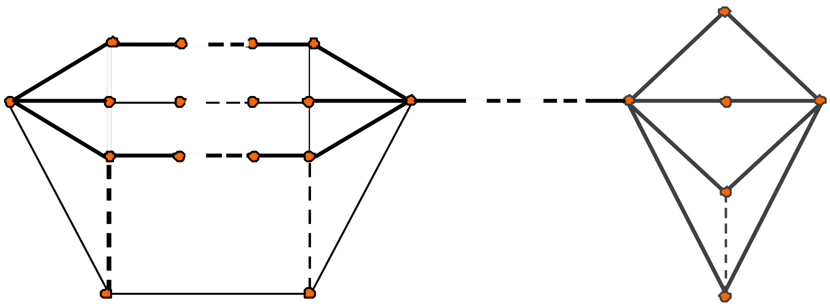

Figure 4. ■

Lemma 2.11: *The triple (x, y, 0) is realizable for* $x \geq 0$; $y = 2m$, *or* $y = 2k + mn$, $k \geq 2$, $m \geq 2$, $n \geq 1$, *or* $y = mn$, $m \geq 2$, $n \geq 1$.

Proof: Consider the graph *G* of *Figure 1*. Join *"x"* pendent edges at any one of the vertices of degree *m*, to get an $G_1'=(x, y, 0)$- graph for $y \geq 2m$, $m \geq 2$. Clearly, contraction of these pendent *"x"* edges does not alter radius of G_1'. Similarly, to each of G_2, G_3, G_4 of above Lemma 2.10, we can join *"x"* pendent edges at any vertex whose degree is not equal to two, to get graphs G_2', G_3', G_4' which are *(x,y,0)* graphs for different values of *"y"*. We note that G_2' is *(x,3m,0)*-graph, G_3' is *(x,mn,0)*-graph and G_4' is *(x,2k+mn,0)*- graph.■

Lemma 2.12: *The triple (0,0,1) is not realizable.*

Proof: Suppose, *(0,0 ,1)* is realizable, let *G* be the realizing graph. In *G*, let *"e"* be the only edge such that $rad(G\text{-}e) = rad(G) = rad(G/e)$. *G* cannot contain only one edge as K_2 is neither deletable nor contractible. Hence all other edges of *G* must be vital. Since, $rad(G/e) = rad(G)$, for some central vertex, *say u*, there are at least two eccentric vertices say u_1 and u_2, joined by disjoint paths. Hence if *"e"* lies on any one path, say $u\text{-}u_1$ path, then any other edge of $u\text{-}u_2$ path can also be contracted without altering the radius of *G*. This contradicts the fact that *G* contains only one radius-vital edge and hence the result.■

Lemma 2.13: *The triple (0, 0, z) is realizable except for z = 1.*

Proof: From *Proposition 2.1, (0, 0, 0)* is realizable. From above lemma, *(0, 0, 1)* is not realizable. For $z \geq 2$, consider a graph *G* obtained by identification of each end vertex of a path P_n with each one central vertex of a path P_{2n-4}. The graph so obtained is as in Figure 5.

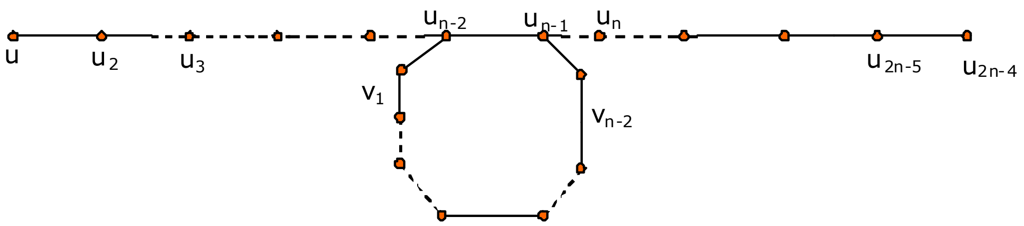

Figure 5

Label the vertices of G as in Figure 5. So $rad(G) = n-2$. Clearly, the edges of the form u_iu_{i+1}, $1 \leq i \leq 2n-4$, are vital. Edges of the form v_iv_{i+1}, v_1u_{n-2}, $v_{n-2}u_{n-1}$, belong to $E_{cd}(G)$ and there is no edge belonging to $E_c(G)$ and $E_d(G)$. Hence by taking $z=n-1$, the triple $(0, 0, z)$ is realizable, for $z \geq 2$.■

<u>Lemma 2.14:</u> *The triple (x, 0, z) is realizable for all* $x \geq 0, z \geq 0$.

<u>Proof</u>: For $x = z = 0$, the realizability of $(x, 0, z)$-graph is ensured by *Proposition 2.1*. For $x = 0$, the graph constructed in the above lemma serves the purpose for $z \geq 2$. For $x \geq 1$, consider a $(0, 0, z)$-graph constructed in above lemma. Join *"x"* pendent edges at either u_{n-2} or u_{n-1} of G of *Figure 5*. Clearly these *"x"* edges belong to $E_c(G)$ and there is no edge belonging to $E_d(G)$. Hence, $(x ,0, z)$ is realizable, $x \geq 0$, $z \geq 2$. For $z = 1$, the graph of the Figure 6 is the realizer.

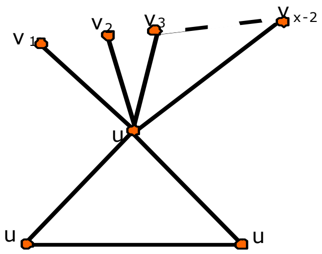

Figure 6.

Clearly, the edge u_1u_2 is the only edge of $E_{cd}(G)$ and rest all belong to $E_c(G)$. Hence this is $(x, 0, 1)$ -graph.■

<u>Theorem 2.15:</u> *The triple (x, y, z) is realizable if* $x \geq 0$; $y = 2m$, $m \geq 2$, *or* $y = 2k + mn$, $k \geq 2$, $m \geq 2$, $n \geq 1$, *or* $y = mn$, $m \geq 2$, $n \geq 1$; $z \neq 1$.

<u>Proof:</u> Proof follows from *Lemma 2.1 to Lemma 2.14*. The realizing graph G is as in Figure 7.

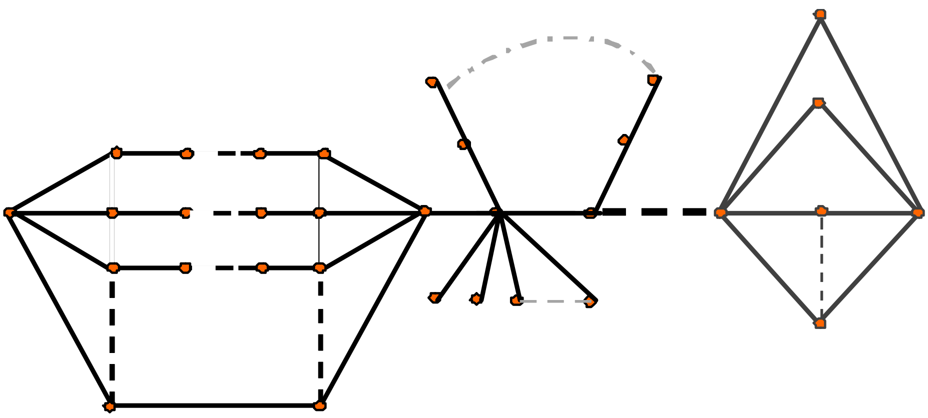

Figure 7.

Clearly, edges of the form $u\, u_i^1$, $u_i^j\, u_{i+1}^j$, $u_i^n v_1$, $1 \le i \le m$, $1 \le j \le n$; $v_{2n-4}x_i$, xx_i, $1 \le i \le 1$ belong to $E_d(G)$. Edges of the form $v_{n-2}y_i$, $1 \le i \le x$, belong to $E_c(G)$ and edges of the form $v_{n-2}w_1$, $v_{n-1}w_{n-2}$, $w_i w_{i+1}$; $2 \le i \le n-3$, belong to $E_{cd}(G)$, rest all edges are vital. Hence G is an (x, y, z)-graph for the values given in the hypothesis. ■

REFERENCES:

1. F. Buckley and F. Harary, "Distance in Graphs*", Addison -- Wesley Publishing Co*., (1990).
2. F. Gliviak, "On radially critical graphs", in *Recent advances in Graph Theory, Proc. Sympos, Prague, Academia Praha, Prague* (1975) 207-221.
3. F. Harary, "Graph Theory", *Addison-Wesley,* Reading Mass. (1969).
4. J.G. Oxley and H. Wu, "The 3-connected graphs with three non-essential edges". (Preprint).
5. W.T. Tutte, " A theory of 3 - connected graphs", *Nederl. Akad. Wetensch. Proc.Ser.A.*64, 441-455 (1961).
6. T.J. Reid and Haidong Wu, "On Non-Essential Edges in 3 -- connected Graphs", *Graphs and Combinatorics* (2000) 16 : 337 -- 354.
7. H.B.Walikar, Fred Buckley, M.K.Itagi, " Diameter-essential edges in a graph ", *Discrete Mathematics*, 259(2002) 211-225.
8. H.B.Walikar, Fred Buckley, M.K.Itagi, "Radius-edge-invariant and diameter-edge-invariant graphs", *Discrete Mathematics*, 272 (2003) 119-126.
9. H.B.Walikar, Fred Buckley, M.K.Itagi, " Radius-essential edges in a graph ", *Journal of Combinatorial Mathematics and Combinatorial Computing*, 53 (2005) 209-220.