

Majorana representation of symmetric multiqubit states

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Abstract As early as 1932, Majorana had proposed that a pure permutation symmetric state of N spin- $\frac{1}{2}$ particles can be represented by N spinors, which correspond geometrically to N points on the Bloch sphere. Several decades after its conception, the Majorana representation has recently attracted a great deal of attention in connection with multiparticle entanglement. A novel use of this representation led to the classification of entanglement families of permutation symmetric qubits—based on the number of distinct spinors and their arrangement in constituting the multiqubit state. An elegant approach to explore how correlation information of the *whole* pure symmetric state gets imprinted in its *parts* is developed for specific entanglement classes of symmetric states. Moreover, an elegant and simplified method to evaluate geometric measure of entanglement in N -qubit states obeying exchange symmetry has been developed based on the distribution of the constituent Majorana spinors over the unit sphere. Multiparticle entanglement being a key resource in several quantum information processing tasks, its deeper understanding is essential. In this review, we present a detailed description of the Majorana representation of pure symmetric states and its applicability in investigating various aspects of multiparticle entanglement.

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1 Introduction

Permutation symmetric multiqubit states, which remain invariant under exchange of any pair of particles, form an important class among quantum states due to their experimental significance and mathematical elegance [1–5]. The class of symmetric states comprises of the well-known Greenberger–Horne–Zeilinger (GHZ) [6], W, and Dicke states [7] etc. Mathematical simplicity in addressing N -qubit states obeying permutation symmetry results because the states are confined to the $N + 1$ dimensional subspace of the 2^N dimensional Hilbert space. The $N + 1$ dimensional subspace is spanned by the Dicke states, $\{|N/2, N/2 - l\rangle, l = 0, 1, 2, \dots, N\}$, which are the simultaneous eigenstates of the squared collective angular momentum operator J^2 and its z -component J_z . An elegant geometrical representation for multiqubit symmetric states in terms of N -points on the Bloch sphere S^2 was proposed by Majorana [8] as early as 1932. The representation of multiqubit states based on their characteristic N -qubits (spinors), the so-called *Majorana representation* (MR) for symmetric states [8–11] has been immensely useful in diverse branches of physics [9, 12–15] in general and in quantum information science [16–23] in particular. The significance of MR in characterizing entanglement in multiqubit symmetric states has been realized in recent years and the avenues appear to be expanding. While the SLOCC classification of symmetric states in terms of the distinct spinors characterizing the state has been accomplished using the MR [16, 17], the reducibility/irreducibility features of multiparty correlations in several important classes of states could well be captured [18, 19] using it. In fact, an ingenious use of this representation allows one to characterize how the multiparty correlation is imprinted in its parts for a class of non-symmetric states too, which is a generalized set of states related to a particular symmetric class [18]. Quantification of multiparty entanglement is another important aspect where the MR finds its applicability. Geometric measure of entanglement [24, 25]—a useful measure of entanglement for multiqubit pure states—has been realized to have a natural interpretation [20, 21] in terms of the arrangement of N -spinors on the Bloch sphere, as given by MR. Identifying maximally entangled symmetric states for each N has been possible utilizing this feature [20–23] and this has paved way for arriving at some novel results while analyzing highly entangled states in the so-called platonic solids [23]. This review aims at capturing the essence of MR and how it has been put to use towards the understanding of multiqubit entanglement.

The organization of the article is as under: Sect. 2 gives a detailed description of MR of symmetric multiqubit states, the Majorana spinors characterizing the state and their geometric representation. The SLOCC classification of symmetric states based on the number and arrangement of spinors constituting the state is given in Sect. 3. We discuss the “whole and its parts” issue in the context of some specific SLOCC class of symmetric states in Sect. 4. More specifically, we show that not all states that are interconvertible into one another through SLOCC operations exhibit the same reducibility/irreducibility of correlations, with the help of an explicit three qubit example. We also demonstrate that only two of the $N - 1$ qubit reduced density matrices uniquely

determine the family of N qubit pure symmetric states, comprised of two distinct Majorana spinors and also another related class of non-symmetric states. A brief review on geometric measure of entanglement and how the geometric representation of symmetric states given by Majorana leads to a quantification of their entanglement, characterized through geometric measure, is given in Sect. 5. Section 6 contains a brief summary.

2 Majorana representation

In his novel 1932 paper [8] (which had not received much attention at that time) Ettore Majorana proposed that a pure spin $j = \frac{N}{2}$ quantum state can essentially be represented as a *symmetrized* combination of N constituent spinors as follows:

$$|\Psi_{\text{sym}}\rangle = \mathcal{N} \sum_P \hat{P} \{|\epsilon_1, \epsilon_2, \dots, \epsilon_N\rangle\}, \quad (1)$$

where

$$|\epsilon_l\rangle = \cos(\beta_l/2) e^{-i\alpha_l/2} |0\rangle + \sin(\beta_l/2) e^{i\alpha_l/2} |1\rangle, \quad l = 0, 1, 2, \dots, N, \quad (2)$$

denote the spinors constituting the symmetric state $|\Psi_{\text{sym}}\rangle$; \hat{P} corresponds to the set of all $N!$ permutations of the spinors (qubits) and \mathcal{N} corresponds to an overall normalization factor.

For example, two and three qubit symmetric pure states have the following representations in terms of the Majorana spinors:

$$|\Psi_{\text{sym}}^{(2)}\rangle = \mathcal{N} [|\epsilon_1, \epsilon_2\rangle + |\epsilon_2, \epsilon_1\rangle] \quad (3)$$

$$|\Psi_{\text{sym}}^{(3)}\rangle = \mathcal{N} [|\epsilon_1, \epsilon_2, \epsilon_3\rangle + |\epsilon_3, \epsilon_1, \epsilon_2\rangle + |\epsilon_2, \epsilon_3, \epsilon_1\rangle \\ + |\epsilon_2, \epsilon_1, \epsilon_3\rangle + |\epsilon_3, \epsilon_2, \epsilon_1\rangle + |\epsilon_1, \epsilon_3, \epsilon_2\rangle]. \quad (4)$$

Equation (1) corresponds to the *Majorana representation* of an arbitrary symmetric state $|\Psi_{\text{sym}}\rangle$ of N qubits in terms of the constituent spinors $|\epsilon_l\rangle$, $l = 1, 2, \dots, N$.

On the other hand, states of N -qubits obeying exchange symmetry get restricted to a $(N + 1)$ dimensional Hilbert space spanned by the collective basis vectors $\{|\frac{N}{2}, l - \frac{N}{2}\rangle, l = 0, 1, 2, \dots, N\}$ where,

$$\left| \frac{N}{2}, l - \frac{N}{2} \right\rangle = \frac{1}{\sqrt{{}^N C_l}} [|\underbrace{0, 0, \dots}_l, \underbrace{1, 1, \dots}_{N-l}\rangle + \text{Permutations}] \quad (5)$$

are the $N + 1$ Dicke states—expressed in the standard qubit basis $|0\rangle$, $|1\rangle$ and ${}^N C_l = \frac{N!}{l!(N-l)!}$ denotes the binomial coefficient. An arbitrary pure symmetric state of N qubits obeying exchange symmetry may thus be expressed as,

$$|\Psi_{\text{sym}}\rangle = \sum_{l=0}^N c_l \left| \frac{N}{2}, l - \frac{N}{2} \right\rangle, \quad (6)$$

and is completely specified by the $(N + 1)$ complex coefficients c_l . Eliminating an overall phase and normalizing the state implies that N complex parameters are required to completely characterize a pure symmetric state (6) of N qubits.

While (6) offers a suitable parametrization of the symmetric multiqubit system in terms of the collective parameters c_l , the MR (1) leads to an intrinsic geometric picture of the system in terms of N -points on the unit sphere S^2 . (Note that the spinors $|\epsilon_l\rangle$, $l = 1, 2, \dots, N$ of (2) correspond geometrically to N points on the so-called Majorana sphere S^2 [11, 12, 20–23])—with the pair of angles (α_l, β_l) determining the orientation of each point on the sphere).

The equivalence between the parameters c_l of the collective representation (6) and that of the MR (1) can be established in an elegant manner [18] as detailed in the following.

1. A symmetric pure state is transformed into another symmetric pure state under identical rotations $R \otimes R \otimes \dots \otimes R$ on all the spinors of (1) (which corresponds to an equivalent collective rotation \mathcal{R} on the state (6) in the $(N + 1)$ dimensional symmetric sub-space).
2. Under identical rotation through $R^{-1}(\alpha_s, \beta_s, 0) \otimes R^{-1}(\alpha_s, \beta_s, 0) \otimes \dots$, where α_s, β_s correspond to the orientation of any one of the spinors in (1), it may be identified that

$$\langle 1, 1, \dots, 1 | R^{-1}(\alpha_s, \beta_s, 0) \otimes R^{-1}(\alpha_s, \beta_s, 0) \otimes \dots | \Psi_{\text{sym}} \rangle \equiv 0. \tag{7}$$

This is because the rotation $R_s^{-1} \otimes R_s^{-1} \dots \otimes R_s^{-1}$ takes one of the spinors $|\epsilon_s\rangle$ with orientation angles (α_s, β_s) to $|0\rangle$ i.e., it aligns the spinor $|\epsilon_s\rangle$ in the positive z -direction. Then, every term in the superposition (1) of the rotated state has at least one $|0\rangle$ and so, the projection $\langle 1, 1, \dots, 1 | R_s^{-1} \otimes R_s^{-1} \otimes \dots \otimes R_s^{-1} | \Psi_{\text{Sym}}^{(N)} \rangle$ of the rotated state in the ‘all-down’ direction vanishes.

3. Equation (7) holds good for collective rotations $\mathcal{R}_s^{-1} = R_s^{-1} \otimes R_s^{-1} \otimes \dots \otimes R_s^{-1}$, $s = 1, 2, \dots, N$, which orient any one of the constituent spinors $|\epsilon_s\rangle$ in the positive z -direction. In other words, there exist N rotations $R_s^{-1} = R^{-1}(\alpha_s, \beta_s, 0)$, $s = 1, 2, \dots, N$ —in general—which lead to the same result (7).
4. In terms of the alternate representation (6) of the symmetric state $|\Psi_{\text{sym}}\rangle$, (7) leads to

$$\begin{aligned} & \left\langle \frac{N}{2}, -\frac{N}{2} \left| R^{-1}(\alpha_s, \beta_s, 0) \otimes R^{-1}(\alpha_s, \beta_s, 0) \otimes \dots \otimes R^{-1}(\alpha_s, \beta_s, 0) \right| \Psi_{\text{sym}} \right\rangle = 0 \\ \Rightarrow & \left\langle \frac{N}{2}, -\frac{N}{2} \left| \mathcal{R}_s^{-1}(\alpha_s, \beta_s, 0) \left\{ \sum_{l=0}^N c_l \left| \frac{N}{2}, l - \frac{N}{2} \right\rangle \right\} \right. \right\rangle = 0 \\ \text{i.e.,} & \sum_{l=0}^N c_l D_{l-N/2, -N/2}^{N/2*}(\alpha_s, \beta_s, 0) = 0, \tag{8} \end{aligned}$$

where we have denoted $R^{-1}(\alpha_s, \beta_s, 0) \otimes R^{-1}(\alpha_s, \beta_s, 0) \otimes \dots \otimes R^{-1}(\alpha_s, \beta_s, 0) = \mathcal{R}_s^{-1}(\alpha_s, \beta_s, 0)$ in the collective $(N + 1)$ dimensional symmetric subspace of N qubits and

$$D_{-N/2, l-N/2}^{N/2\dagger} = \langle N/2, -N/2 | \mathcal{R}_l^{-1} | N/2, l - N/2 \rangle,$$

represents the collective rotation in the Wigner- D representation [26]. Substituting the explicit form of the D -matrix [26], i.e.,

$$\begin{aligned} \left[D^{N/2\dagger}(\alpha, \beta, 0) \right]_{-N/2, l-N/2} &= D_{l-N/2, -N/2}^{N/2*}(\alpha, \beta, 0) \\ &= \sqrt{C_l} \left[\cos\left(\frac{\beta}{2}\right) \right]^{N-l} \left[-\sin\left(\frac{\beta}{2}\right) \right]^l e^{i\left(l-\frac{N}{2}\right)\alpha}, \end{aligned} \quad (9)$$

and on subsequent simplification we obtain,

$$\mathcal{A} \sum_{l=0}^N (-1)^l \sqrt{C_l} c_l z^l = 0 \quad (10)$$

where $z = \tan\left(\frac{\beta}{2}\right) e^{i\alpha}$ and the overall coefficient $\mathcal{A} = \cos^N\left(\frac{\beta}{2}\right) e^{-i\frac{N\alpha}{2}}$.

In other words, the N roots $z_l = \tan\left(\frac{\beta_l}{2}\right) e^{i\alpha_l}$, $l = 1, 2, \dots, N$ of the Majorana polynomial $P(z)$

$$P(z) = \sum_{l=0}^N (-1)^l \sqrt{C_l} c_l z^l \quad (11)$$

determine the orientations (α_l, β_l) of the spinors constituting the N -qubit symmetric state, in terms of the collective parameters c_l .

It may be noted that the orientations of all the N constituent spinors may not be determined in the cases where the Majorana Polynomial $P(z)$ is of degree $r < N$ (i.e., when some of the coefficients c_l , $r < l \leq N$ are zero). To see this, let us consider the example of Dicke states (5). We have only one of the coefficients non-zero i.e., $c_l = \delta_{l,r}$. The corresponding Majorana polynomial $P(z)$ reduces to $P(z) = (-1)^r \sqrt{C_r} c_r z^r$. The r -fold degenerate root $z = 0$ of the polynomial leads to the specification of the spinor orientation angles $\beta_l = 0$, $\alpha_l = \text{arbitrary}$, $l = 1, 2, \dots, r$ leading to the identification $|\epsilon_l\rangle \sim |0\rangle$, $l = 1, 2, \dots, r$ (up to an overall phase) of the constituent spinors. There is no further information about the remaining $N - r$ spinors constituting the state in terms of the Majorana Polynomial (11). It is convenient to recast the polynomial in terms of $z' = \frac{1}{z} = \cot\left(\frac{\beta_l}{2}\right) e^{-i\alpha_l}$ and following the same procedure outlined above, we obtain

$$\mathcal{A}' \sum_{l=0}^N (-1)^l \sqrt{C_l} c_{\frac{N}{2}-l} z'^{N-l} = 0, \quad (12)$$

where $\mathcal{A}' = \sin^N\left(\frac{\beta_l}{2}\right) e^{i\alpha_l N/2}$.

We thus obtain,

$$P(z') = \sum_{l=0}^N (-1)^{N-l} \sqrt{C_l} c_{\frac{N}{2}-l} z'^{N-l}. \quad (13)$$

The $N - r$ roots of the polynomial (13) determine the orientations of the remaining $N - r$ spinors $|\epsilon_l\rangle, l = r + 1, r + 2, \dots, N$, constituting the state $|\Psi_{\text{sym}}\rangle$. In particular, for Dicke states $|\frac{N}{2}, r - \frac{N}{2}\rangle$, it is easy to see that (13) leads to $(N - r)$ -fold degenerate root $z' = 0$ which in turn corresponds to $\beta_l = \pi/2, \alpha_l = \text{arbitrary}$ i.e., $|\epsilon_l\rangle \equiv |1\rangle, l = r + 1, r + 2, \dots, N$. It may be readily seen that except for the *all-up* (*all down*) N -qubit Dicke states $|\frac{N}{2}, \frac{N}{2}\rangle \equiv |0, 0, \dots, 0\rangle$ ($|\frac{N}{2}, -\frac{N}{2}\rangle \equiv |1, 1, \dots, 1\rangle$), for which the Majorana Polynomial $P(z) = (-1)^N z^N$ ($P(z') = (-1)^N z'^N$) results in N -fold degenerate root, all the other Dicke states $|\frac{N}{2}, l - \frac{N}{2}\rangle, l \neq 0, N$ are characterized by two distinct spinors, $|0\rangle, |1\rangle$ each occurring r and $N - r$ times respectively.

The GHZ state $\frac{1}{\sqrt{2}} [|\frac{N}{2}, \frac{N}{2}\rangle + |\frac{N}{2}, -\frac{N}{2}\rangle]$ of N qubits satisfy the polynomial equation $1 + (-1)^N z^N = 0$, solutions of which are N th roots of unity (when N is odd) $z_l = e^{\frac{2\pi i l}{N}}; l = 0, 1, 2, \dots, N - 1$ (when $N = \text{even}$, we have $z_l = e^{\frac{2\pi i}{N}(l - \frac{1}{2})}$). The associated Majorana spinors are given by, $|\epsilon_l\rangle = \sqrt{\frac{z_l}{2}} [|0\rangle + z_l |1\rangle], l = 0, 1, 2, \dots, N - 1$.

We list some examples of symmetric states of two and three qubits and the corresponding constituent spinors in Table 1.

3 Entanglement classification of multiqubit symmetric states

Multiparticle entanglement can be of different kinds [27]. Two multiparty states have the same kind of entanglement if they can be obtained from each other via stochastic local operations and classical communication (SLOCC) with nonzero probability. It is well-known that three qubit GHZ and W states are inequivalent under SLOCC and are representatives of inequivalent three party entanglement. Understanding inequivalent classes of multiparticle entanglement, which are not interconvertible into each other under SLOCC operations is of fundamental importance [27–29, 16]. It has been identified that the number of inequivalent multiparticle entanglement classes grows rapidly with the increase of the number of parties [27, 29, 16]. This poses increasing algebraic complexity in the identification of inequivalent entanglement classes as the number of parties increase. However, when one restricts to the set of permutation symmetric multiqubit states, the MR, discussed in Sect. 2, offers an elegant approach towards the SLOCC classification of entanglement families—based entirely on the number and arrangement of the independent spinors (qubits) constituting the pure symmetric multiqubit state [16]. More recently, innovative experimental schemes have been proposed to generate a large variety of symmetric multiqubit photonic states [30, 31]. In the following, we outline the approach of Bastin et al. [16] in identifying the SLOCC classification of symmetric multiqubit pure states based on the MR.

3.1 SLOCC classification of symmetric multiqubit pure states

Any two N -party *pure* states $|\phi\rangle, |\psi\rangle$ are interconvertible, with non-zero probability of success, by means of SLOCC if and only if there exists an invertible local operation (ILO) [27] $\mathbf{A}_1 \otimes \mathbf{A}_2 \otimes \dots \otimes \mathbf{A}_N$ such that $|\phi\rangle = (\mathbf{A}_1 \otimes \mathbf{A}_2 \otimes \dots \otimes \mathbf{A}_N)|\psi\rangle$. Restricting ourselves to the set of permutation symmetric multiqubit states, it suffices to consider transformations of the form $\mathbf{A}^{\otimes N} = \mathbf{A} \otimes \mathbf{A} \otimes \dots \otimes \mathbf{A}$, comprising only *identical*

Table 1 Majorana spinors for some two and three qubit symmetric states

Symmetric state in the collective basis $ \frac{N}{2}, l - \frac{N}{2}\rangle$	Polynomial equation and solutions	Majorana spinors	Symmetrization of Majorana spinors as in Eq. (1) (expressed in the standard qubit basis)
$N = 2$			
$\frac{ 1,1\rangle+ 1,-1\rangle}{\sqrt{2}}$	$1 + z^2 = 0,$ $z_{1,2} = e^{\pm i\frac{\pi}{2}}, \beta_{1,2} = \frac{\pi}{2}, \alpha_{1,2} = \pm \frac{\pi}{2}$	$ \epsilon_1\rangle = \frac{e^{-i\frac{\pi}{2}}}{\sqrt{2}} (0\rangle + i 1\rangle)$ $ \epsilon_2\rangle = \frac{e^{i\frac{\pi}{2}}}{\sqrt{2}} (0\rangle - i 1\rangle)$	$\frac{ 0,0\rangle+ 1,1\rangle}{\sqrt{2}}$
$\frac{ 1,1\rangle- 1,-1\rangle}{\sqrt{2}}$	$z^2 - 1 = 0,$ $z_{1,2} = \pm 1, \beta_{1,2} = \frac{\pi}{2},$ $\alpha_{1,2} = 0$	$ \epsilon_1\rangle = \frac{1}{\sqrt{2}} (0\rangle + 1\rangle)$ $ \epsilon_2\rangle = \frac{1}{\sqrt{2}} (0\rangle - 1\rangle)$	$\frac{ 0,0\rangle- 1,1\rangle}{\sqrt{2}}$
$ 1, 0\rangle$	$z = 0, z^{-1} = 1$ $z_1 = 0, z_2 = z^{-1} = 1$ $\beta_{1,2} = 0, \pi, \alpha_{1,2} = \text{arbitrary}$	$ \epsilon_1\rangle = 0\rangle$ $ \epsilon_2\rangle = 1\rangle$	$\frac{ 0,1\rangle+ 1,0\rangle}{\sqrt{2}}$
$N = 3$			
$\frac{ \frac{3}{2}, \frac{3}{2}\rangle+ \frac{3}{2}, -\frac{3}{2}\rangle}{\sqrt{2}}$	$1 - z^3 = 0,$ $z_r = e^{\frac{2\pi i r}{3}},$ $\beta_r = \frac{\pi}{2}, \alpha_r = \frac{2\pi r}{3}, r = 0, 1, 2.$	$ \epsilon_r\rangle = \sqrt{\frac{2}{3}} (0\rangle + z_r 1\rangle),$ $r = 0, 1, 2$	$\frac{ 0,0,0\rangle+ 1,1,1\rangle}{2}$
$\frac{ \frac{3}{2}, \frac{3}{2}\rangle- \frac{3}{2}, -\frac{3}{2}\rangle}{\sqrt{2}}$	$z^3 + 1 = 0,$ $z_r = e^{\frac{2\pi i r}{3}(r-\frac{1}{2})}$ $\beta_r = \frac{\pi}{2}, \alpha_r = \frac{2\pi}{3}(r - \frac{1}{2}), r = 0, 1, 2$	$ \epsilon_r\rangle = \sqrt{\frac{2}{3}} (0\rangle + z_r 1\rangle),$ $r = 0, 1, 2$	$\frac{ 0,0,0\rangle- 1,1,1\rangle}{2}$
$\frac{ \frac{3}{2}, \frac{1}{2}\rangle+ \frac{3}{2}, -\frac{1}{2}\rangle}{\sqrt{2}}$	$z_1 = \pm 1, z_2 = 0, z_3^{-1} = 0;$ $\beta_1 = \pm \frac{\pi}{2}, \alpha_1 = \mp \pi; \beta_2 = 0; \beta_3 = \pi.$ $z = 0, z^{-2} = 0$	$ \epsilon_1\rangle = \frac{1}{\sqrt{2}} (0\rangle \pm 1\rangle),$ $ \epsilon_2\rangle = 0\rangle, \epsilon_3\rangle = 1\rangle$	$\frac{1}{\sqrt{6}} [0, 0, 1\rangle + 0, 1, 0\rangle$ $+ 0, 0, 1\rangle \pm 0, 1, 1\rangle$ $\pm 1, 0, 1\rangle \pm 1, 1, 0\rangle]$
$ \frac{3}{2}, -\frac{1}{2}\rangle$	$z_1 = 0, z_2^{-1} = z_3^{-1} = 0, \beta_1 = 0, \beta_{2,3} = \pi$ $\alpha_{1,2,3} = \text{arbitrary}$	$ \epsilon_1\rangle = 0\rangle, \epsilon_{2,3}\rangle = 1\rangle$	$\frac{1}{\sqrt{3}} [0, 0, 1\rangle + 0, 1, 0\rangle + 0, 0, 1\rangle]$

ILOs on all the qubits to verify the SLOCC equivalence [16,32]. This identification is significant in that MR [8] of symmetric states offers itself to recognize how different entanglement families emerge.

As we have noted in Sect. 2, the roots of the Majorana polynomial (11) (and (13)) could be degenerate and hence not all the N constituent spinors of a pure symmetric N qubit state are distinct. Let $|\epsilon_1\rangle, |\epsilon_2\rangle, \dots, |\epsilon_d\rangle, d \leq N$ be the number of distinct spinors, in a N qubit pure symmetric state (1). Then, the list of numbers

$$\{n_1, n_2, \dots, n_d; n_1 \geq n_2 \geq \dots \geq n_d; n_1 + n_2 + \dots + n_d = N\}$$

corresponds respectively to the number of times the independent spinors $|\epsilon_i\rangle, (i = 1, 2, \dots, d \leq N)$ appear in the symmetric state (1) under consideration. The number $d \leq N$, called the *diversity degree* and the list of numbers $\{n_1, n_2, \dots, n_d; n_1 \geq n_2 \geq \dots \geq n_d, \sum_{i=1}^d n_i = N\}$, called the *degeneracy configuration*, form the key elements in the classification of pure symmetric states [16]. The different classes (based on the number of distinct spinors and their arrangement in a given N -qubit symmetric state) are denoted by $\{\mathcal{D}_{n_1, n_2, \dots, n_d}\}$. An identical ILO $\mathbf{A}^{\otimes N}$ transforms a symmetric state belonging to the class $\{\mathcal{D}_{n_1, n_2, \dots, n_d}\}$ to another state of the *same* class. More explicitly, we have

$$|D_{n_1, n_2, \dots, n_d}\rangle \xrightarrow{\text{ILO}} |D'_{n_1, n_2, \dots, n_d}\rangle = \mathbf{A}^{\otimes N} |D_{n_1, n_2, \dots, n_d}\rangle \tag{14}$$

with the constituent spinors transforming as $|\epsilon'_i\rangle = A |\epsilon_i\rangle, i = 1, 2, \dots, d$. This forms the main basis of the SLOCC classification of symmetric pure states [16].

1. $\{\mathcal{D}_N\}$: When all the N solutions of the Majorana polynomial are identically equal, the corresponding class of symmetric states is given by

$$|D_N\rangle = |\epsilon, \epsilon, \dots, \epsilon\rangle, \tag{15}$$

where the diversity degree $d = 1$; the states belonging to this family of separable symmetric states is denoted by $|D_N\rangle$.

2. $\{\mathcal{D}_{n_1, n_2}; n_1 = N - k, n_2 = k = 1, 2, \dots, [N/2]\}$: The states with two distinct spinors have the form,

$$|D_{N-k, k}\rangle = \mathcal{N} [|\underbrace{\epsilon_1, \epsilon_1, \dots, \epsilon_1}_{N-k}, \underbrace{\epsilon_2, \epsilon_2, \dots, \epsilon_2}_k\rangle + \text{Permutations}] \tag{16}$$

where $k = 1, 2, \dots [N/2]$.

Dicke states $|\frac{N}{2}, k - \frac{N}{2}\rangle$ are the representative states of the entanglement class $\{\mathcal{D}_{N-k, k}\}$ with two independent spinors and clearly, they are all inequivalent under SLOCC (as the degeneracy classification is different for each $k = 1, 2, \dots, [N/2]$).

3. $\{\mathcal{D}_{1, 1, \dots, 1}\}$: When the N roots of the Majorana Polynomial (1) are all distinct, the pure symmetric states constitute the class $\{\mathcal{D}_{1, 1, 1, \dots, 1}\}$ with diversity degree $d = N$. Clearly, the N qubit GHZ state is a representative of this entanglement class.

The number of SLOCC classes grows with the increase in the number of qubits: For $N = 2$, there are only 2 entanglement families given by \mathcal{D}_2 (the separable class) and $\mathcal{D}_{1,1}$; for $N = 3$ there are 3 SLOCC classes given by \mathcal{D}_3 , $\mathcal{D}_{2,1}$ and $\mathcal{D}_{1,1,1}$ etc. In general, the number of entanglement families for a symmetric N -qubit state grows [16], based entirely on the partition of the number N in the arrangement $\{n_1, n_2, \dots, n_d; n_1 \geq n_2 \geq \dots \geq n_d; \sum_{i=1}^d n_i = N\}$. However, the Majorana classes with diversity degree $d \geq 4$ contain a continuous range of SLOCC classes, depending on a continuous parameter and the states with different value of this continuous parameter are not SLOCC convertible into each other [16]. More recently Bastin et al. [17] have also extended the entanglement classification scheme for mixed symmetric multiqubit systems, based on the hierarchical families of different SLOCC classes, successively embedded into each other.

4 Determining the whole pure state from its parts

One among the basic issues of interest in quantum information theory is to learn about how much of the whole quantum state can be known from its subsystems [33–40]. The importance of knowing if higher order correlations in a multipartite system follow entirely from lower order ones involving few parties has been of interest in many body physics [41]. Construction of the many electron state with the knowledge of its two particle reduced density matrices has been discussed in a series of papers [42–46]. While it has been shown by Linden et al [47,48] that N -party entanglement cannot, in general, be reversibly transformed into entanglement of two parties, Linden, Popescu and Wootters [34,35] proved a striking result that reduced states of a smaller fraction of the parties specify most of the *generic* multipartite pure states *uniquely*. Walck and Lyons [37,38] showed that the N party GHZ states and their local unitary equivalents are the only exceptions to this result and the correlations in a multi-qubit GHZ state are *irreducible*. Preeti Parashar and Swapan Rana have shown that N qubit W class states can be uniquely determined by their bipartite marginals [39]. Generalized Dicke-class states is another class of symmetric as well as non-symmetric states that is shown to possess reducible correlations [40]. In this section we discuss determining the whole pure symmetric N qubit state of a specific SLOCC class from its $N - 1$ party reduced states [18,19].

4.1 Irreducibility features of three qubit symmetric states of the class $\{\mathcal{D}_{1,1,1}\}$ with all distinct Majorana spinors

While the equivalence of quantum states under SLOCC is known to indicate that states belonging to the same equivalence class can be used to implement similar quantum information tasks [27], here, we address the question “Do SLOCC interconvertible states possess similar irreducibility features?” by considering specific examples of three qubit symmetric states belonging to the same SLOCC class $\{\mathcal{D}_{1,1,1}\}$. With the help of this example, we demonstrate explicitly that the states of the same Majorana class could exhibit contrasting irreducibility features. GHZ state and its local

unitary equivalent states are the only ones of the Majorana class $\{\mathcal{D}_{1,1,1}\}$, which are undetermined by their two qubit reduced systems [34,37,38].

We consider two specific examples of the SLOCC family $\{\mathcal{D}_{1,1,1}\}$, the first being the three qubit GHZ state,

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}} [|0, 0, 0\rangle + |1, 1, 1\rangle]. \tag{17}$$

The Majorana polynomial equation (1) for this state has a simple structure $1 - z^3 = 0$, solutions of which are cube roots of unity $\omega, \omega^2, \omega^3 = 1$ and the corresponding spinors constituting the state are readily identified to be $|\epsilon_1\rangle = \frac{1}{\sqrt{2}} [|0\rangle + \omega |1\rangle]$, $|\epsilon_2\rangle = \frac{1}{\sqrt{2}} [|0\rangle + \omega^2 |1\rangle]$, $|\epsilon_3\rangle = \frac{1}{\sqrt{2}} [|0\rangle + |1\rangle]$. GHZ state is fragile under the loss of a qubit, with vanishing pairwise concurrence [49,50] for any pairs of two qubit reduced density matrices; but it exhibits genuine three-party entanglement [27,51] with the maximum tangle [52] $\tau = 1$. The state exhibits irreducible three party correlations which can not be determined by its reduced states [34,37,38].

We consider another three qubit state which belong to the same SLOCC class $\{\mathcal{D}_{1,1,1}\}$ of three distinct Majorana spinors:

$$|\eta\rangle = \frac{1}{\sqrt{2}} [|\text{W}\rangle + |\bar{\text{W}}\rangle]. \tag{18}$$

This is a superposition of the three qubit W state $|\text{W}\rangle = \frac{1}{\sqrt{3}} [|0, 0, 1\rangle + |0, 1, 0\rangle + |1, 0, 0\rangle]$ and its obverse state $|\bar{\text{W}}\rangle = \frac{1}{\sqrt{3}} [|1, 1, 0\rangle + |1, 0, 1\rangle + |0, 1, 1\rangle]$. The state $|\eta\rangle$ has genuine three party entanglement, quantified in terms of the tangle $\tau = 1/3$, and it is also robust under the loss of qubits—as reflected through the concurrence $C = 1/3$ for any pairs of two qubits. The three qubit symmetric state $|\eta\rangle$ given by (18) satisfies the Majorana polynomial equation $z(z - 1) = 0$ and the corresponding spinors constituting the state are $|\epsilon'_1\rangle = |1\rangle$, $|\epsilon'_2\rangle = \frac{1}{\sqrt{2}} [|0\rangle + |1\rangle]$, $|\epsilon'_3\rangle = |0\rangle$.

The states $|\text{GHZ}\rangle$ and the W superposition state $|\eta\rangle$ can be *locally* converted from one another, with the help of an identical ILO i.e., $|\text{GHZ}\rangle = A \otimes A \otimes A |\eta\rangle$, where $A = \begin{pmatrix} 1 & \omega \\ 1 & \omega^2 \end{pmatrix}$. The corresponding Majorana spinors of the states $|\eta\rangle$ and $|\text{GHZ}\rangle$ are related to each other up to an overall factor: $A |\epsilon'_1\rangle = \sqrt{2}\omega |\epsilon_1\rangle$, $A |\epsilon'_2\rangle = -\omega^2 |\epsilon_2\rangle$, and $A |\epsilon'_3\rangle = \sqrt{2} |\epsilon_3\rangle$.

We now explicitly show that the higher order correlation in the W superposition state $|\eta\rangle$ is imprinted in its two qubit reduced states. We follow the approach of Linden et al. [34] in demonstrating this feature of the three qubit W superposition state.

Let us suppose that a mixed three qubit state γ too has the same two-qubit reduced system ϱ_{12} , as that of the W superposition state $|\eta\rangle$. Denoting the pure state $|\Gamma\rangle$ to be containing the three qubits and the environment such that

$$\text{Tr}_E[|\Gamma\rangle\langle\Gamma|] = \gamma,$$

the two party reduced state ϱ_{12} can be expressed as

$$\varrho_{12} = \text{Tr}_{3,E} [|\Gamma\rangle\langle\Gamma|].$$

The two qubit reduced system ϱ_{12} of the pure state $|\eta\rangle$ is a rank-2 state given by,

$$\varrho_{12} = |\chi_0\rangle\langle\chi_0| + |\chi_1\rangle\langle\chi_1|, \quad (19)$$

where

$$|\chi_0\rangle = \frac{1}{\sqrt{6}}[|1, 0\rangle + |0, 1\rangle + |1, 1\rangle],$$

$$|\chi_1\rangle = \frac{1}{\sqrt{6}}[|0, 0\rangle + |0, 1\rangle + |1, 0\rangle].$$

Given that the two party reduced state ϱ_{12} also belongs to the extended pure state $|\Gamma\rangle$ (and in turn to the mixed state γ) of the three qubits and the environment, we must have

$$|\Gamma\rangle = |\chi_0\rangle|E_0\rangle + |\chi_1\rangle|E_1\rangle, \quad (20)$$

$$\langle E_i|E_j\rangle = \delta_{i,j}. \quad (21)$$

In terms of the basis states of qubit 3, the states of the environment $|E_0\rangle, |E_1\rangle$ are given by

$$|E_0\rangle = |0\rangle|e_{00}\rangle + |1\rangle|e_{01}\rangle,$$

$$|E_1\rangle = |0\rangle|e_{10}\rangle + |1\rangle|e_{11}\rangle. \quad (22)$$

Thus, (20) takes the following form:

$$|\Gamma\rangle = \frac{1}{\sqrt{6}}[(|1_1, 0_2, 0_3\rangle + |0_1, 1_2, 0_3\rangle + |1_1, 1_2, 0_3\rangle)|e_{00}\rangle + (|1_1, 0_2, 1_3\rangle + |0_1, 1_2, 1_3\rangle + |1_1, 1_2, 1_3\rangle)|e_{01}\rangle + (|0_1, 0_2, 0_3\rangle + |0_1, 1_2, 0_3\rangle + |1_1, 0_2, 0_3\rangle)|e_{10}\rangle + (|0_1, 0_2, 1_3\rangle + |0_1, 1_2, 1_3\rangle + |1_1, 0_2, 1_3\rangle)|e_{11}\rangle] \quad (23)$$

Now, demanding that the reduced system ϱ_{13} of $|\eta\rangle$ is also shared by $|\Gamma\rangle$ leads to further constraints.

1. First we compare $\langle 0, 1|\varrho_{13}|0, 1\rangle$, from the states (18) and (20): We have, $\langle 0, 1|\text{Tr}_2 [|\eta\rangle\langle\eta|]|0, 1\rangle = \frac{1}{3}$ and $\langle 0, 1|\text{Tr}_{2,E} [|\Gamma\rangle\langle\Gamma|]|0, 1\rangle = \frac{1}{6}\langle e_{01}|e_{01}\rangle + \frac{1}{3}\langle e_{11}|e_{11}\rangle$ leading to $\langle e_{01}|e_{01}\rangle + 2\langle e_{11}|e_{11}\rangle = 2$.
2. Next, we compare $\langle 1, 1|\varrho_{13}|1, 1\rangle$ evaluated from the states $|\eta\rangle$ and $|\Gamma\rangle$: We get, $\langle 1, 1|\text{Tr}_{2,E} [|\Gamma\rangle\langle\Gamma|]|1, 1\rangle = \frac{1}{3}\langle e_{01}|e_{01}\rangle + \frac{1}{6}\langle e_{11}|e_{11}\rangle$ and $\langle 1, 1|\text{Tr}_2 [|\eta\rangle\langle\eta|]|1, 1\rangle = \frac{1}{6}$ implying, $2\langle e_{01}|e_{01}\rangle + \langle e_{11}|e_{11}\rangle = 1$. From these relations we obtain $\langle e_{11}|e_{11}\rangle = 1$, $\langle e_{01}|e_{01}\rangle = 0$ (or $|e_{01}\rangle \equiv 0$). Further, from the orthonormality (21) it follows that $\langle e_{00}|e_{00}\rangle = 1$, and $|e_{10}\rangle \equiv 0$.

3. Finally, a comparison of the matrix elements $\langle 0, 0 | \text{Tr}_{2,E} [|\Gamma\rangle\langle\Gamma|] |0, 1\rangle = \frac{1}{6} \langle e_{00} | e_{11} \rangle$ and $\langle 0, 0 | \text{Tr}_2 [|\eta\rangle\langle\eta|] |0, 1\rangle = \frac{1}{6}$ lead to $\langle e_{00} | e_{11} \rangle = 1$ or $|e_{11}\rangle \equiv |e_{00}\rangle$. Thus, the extended pure state (20) should take the form $|\Gamma\rangle \equiv |\eta\rangle |e_{00}\rangle$.

In other words, the three qubit pure state $|\eta\rangle$ is uniquely determined by its two-qubit reduced systems and is therefore, reducible.

This illustrative example of three qubits supports (with the help of an independent and non-trivial proof) the already existing general result [37,38] that only the N qubit GHZ state and its local unitarily equivalents remain undetermined by their reduced density matrices. Moreover, this clearly projects out the contrasting irreducibility features of two SLOCC interconvertible states (17) and (18) of the same entanglement family $\{\mathcal{D}_{1,1,1}\}$.

4.2 Determining $\{\mathcal{D}_{N-k,k}\}$ SLOCC class of pure symmetric N qubit states from its $N - 1$ qubit reduced density matrices

While it has been proved [37] that except for the N qubit GHZ states and their unitary equivalents all other pure states are uniquely determined by their $N - 1$ party marginals, *how many* $N - 1$ party marginals are required to uniquely specify a given N -qubit state—not unitarily equivalent to GHZ state—was not known.

Let us consider the example of 4-qubit pure states considered by Ref. [37]: $|\chi_1\rangle = \frac{1}{\sqrt{3}} (|0000\rangle + |0001\rangle + |1111\rangle)$ and $|\chi_2\rangle = \frac{1}{\sqrt{3}} (|0000\rangle + |0001\rangle - |1111\rangle)$. These two states are not unitarily equivalent to the 4-qubit GHZ state; they both share the same 3-qubit reduced density matrices, when partial trace over first, second and third qubits are taken—whereas the partial trace over 4th qubit leads to distinct 3-qubit marginals. In other words, all the four 3-qubit reduced density matrices are required to uniquely specify each of them. Examples of N qubit states determined by smaller numbers of $N - 1$ qubit reduced density matrices are therefore of interest.

It may be mentioned here that Preeti Parashar and Swapan Rana [40] focussed on identifying the class of states which can be uniquely determined by reduced density matrices of smaller than $N - 1$ parties. The present authors [18] showed that the $\{\mathcal{D}_{N-k,k}\}$ SLOCC class of pure symmetric N qubit states containing two distinct Majorana spinors and another related class of non-symmetric states can be determined with the help of *only two* of their $N - 1$ qubit marginals. We outline the approach of Ref. [18] in the following.

Let us consider a representative symmetric state (16) of the entanglement family $\{\mathcal{D}_{N-k,k}\}$

$$\begin{aligned}
 |D_{N-k,k}\rangle &= \mathcal{N} \sum_P \hat{P} \left\{ \underbrace{|\epsilon_1, \epsilon_1, \dots, \epsilon_1\rangle}_{N-k}; \underbrace{|\epsilon_2, \epsilon_2, \dots, \epsilon_2\rangle}_k \right\} \\
 &= \mathcal{N} R_1^{\otimes N} \sum_P \hat{P} \left\{ \underbrace{|0, 0, \dots, 0\rangle}_{N-k}; \underbrace{|\epsilon'_2, \epsilon'_2, \dots, \epsilon'_2\rangle}_k \right\}, \quad (24)
 \end{aligned}$$

where $|\epsilon_1\rangle = R_1|0\rangle$ and $|\epsilon_2\rangle = R_2|0\rangle$, and

$$|\epsilon'_2\rangle = R_1^{-1}R_2|0\rangle = d_0|0\rangle + d_1|1\rangle, \quad |d_0|^2 + |d_1|^2 = 1, \quad d_1 \neq 0. \quad (25)$$

Substituting (25) in (24) and upon simplification, we obtain,

$$|D_{N-k,k}\rangle = R_1^{\otimes N} \sum_{r=0}^k \sqrt{{}^N C_r} \alpha_r \left| \frac{N}{2}, \frac{N}{2} - r \right\rangle,$$

where $\alpha_r = \mathcal{N} \frac{(N-r)!}{(N-k)!(k-r)!} d_0^{k-r} d_1^r$. (26)

In other words, all symmetric states $|D_{N-k,k}\rangle$, constituted by two distinct Majorana spinors are equivalent (under local unitary transformations) to

$$|D'_{N-k,k}\rangle = R_1^{-1 \otimes N} |D_{N-k,k}\rangle = \sum_{r=0}^k \sqrt{{}^N C_r} \alpha_r \left| \frac{N}{2}, \frac{N}{2} - r \right\rangle. \quad (27)$$

As $d_1 \neq 0$ (see Eq. (25)), the coefficients α_r , ($r = 0, 1, 2, \dots, k$) are non-zero, except when $d_1 = 1, d_0 = 0$ —in which case the state $|D'_{N-k,k}\rangle$ reduces to the Dicke state $|\frac{N}{2}, \frac{N}{2} - k\rangle$ itself and then we have, $\alpha_r = \delta_{k,r}$. We proceed to show that only two of the $N - 1$ qubit reduced density matrices determine the whole state $|D'_{N-k,k}\rangle$, following an analogous procedure as in Ref [34].

We express the state $|D'_{N-k,k}\rangle$ in the qubit basis as

$$\begin{aligned} |D'_{N-k,k}\rangle &= \alpha_0 |0_1, 0_2, \dots, 0_N\rangle + \alpha_1 \sum_P \hat{P}\{|1_1, 0_2, \dots, 0_{N-1}, 0_N\rangle\} \\ &\quad + \alpha_2 \sum_P \hat{P}\{|1_1, 1_2, 0_3, \dots, 0_N\rangle\} + \dots \\ &\quad + \alpha_k \sum_P \hat{P}\{|1_1, 1_2, \dots, 1_k, 0_{k+1}, \dots, 0_N\rangle\} \\ &= |\phi_0\rangle |0_N\rangle + |\phi_1\rangle |1_N\rangle \end{aligned} \quad (28)$$

where

$$\begin{aligned} |\phi_0\rangle &= \alpha_0 |0_1, 0_2, \dots, 0_{N-1}\rangle + \alpha_1 \sum_P \hat{P}\{|1_1, 0_2, \dots, 0_{N-1}\rangle\} \\ &\quad + \alpha_2 \sum_P \hat{P}\{|1_1, 1_2, 0_3, \dots, 0_{N-1}\rangle\} \\ &\quad + \dots + \alpha_k \sum_P \hat{P}\{|1_1, 1_2, 1_3, \dots, 1_k, 0_{k+1}, \dots, 0_{N-1}\rangle\} \end{aligned} \quad (29)$$

$$\begin{aligned}
 |\phi_1\rangle &= \alpha_1|0_1, 0_2, \dots, 0_{N-1}\rangle + \alpha_2 \sum_P \hat{P}\{|1_1, 0_2, \dots, 0_{N-1}\}\} \\
 &+ \alpha_3 \sum_P \hat{P}\{|1_1, 1_2, 0_3, \dots, 0_{N-1}\}\} \\
 &+ \dots + \alpha_k \sum_P \hat{P}\{|1_1, 1_2, 1_3, \dots, 1_{k-1}, 0_k, \dots, 0_{N-1}\}\}. \quad (30)
 \end{aligned}$$

It is not difficult to see that the $N - 1$ qubit reduced density matrix $\rho_{1,2,\dots,N-1}$, obtained by tracing out the N th qubit from the state $|D'_{N-k,k}\rangle$, is a rank-2 mixed state given by,

$$\rho_{1,2,\dots,N-1} = \text{Tr}_N[|D'_{N-k,k}\rangle\langle D'_{N-k,k}|] = |\phi_0\rangle\langle\phi_0| + |\phi_1\rangle\langle\phi_1|. \quad (31)$$

On supposing that a mixed N qubit state ω_N too shares the same $N - 1$ qubit reduced system $\rho_{1,2,\dots,N-1}$ one has

$$\begin{aligned}
 \rho_{1,2,\dots,N-1} &= \text{Tr}_N[|D'_{N-k,k}\rangle\langle D'_{N-k,k}|] = \text{Tr}_N[\omega_N] \\
 &= |\phi_0\rangle\langle\phi_0| + |\phi_1\rangle\langle\phi_1|. \quad (32)
 \end{aligned}$$

Considering the purification of the mixed state ω_N , i.e., considering ω_N as a reduced system of an extended *pure* state $|\Omega_{NE}\rangle$ consisting of N qubits and an environment E , one has

$$\text{Tr}_E[|\Omega_{NE}\rangle\langle\Omega_{NE}|] = \omega_N. \quad (33)$$

In order that the pure state $|\Omega_{NE}\rangle$ (or the mixed state ω_N) too shares the same $N - 1$ qubit reduced density matrix $\rho_{1,2,\dots,N-1}$, one must have

$$|\Omega_{NE}\rangle = |\phi_0\rangle|E_0\rangle + |\phi_1\rangle|E_1\rangle, \quad (34)$$

$$\langle E_i|E_j\rangle = \delta_{i,j}. \quad (35)$$

Here, the states $|E_0\rangle, |E_1\rangle$ are the ones containing the qubit labelled N , and the environment E . Expanding $|E_0\rangle, |E_1\rangle$ in the basis states of the qubit N as,

$$\begin{aligned}
 |E_0\rangle &= |0_N\rangle|e_{00}\rangle + |1_N\rangle|e_{01}\rangle \\
 |E_1\rangle &= |0_N\rangle|e_{10}\rangle + |1_N\rangle|e_{11}\rangle, \quad (36)
 \end{aligned}$$

the state $|\Omega_{NE}\rangle$ can be re-expressed using (34), (36) as

$$|\Omega_{NE}\rangle = |\phi_0\rangle|0_N\rangle|e_{00}\rangle + |\phi_0\rangle|1_N\rangle|e_{01}\rangle + |\phi_1\rangle|0_N\rangle|e_{10}\rangle + |\phi_1\rangle|1_N\rangle|e_{11}\rangle. \quad (37)$$

The states $|e_{ij}\rangle, i, j = 0, 1$ correspond to those of the environment and they are not assumed to be orthonormal *a priori*.

As both $|D'_{N-k,k}\rangle$ and $|\Omega_{NE}\rangle$ are sharing a common reduced density matrix $\rho_{1,2,\dots,N-1}$, we wish to check the form of $|\Omega_{NE}\rangle$ so that it shares another reduced density matrix $\rho_{2,3,\dots,N}$ of $|D'_{N-k,k}\rangle$.

1. We first compare the matrix elements of $\rho_{2,\dots,N} = \text{Tr}_1 |D'_{N-k,k}\rangle\langle D'_{N-k,k}|$ with $\text{Tr}_{1,E} [|\Omega_{NE}\rangle\langle\Omega_{NE}|]$ to find that

$$\begin{aligned} \langle 0_2, 0_3 \dots, 0_{N-k-1}, 1_{N-k}, \dots, 1_N | \text{Tr}_{1,E} [|\Omega_{NE}\rangle\langle\Omega_{NE}|] | 0_2, 0_3 \dots, \\ 0_{N-k-1}, 1_{N-k}, \dots, 1_N \rangle &= |\alpha_k|^2 \langle e_{01} | e_{01} \rangle \\ \langle 0_2, 0_3 \dots, 0_{N-k-1}, 1_{N-k}, \dots, 1_N | \text{Tr}_1 [|D'_{N-k,k}\rangle\langle D'_{N-k,k}|] | 0_2, 0_3 \dots, \\ 0_{N-k-1}, 1_{N-k}, \dots, 1_N \rangle &= 0 \end{aligned} \quad (38)$$

As $\alpha_k \neq 0$ for the states $|D'_{N-k,k}\rangle$, we must have $|e_{01}\rangle \equiv 0$. The simpler form of $|\Omega_{NE}\rangle$ obtained on putting $|e_{01}\rangle \equiv 0$ in (37) is given by

$$|\Omega_{NE}\rangle = |\phi_0\rangle|0_N\rangle|e_{00}\rangle + |\phi_1\rangle[|0_N\rangle|e_{10}\rangle + |1_N\rangle|e_{11}\rangle] \quad (39)$$

On making use of the orthonormality relations $\langle E_0 | E_0 \rangle = 1$, $\langle E_0 | E_1 \rangle = 0$, we get

$$\langle e_{00} | e_{00} \rangle = 1, \quad \langle e_{00} | e_{10} \rangle = 0. \quad (40)$$

2. We now equate another matrix element of $\rho_{2,\dots,N}$ obtained from both the states $|D'_{N-k,k}\rangle$ and $|\Omega_{NE}\rangle$:

$$\begin{aligned} \langle 0_2, 0_3 \dots, 0_{N-k}, 1_{N-k+1}, \dots, 1_N | \text{Tr}_{1,E} [|\Omega_{NE}\rangle\langle\Omega_{NE}|] | 1_2, \dots, \\ 1_{k+1}, 0_{k+2}, 0_3 \dots, 0_N \rangle &= |\alpha_k|^2 \langle e_{11} | e_{00} \rangle, \end{aligned} \quad (41)$$

$$\begin{aligned} \langle 0_2, 0_3 \dots, 0_{N-k}, 1_{N-k+1}, \dots, 1_N | \text{Tr}_1 [|D'_{N-k,k}\rangle\langle D'_{N-k,k}|] | 1_2, \dots, \\ 1_{k+1}, 0_{k+2}, 0_3 \dots, 0_N \rangle &= |\alpha_k|^2. \end{aligned} \quad (42)$$

This leads to the identification, $\langle e_{11} | e_{00} \rangle = 1$, as $\alpha_k \neq 0$. In view of (40), we obtain $|e_{11}\rangle = |e_{00}\rangle + |e_{00}^\perp\rangle$. Substituting this in the orthonormality condition $\langle E_1 | E_1 \rangle = 1$, we readily obtain

$$\langle e_{10} | e_{10} \rangle + \langle e_{00}^\perp | e_{00}^\perp \rangle = 0 \Rightarrow |e_{10}\rangle = 0, |e_{00}^\perp\rangle = 0$$

which in turn implies that

$$|e_{11}\rangle \equiv |e_{00}\rangle.$$

Thus, we obtain

$$\begin{aligned} |\Omega_{NE}\rangle &= (|\phi_0\rangle|0_N\rangle + |\phi_1\rangle|1_N\rangle)|e_{00}\rangle \\ &= |D'_{N-k,k}\rangle|e_{00}\rangle. \end{aligned} \quad (43)$$

This implies that the state $|D'_{N-k,k}\rangle$ is the unique *whole* pure state that is consistent with its $N - 1$ qubit reduced density matrices. We have employed only *two* of the $N - 1$ reduced density matrices $\rho_{1,2,\dots,N-1}, \rho_{2,3,\dots,N}$ to arrive at this result. It may also be noted here that any other choice of the second $N - 1$ qubit reduced density matrix (obtained by tracing over *any* of one the qubits) would have led us to the same result, though with appropriate choices of matrix elements in (38), (41).

4.3 Unique determination of a general class of non-symmetric N qubit states through its parts

The method illustrated for symmetric states $|D_{N-k,k}\rangle$ suggests a natural extension to a generalized family $\mathcal{D}^G_{N-k,k}$ of non-symmetric states. This family consists of states $|D^G_{N-k,k}\rangle$ which are a superpositions of the so-called generalized Dicke states

$$\sum_i a_i^{(r)} \left[|1_{P(i_1)}, 1_{P(i_2)}, \dots, 1_{P(i_r)}, 0_{P(i_{r+1})}, \dots, 0_{P(i_N)}\rangle \right]$$

obtained on associating an arbitrary coefficient $a_i^{(r)}$ with each term of the Dicke state (5).

As we have shown in the previous section that the states $|D_{N-k,k}\rangle$ are local unitarily equivalent to the state

$$|D'_{N-k,k}\rangle = \sum_{r=0}^k \sqrt{{}^N C_r} \alpha_r \left| \frac{N}{2}, \frac{N}{2} - r \right\rangle,$$

we construct the generalized non-symmetric N qubit pure states from the state $|D'_{N-k,k}\rangle$ as follows:

$$\begin{aligned} |D^G_{N-k,k}\rangle &= \alpha_0 a_0^{(0)} |0_1, 0_2, \dots, 0_N\rangle \\ &+ \sum_{r=1}^k \alpha_r \left\{ \sum_{i=1}^{{}^N C_r} a_i^{(r)} \left[|1_{P(i_1)}, 1_{P(i_2)}, \dots, 1_{P(i_r)}, 0_{P(i_{r+1})}, \dots, 0_{P(i_N)}\rangle \right] \right\} \\ &= \alpha_0 a_0^{(0)} |0_1, 0_2, \dots, 0_N\rangle + \alpha_1 \left\{ a_1^{(1)} |1_1, 0_2, \dots, 0_N\rangle \right. \\ &\quad \left. + a_2^{(1)} |0_1, 1_2, \dots, 0_N\rangle + \dots + a_N^{(1)} |0_1, 0_2, \dots, 0_{N-1}, 1_N\rangle \right\} \\ &\quad + \alpha_2 \left\{ a_1^{(2)} |1_1, 1_2, 0_3, \dots, 0_N\rangle + a_2^{(2)} |1_1, 0_2, 1_3, 0_4, \dots, 0_N\rangle \right. \\ &\quad \left. + \dots + a_{\frac{(N-1)(N-2)}{2}}^{(2)} |0_1, \dots, 0_{N-3}, 1_{N-2}, 1_{N-1}, 0_N\rangle \right. \\ &\quad \left. + a_{\frac{(N-1)(N-2)}{2}+1}^{(2)} |1_1, 0_2, 0_3, \dots, 0_{N-1}, 1_N\rangle + \dots \right. \\ &\quad \left. + a_{\frac{N(N-1)}{2}}^{(2)} |0_1, 0_2, \dots, 0_{N-2}, 1_{N-1}, 1_N\rangle \right\} + \dots \end{aligned}$$

$$\begin{aligned}
& +\alpha_k \left\{ a_1^{(k)} |1_1, 1_2, \dots, 1_k, 0_{k+1}, \dots, 0_N\rangle + \dots \right. \\
& + a_{N-1C_k}^{(k)} |0_1, \dots, 0_{N-k-1}, 1_{N-k}, 1_{N-k+1}, \dots, 1_{N-1}, 0_N\rangle \\
& + a_{N-1C_{k+1}}^{(k)} |1_1, 1_2, \dots, 1_{k-1}, 0_k, \dots, 0_{N-1}, 1_N\rangle + \dots \\
& \left. + a_{N C_k}^{(k)} |0_1, \dots, 0_{N-k}, 1_{N-k+1}, \dots, 1_N\rangle \right\}. \quad (44)
\end{aligned}$$

Here α_r 's are as given in (26) and $\sum_i a_i^{(r)} \left[|1_{P(i_1)}, 1_{P(i_2)}, \dots, 1_{P(i_r)}, 0_{P(i_{r+1})}, \dots, 0_{P(i_N)}\rangle \right]$ are the generalized Dicke states. We show that no other (pure or mixed) N qubit state, can share the same $N - 1$ qubit subsystem density matrices as that of $|D_{N-k,k}^G\rangle$. The procedure adopted for this purpose is same as that employed in the Sect. 4.2.

The state $|D_{N-k,k}^G\rangle$ can be expressed as

$$|D_{N-k,k}^G\rangle = |\phi_0^G\rangle|0\rangle_N + |\phi_1^G\rangle|1\rangle_N, \quad (45)$$

where

$$\begin{aligned}
|\phi_0^G\rangle &= \alpha_0 a_0^{(0)} |0_1, 0_2, \dots, 0_{N-1}\rangle \\
&+ \sum_{r=1}^k \alpha_r \left\{ \sum_{i=1}^{N-1C_r} a_i^{(r)} \left[|1_{P(i_1)}, 1_{P(i_2)}, \dots, 1_{P(i_r)}, 0_{P(i_{r+1})}, \dots, 0_{P(i_{N-1})}\rangle \right] \right\} \\
&= \alpha_0 a_0^{(0)} |0_1, 0_2, \dots, 0_{N-1}\rangle + \alpha_1 \left\{ a_1^{(1)} |1_1, 0_2, \dots, 0_{N-1}\rangle + \dots \right. \\
&\quad \left. + a_{N-1}^{(1)} |0_1, \dots, 0_{N-2}, 1_{N-1}\rangle \right\} \\
&+ \alpha_2 \left\{ a_1^{(2)} |1_1, 1_2, 0_3, \dots, 0_{N-1}\rangle + a_2^{(2)} |1_1, 0_2, 1_3, 0_4, \dots, 0_{N-1}\rangle + \dots \right. \\
&\quad \left. + a_{\frac{(N-1)(N-2)}{2}}^{(2)} |0_1, \dots, 0_{N-3}, 1_{N-2}, 1_{N-1}\rangle \right\} + \dots \\
&+ \alpha_k \left\{ a_1^{(k)} |1_1, 1_2, \dots, 1_k, 0_{k+1}, \dots, 0_{N-1}\rangle \right. \\
&\quad + a_2^{(k)} |1_1, 1_2, \dots, 1_{k-1}, 0_k, 1_{k+1}, 0_{k+2}, \dots, 0_{N-1}\rangle \\
&\quad + a_3^{(k)} |1_1, 1_2, \dots, 1_{k-2}, 0_{k-1}, 1_k, 1_{k+1}, 0_{k+2}, \dots, 0_{N-1}\rangle + \dots \\
&\quad + a_k^{(k)} |0_1, 1_2, \dots, 1_{k+1}, 0_{k+2}, \dots, 0_{N-1}\rangle + \dots \\
&\quad \left. + a_{N-1C_k}^{(k)} |0_1, \dots, 0_{N-k-1}, 1_{N-k}, \dots, 1_{N-1}\rangle \right\}, \quad (46)
\end{aligned}$$

and

$$\begin{aligned}
|\phi_1^G\rangle &= \sum_{r=0}^{k-1} \alpha_{r+1} \left\{ \sum_{i=N-1C_{r+1}}^{N C_{r+1}} a_i^{(r+1)} \left[|1_{P(i_1)}, 1_{P(i_2)}, \dots, 1_{P(i_r)}, 0_{P(i_{r+1})}, \dots, 0_{P(i_{N-1})}\rangle \right] \right\} \\
&= \alpha_1 a_N^{(1)} |0_1, 0_2, \dots, 0_{N-1}\rangle + \alpha_2 \left\{ a_{\frac{(N-1)(N-2)}{2}+1}^{(2)} |1_1, 0_2, 0_3, \dots, 0_{N-1}\rangle + \dots \right.
\end{aligned}$$

$$\begin{aligned}
 &+ a_{\frac{N(N-1)}{2}}^{(2)} |0_1, 0_2, \dots, 0_{N-2}, 1_{N-1}\rangle \} + \dots \\
 &+ \alpha_k \left\{ a_{N-1C_k+1}^{(k)} |1_1, 1_2, \dots, 1_{k-1}, 0_k, \dots, 0_{N-1}\rangle \right. \\
 &+ \dots + a_{N C_k}^{(k)} |0_1, \dots, 0_{N-k}, 1_{N-k+1}, \dots, 1_{N-1}\rangle \left. \right\}. \tag{47}
 \end{aligned}$$

It is to be noticed that the coefficients $a_i^{(r)}$ in (44) are labeled such that $a_i^{(r)}, i = 1, 2, \dots, {}^{N-1}C_r$ are associated with the states that have their r spin-down qubits $|1\rangle$, permuted in the first $N - 1$ positions, leaving the N th position to $|0\rangle$. The remaining coefficients $a_{i'}^{(r)}, i' = {}^{N-1}C_r + 1, {}^{N-1}C_r + 2, \dots, {}^N C_r$ are associated with the states having their N th position occupied by $|1\rangle$. Thus, $|\phi_0^G\rangle$ contains coefficients $a_i^{(r)}, i = 1, 2, \dots, {}^{N-1}C_r$ whereas $|\phi_1^G\rangle$ contains coefficients $a_{i'}^{(r)}, i' = {}^{N-1}C_r + 1, {}^{N-1}C_r + 2, \dots, {}^N C_r$. led such that $a_i^{(r)}, i = 1, 2, \dots, {}^{N-1}C_r$ are associated with the states that have their r spin-down qubits $|1\rangle$, permuted in the first $N - 1$ positions, leaving the N th position to $|0\rangle$. The remaining coefficients $a_{i'}^{(r)}, i' = {}^{N-1}C_r + 1, {}^{N-1}C_r + 2, \dots, {}^N C_r$ are associated with the states having their N th position occupied by $|1\rangle$. Thus, $|\phi_0^G\rangle$ contains coefficients $a_i^{(r)}, i = 1, 2, \dots, {}^{N-1}C_r$ whereas $|\phi_1^G\rangle$ contains coefficients $a_{i'}^{(r)}, i' = {}^{N-1}C_r + 1, {}^{N-1}C_r + 2, \dots, {}^N C_r$.

From (45), it can be readily seen that the $N - 1$ qubit reduced density matrix $\rho_{1,2,\dots,N-1}^G$ of the state $|D_{N-k,k}^G\rangle$ has the form

$$\rho_{1,2,\dots,N-1}^G = |\phi_0^G\rangle\langle\phi_0^G| + |\phi_1^G\rangle\langle\phi_1^G|. \tag{48}$$

where $|\phi_0^G\rangle, |\phi_1^G\rangle$ are as given in (46), (47) respectively. If we demand that an N qubit mixed state ω_N^G possesses the same $N - 1$ qubit reduced state (48) then, there should exist an extended pure state $|\Omega_{NE}^G\rangle$ of N qubits, appended with an environment E in such a way that $\text{Tr}_E[|\Omega_{NE}^G\rangle\langle\Omega_{NE}^G|] = \omega_N^G$ and

$$|\Omega_{NE}^G\rangle = |\phi_0^G\rangle|E_0^G\rangle + |\phi_1^G\rangle|E_1^G\rangle. \tag{49}$$

The states $|E_0^G\rangle, |E_1^G\rangle$ are comprised of the N th qubit and the environment

$$\begin{aligned}
 |E_0^G\rangle &= |0_N\rangle |e_{00}^G\rangle + |1_N\rangle |e_{01}^G\rangle \\
 |E_1^G\rangle &= |0_N\rangle |e_{10}^G\rangle + |1_N\rangle |e_{11}^G\rangle
 \end{aligned}$$

and they obey the orthonormality relations,

$$\langle E_i^G | E_j^G \rangle = \delta_{i,j}. \tag{50}$$

The extended pure state (49) takes the following form:

$$|\Omega_{NE}^G\rangle = |\phi_0^G\rangle|0_N\rangle |e_{00}^G\rangle + |\phi_0^G\rangle|1_N\rangle |e_{01}^G\rangle + |\phi_1^G\rangle|0_N\rangle |e_{10}^G\rangle + |\phi_1^G\rangle|1_N\rangle |e_{11}^G\rangle. \tag{51}$$

Having ascertained that $|D_{N-k,k}^G\rangle$ and ω_N^G possess a common reduced density matrix $\rho_{1,2,\dots,N-1}^G$, we now impose that another $N-1$ qubit reduced density matrix $\rho_{2,3,\dots,N}$ of $|D_{N-k,k}^G\rangle$ too is shared by ω_N^G , or equivalently on $|\Omega_{NE}^G\rangle$. To verify the restrictions on ω_N^G , or on $|\Omega_{NE}^G\rangle$ due to this, we compare the matrix elements of $\rho_{2,3,\dots,N} = \text{Tr}_1 [|D_{N-k,k}^G\rangle\langle D_{N-k,k}^G|]$ with that obtained by tracing the 1st qubit, environment from $|\Omega_{NE}^G\rangle$.

1. We first compare the following matrix elements:

$$\begin{aligned} \langle 0_2, 0_3, \dots, 0_{N-k-1}, 1_{N-k}, \dots, 1_N | \text{Tr}_{1,E} [|\Omega_{NE}^G\rangle\langle \Omega_{NE}^G|] | 0_2, 0_3, \dots, \\ 0_{N-k-1}, 1_{N-k}, \dots, 1_N \rangle &= |\alpha_k|^2 |a_{(N-1)C_k}^{(k)}|^2 \langle e_{01}^G | e_{01}^G \rangle, \\ \langle 0_2, 0_3, \dots, 0_{N-k-1}, 1_{N-k}, \dots, 1_N | \text{Tr}_1 [|D_{N-k,k}^G\rangle\langle D_{N-k,k}^G|] | 0_2, 0_3, \dots, \\ 0_{N-k-1}, 1_{N-k}, \dots, 1_N \rangle &= 0. \end{aligned} \quad (52)$$

Let us suppose that the coefficient $|a_{(N-1)C_k}^{(k)}|$ is non-zero. We may then deduce that $|e_{01}^G\rangle \equiv 0$ (note that $\alpha_k \neq 0$). The orthonormality relations $\langle E_0^G | E_0^G \rangle = 1$, $\langle E_0^G | E_1^G \rangle = 0$ would then lead to

$$\langle e_{00}^G | e_{00}^G \rangle = 1, \quad \langle e_{00}^G | e_{10}^G \rangle = 0. \quad (53)$$

2. Comparing yet another matrix element of $\rho_{2,3,\dots,N}$ from both the pure states $|D_{N-k,k}^G\rangle$ and $|\Omega_{NE}^G\rangle$ (see Eqs. (44)–(47), (51)), we obtain,

$$\begin{aligned} \langle 0_2, 0_3, \dots, 0_{N-k}, 1_{N-k+1}, \dots, 1_N | \text{Tr}_{1,E} [|\Omega_{NE}^G\rangle\langle \Omega_{NE}^G|] | 0_2, 0_3, \dots, \\ 0_{N-k-1}, 1_{N-k}, \dots, 1_{N-1}, 0_N \rangle &= |\alpha_k|^2 a_{NC_k}^{(k)} a_{N-1C_k}^{(k)*} \langle e_{11}^G | e_{00}^G \rangle, \\ \langle 0_2, 0_3, \dots, 0_{N-k}, 1_{N-k+1}, \dots, 1_N | \text{Tr}_1 [|D_{N-k,k}^G\rangle\langle D_{N-k,k}^G|] | 0_2, 0_3, \dots, \\ 0_{N-k-1}, 1_{N-k}, \dots, 1_{N-1}, 0_N \rangle &= |\alpha_k|^2 a_{NC_k}^{(k)} a_{N-1C_k}^{(k)*}. \end{aligned} \quad (54)$$

As $|a_{(N-1)C_k}^{(k)}| \neq 0$, and assuming that $a_{NC_k}^{(k)} \neq 0$, (54) results in the condition,

$$\langle e_{11}^G | e_{00}^G \rangle = 1. \quad (55)$$

This, together with the relation $\langle e_{00}^G | e_{00}^G \rangle = 1$ yields

$$|e_{11}^G\rangle \equiv |e_{00}^G\rangle. \quad (56)$$

It is not difficult to see that

$$\begin{aligned}
 \langle E_1^G | E_1^G \rangle = 1 &\Rightarrow \langle e_{10}^G | e_{10}^G \rangle + \langle e_{11}^G | e_{11}^G \rangle = 1 \\
 &\Rightarrow \langle e_{10}^G | e_{10}^G \rangle + \langle e_{00}^G | e_{00}^G \rangle = 1 \\
 &\Rightarrow \langle e_{10}^G | e_{10}^G \rangle = 0 \quad (\text{as } \langle e_{00}^G | e_{00}^G \rangle = 1) \\
 &\Rightarrow |e_{10}^G\rangle = 0.
 \end{aligned}
 \tag{57}$$

Finally, on substituting $|e_{01}^G\rangle \equiv |e_{10}^G\rangle = 0$, $|e_{00}^G\rangle \equiv |e_{11}^G\rangle = 1$ in (51), we get the explicit form of the state $|\Omega_{NE}^G\rangle$ as,

$$\begin{aligned}
 |\Omega_{NE}^G\rangle &= |\phi_0^G\rangle|0_N\rangle |e_{00}^G\rangle + |\phi_1^G\rangle|1_N\rangle |e_{00}^G\rangle \\
 &= |D_{N-k,k}^G\rangle |e_{00}^G\rangle
 \end{aligned}
 \tag{58}$$

with $|D_{N-k,k}^G\rangle = |\phi_0^G\rangle|0\rangle_N + |\phi_1^G\rangle|1\rangle_N$ (see (45)).

We thus come to the conclusion, by employing only *two* of the $N - 1$ qubit reduced density matrices, that the generalized states $|D_{N-k,k}^G\rangle$ are uniquely determined by their $N - 1$ party marginals.

It is important to note here that while the above result perfectly holds good for the class of states $\{|D_{N-k,k}^G\rangle\}$ of Eq. (44) when all the coefficients α_r , $a_i^{(r)}$, $i = 0, 1, 2, \dots, N - C_r$, $r = 0, 1, \dots, k$ are non-zero, it is valid if at least the coefficients $a_{N-C_k}^{(k)}$ and $a_{N-1-C_k}^{(k)}$, in Eq. (44) are non-zero (because the matrix elements of $N - 1$ qubit reduced states given in Eqs. (52), (54) vanish if the coefficients $a_{N-C_k}^{(k)}$ and $a_{N-1-C_k}^{(k)}$ are zero and therefore the inferences drawn from the Eqs. (52), (54) do not hold good in such cases). Based on the possibility of different choices of $N - 1$ qubit reduced matrix elements to arrive at the same result (58), we arrive at the conclusion that a unique specification of the generalized class of states $\{D_{N-k,k}^G\}$ —using only *two* of their $N - 1$ qubit marginals—is possible provided both the conditions given below are satisfied:

- among the set of coefficients $\{a_s^{(k)}, s = 1, 2, \dots, N-1 - C_k\}$ (see Eq. (44)) for a given $k=0,1,2,\dots,[N/2]$, at least one coefficient—which *does not* contain $|1\rangle$ in the first qubit position—is non zero.
- among the remaining coefficients in $\{a_{s'}^{(k)}, s' = N-1 - C_k + 1, N-1 - C_k + 2 \dots N - C_k\}$ at least one coefficient—with its first qubit position *not occupied* by $|1\rangle$ —is non-vanishing.

Excluding the class of states not obeying the above two conditions, all other states belonging to the generalized class of states $\{D_{N-k,k}^G\}$ belong uniquely to their $N - 1$ party marginals. It is illuminating to note that in spite of the generality of this class of non-symmetric states, only *two* of the $N - 1$ qubit marginals suffice for their unique determination.

5 Geometric measure of entanglement

Quantification of multiparty entanglement forms one of the central themes underlying quantum information theory. Several entanglement measures have been proposed in this context [53, 54], though they suffer because of the optimization involved in their evaluation. Natural strategy towards this end has been to restrict to certain class of states obeying specific symmetries in order to carry out such optimization procedures.

Let us consider the widely employed geometric measure of entanglement [24, 25] associated with a multiparty pure state $|\psi\rangle$:

$$E_G(|\psi\rangle) = 1 - \max_{\{\epsilon^{\text{prod}}\}} |\langle \epsilon^{\text{prod}} | \psi \rangle|^2 \quad (59)$$

where $\{\epsilon^{\text{prod}}\}$ is the set of *all* pure separable (product) states. Another equivalent quantification of the geometric measure [21] is given by,

$$\mathcal{E}_G(\psi) = -\log_2 \left[\max_{\{\epsilon^{\text{prod}}\}} |\langle \epsilon^{\text{prod}} | \psi \rangle|^2 \right]. \quad (60)$$

For N qubit GHZ states $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}[|0, 0, \dots, 0\rangle + |1, 1, \dots, 1\rangle]$ the geometric measure $E_G(|\text{GHZ}\rangle) = \frac{1}{2}$ and the logarithmic geometric measure $\mathcal{E}_G(|\text{GHZ}\rangle) = 1$ —independent of the number of qubits [25]. The geometric measure for the Dicke states (5) is found to be [25]

$$E_G\left(\left|\frac{N}{2}, \frac{N}{2} - l\right\rangle\right) = 1 - {}^N C_l \left(\frac{l}{N}\right)^l \left(\frac{N-l}{N}\right)^{N-l}, \quad (61)$$

which takes its maximum value when l is closest to $N/2$.

The optimization procedure in evaluating the geometric measure (59) is non-trivial in the case of general multiparty states. In this connection, a great deal of attention has been drawn to address the question: “Is the closest separable state of an arbitrary symmetric multiparty state $|\psi_{\text{sym}}\rangle$ itself a symmetric product state?” [25, 55–57]. It is only very recently [58] that it has been established that the optimal state (closest separable state) maximizing the geometric measure $E_G(|\psi\rangle)$ of (59) is necessarily symmetric for three or more party states obeying exchange symmetry [57]. This identification amounts to considerable simplification in the evaluation of the geometric measure of entanglement of pure permutation symmetric multiqubit states.

The closest product state associated with the Dicke states (5) leading to optimization of the geometric measure is found to be [55]

$$|\epsilon_{N,l}^{\text{prod}}\rangle = \left(\sqrt{\frac{N-l}{N}} |0\rangle + \sqrt{\frac{l}{N}} |1\rangle \right)^{\otimes N}. \quad (62)$$

and this yields the amount of geometric entanglement given by (61).

The MR (1) of symmetric multiqubit states is very useful to obtain a simplified structure for the geometric measure of entanglement. Substituting Eq. (1) into (59) and considering that the maximization is only required over the set of symmetric separable states $|\epsilon, \epsilon, \dots, \epsilon\rangle$ leads to

$$E_G(|\Psi_{\text{sym}}\rangle) = 1 - \mathcal{N}^2 N!^2 \max_{\{|\epsilon\rangle\}} \prod_{i=1}^N |\langle \epsilon | \epsilon_i \rangle|^2 \quad (63)$$

We recall that MR maps every permutation symmetric state $|\Psi_{\text{sym}}\rangle$ to N points on the unit sphere (these points are referred to as the Majorana points (MP) [21]). The point on the Majorana sphere corresponding to the state $|\epsilon\rangle$ which optimizes the geometric measure $E_G(|\Psi_{\text{sym}}\rangle)$ in (63) is called the closest product point (CPP) [21].

Aulbach et al. [21] evaluated the geometric measure of entanglement for some well-known two and three qubit symmetric states by making use of the MR as follows: Any identical local unitary operation on each spinor of the two qubit state $|\Psi_{\text{sym}}\rangle = \mathcal{N} [|\epsilon_1, \epsilon_2\rangle + |\epsilon_2, \epsilon_1\rangle]$ is equivalent to a rotation of MPs about a common axis on the Majorana sphere. Making use of identical local unitary transformation, a given distribution of two MPs can be rotated on the Majorana sphere in such a way that both the points lie in the positive hemisphere. In other words, the two spinors constituting a two-qubit symmetric state can be rotated so as to obtain $|\epsilon'_1\rangle = |0\rangle$ and $|\epsilon'_2\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} |1\rangle$, $0 \leq \theta \leq \pi$. The closest separable state of the two qubit symmetric state constituted by the spinors $|\epsilon'_1\rangle$, $|\epsilon'_2\rangle$ is identified as $|\epsilon\rangle = \cos \frac{\theta}{4} |0\rangle + \sin \frac{\theta}{4} |1\rangle$. (i) When $\theta = 0$, one gets the separable state $|0, 0\rangle$ and the geometric measure (see (63) vanishes. (ii) Choosing $\theta = \pi$, one obtains the Bell state $|\Psi^+\rangle = \frac{1}{\sqrt{2}} (|0, 1\rangle + |1, 0\rangle)$. The corresponding closest separable state is given by, $|\epsilon\rangle = \frac{1}{\sqrt{2}} [|0\rangle + |1\rangle]$ (note that the entire set of states $|0\rangle + e^{i\phi} |1\rangle$, which form a continuous ring around the equator on the Majorana sphere correspond to closest separable states of the Bell state $|\Psi^+\rangle$) and the geometric measure of entanglement is given by $E_G(|\Psi^+\rangle) = \frac{1}{2}$ (the logarithmic geometric measure $\mathcal{E}_G(|\Psi^+\rangle) = 1$).

The Majorana spinors constituting the 3 qubit GHZ state are given by (see Table 1) $|\epsilon_1\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$, $|\epsilon_2\rangle = \frac{1}{\sqrt{2}} (|0\rangle + e^{2i\pi/3} |1\rangle)$, $|\epsilon_3\rangle = \frac{1}{\sqrt{2}} (|0\rangle + e^{4i\pi/3} |1\rangle)$ (upto an overall phase); the CPP states are identified [21] to be $|0\rangle, |1\rangle$. Thus, the geometric measure of entanglement is readily evaluated to obtain $E_G(|\text{GHZ}\rangle) = \frac{1}{2}$ and the associated logarithmic measure (see (60) $\mathcal{E}_G(|\text{GHZ}\rangle) = 1$.

The Majorana spinors associated with the 3-qubit W state $|W\rangle = \frac{1}{\sqrt{3}} [|0, 0, 1\rangle + |0, 1, 0\rangle + |1, 0, 0\rangle]$ are given by (see Table 1) $|\epsilon_1\rangle = |0\rangle$, $|\epsilon_2\rangle = \epsilon_3 = |1\rangle$ and the CPP state is given by $|\epsilon\rangle = \sqrt{\frac{2}{3}} |0\rangle + \sqrt{\frac{1}{3}} |1\rangle$. A continuous ring given by $\sqrt{\frac{2}{3}} |0\rangle + e^{i\phi} \sqrt{\frac{1}{3}} |1\rangle$ forms a set of CPP states [21] of the three qubit W state, entanglement of which is therefore found to be $E_G(|W\rangle) = \frac{5}{9}$.

An approach to evaluate the geometric measure of entanglement of symmetric multiqubit states (by identifying CPP states geometrically) and to identify maximally entangled symmetric states—by exploiting the symmetries of the MP distribution—has been discussed in detail in Ref. [21]. Further, an improved asymptotic trend of the geometric measure of maximally entangled symmetric states (compared to that of highly entangled Dicke states (5) with $l = [N/2]$) has also been identified [22].

Instead of identifying CPP states geometrically [21], we propose to use the collective representation of the symmetric states as follows: The set of all symmetric N -qubit product states $\{|\phi_{\text{sym}}^{\text{prod}}\rangle = |\epsilon, \epsilon, \dots, \epsilon\rangle\}$ can be collectively represented as the spin coherent states [59] i.e.,

$$\begin{aligned} |\phi_{\text{sym}}^{\text{prod}}\rangle &\equiv |\alpha, \beta\rangle = e^{\tau J_+ - \tau^* J_-} \left| \frac{N}{2}, -\frac{N}{2} \right\rangle \\ &= \sum_{r=0}^N \sqrt{{}^N C_r} \left(\cos \frac{\beta}{2} \right)^r \left(\sin \frac{\beta}{2} \right)^{N-r} e^{-i(N-r)\alpha} \left| \frac{N}{2}, \frac{N}{2} - r \right\rangle. \end{aligned} \quad (64)$$

where $\tau = \frac{\beta}{2} e^{i\alpha}$, $0 \leq \alpha \leq 2\pi$, $0 \leq \beta \leq \pi$; $J_{\pm} = \frac{1}{2} \sum_{i=1}^N \sigma_{i\pm}$ are the collective spin ladder operators, and $\sigma_{i\pm} = \sigma_{ix} \pm \sigma_{iy}$ denote the Pauli operators of the i th qubit. Employing this collective spin coherent state representation of symmetric product states, we may express the geometric measure of entanglement for permutation symmetric states (6) as,

$$E_G(|\Psi_{\text{sym}}\rangle) = 1 - \max_{\{\alpha, \beta\}} F(\alpha, \beta), \quad (65)$$

$$F(\alpha, \beta) = \left| \sum_l c_l \left\langle \alpha, \beta \left| \frac{N}{2}, \frac{N}{2} - l \right\rangle \right|^2. \quad (66)$$

Here, the optimization is done over the set of angles α, β of the collective spin coherent states (64)—thus offering a novel method for evaluating the geometric entanglement of symmetric states.

For example, let us consider the Dicke states Eq. (5): We simplify the maximum value of $F(\alpha, \beta)$ as follows:

$$\begin{aligned} \max_{\{\alpha, \beta\}} F(\alpha, \beta) &= \max_{\{\alpha, \beta\}} \left| \left\langle \alpha, \beta \left| \frac{N}{2}, \frac{N}{2} - l \right\rangle \right|^2 \\ &= \max_{\{\beta\}} F_{N,l}(\beta) = \max_{\{\beta\}} \left[{}^N C_l \left(\cos \frac{\beta}{2} \right)^{2l} \left(\sin \frac{\beta}{2} \right)^{2(N-l)} \right]. \end{aligned} \quad (67)$$

In order to obtain the maximum value of the function $F_{N,l}(\beta)$ in Eq. (67) we consider $\log [F_{N,l}(\beta)]$ and set $\left. \frac{d \log [F_{N,l}(\beta)]}{d\beta} \right|_{\beta_M} = 0$. We obtain,

$$\begin{aligned} \left. \frac{d \log [F_{N,l}(\beta)]}{d\beta} \right|_{\beta_M} &= -l \tan \frac{\beta_M}{2} + (N-l) \cot \frac{\beta_M}{2} = 0 \\ \Rightarrow \tan \frac{\beta_M}{2} &= \pm \sqrt{\frac{N-l}{l}}. \end{aligned} \quad (68)$$

We thus obtain,

$$F_{N,l}(\beta_M) = {}^N C_l \left(\frac{l}{N}\right)^l \left(\frac{N-l}{N}\right)^{N-l} \quad (69)$$

which readily leads to the geometric measure of entanglement (61) of Dicke states and also to the identification of their closest product states (62).

For the N -qubit GHZ state

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}} \left[\left| \frac{N}{2}, \frac{N}{2} \right\rangle + \left| \frac{N}{2}, -\frac{N}{2} \right\rangle \right] \quad (70)$$

we obtain

$$\begin{aligned} \max_{\{\alpha, \beta\}} F_{\text{GHZ}}(\alpha, \beta) &= \max_{\{\alpha, \beta\}} |\langle \alpha, \beta | \text{GHZ} \rangle|^2 = \frac{1}{2} \left[\left(\sin \frac{\beta}{2} \right)^{2N} + \left(\cos \frac{\beta}{2} \right)^{2N} \right. \\ &\quad \left. + 2 \left(\cos \frac{\beta}{2} \sin \frac{\beta}{2} \right)^N \cos(N\alpha) \right]. \end{aligned} \quad (71)$$

Optimal value of the function $F_{\text{GHZ}}(\alpha, \beta)$ is obtained for $\alpha_M = \text{arbitrary}$, $\beta_M = 0$ leading to $F_{\text{GHZ}}(\alpha_M, \beta_M) = \frac{1}{2}$ —in agreement with the earlier result [25]. Note that the geometric measure of entanglement of GHZ state is less than that of the Dicke state (5) with $l = [N/2]$, indicating different hierarchies of multiparticle entanglement (depending on the nature of the measure employed). It would be interesting to explore expansions of N -particle symmetric states in terms of p -particle constituents [60] (in particular, those with $p = 2$ are called geminal expansions) in order to recognize genuine multiparticle entanglement in a physically significant manner.

6 Summary

This article presents a detailed description of the Majorana geometrical representation of symmetric multiqubit states. With the help of the MR, the SLOCC entanglement classification of pure symmetric states in terms of the number and arrangement of the distinct Majorana spinors constituting them is elucidated. Further, uniqueness of the whole pure symmetric N -qubit states belonging to the two distinct spinor family—and also, another related class of non-symmetric states—to their $N - 1$ qubit reduced density matrices is established (by employing only two of the reduced states). Quantification of multiqubit entanglement of permutation symmetric states in terms of geometric measure of entanglement (where the MR has been employed extensively) is detailed.

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References

1. Sackett, C.A. et al.: Experimental entanglement of four particles. *Nature (London)* **404**, 256–259 (2000)
2. Roos, C.F., Riebe, M., Häffner, H., Häsnel, W., Benhelm, J., Lancaster, G.P.T., Becher, C., Schmidt-Kaler, F., Blatt, R.: Control and measurement of three-qubit entangled states. *Science* **304**, 1478–1480 (2004)
3. Leibfried, D. et al.: Creation of a six-atom ‘Schrodinger Cat’ state. *Nature (London)* **438**, 639–642 (2005)
4. Sorensen, A., Duan, L.-M., Cirac, J.I., Zoller, P.: Many-particle entanglement with Bose-Einstein condensates. *Nature (London)* **409**, 63–66 (2001)
5. Usha Devi, A.R., Prabhu, R., Rajagopal, A.K.: Characterizing multiparticle entanglement in symmetric N-qubit states via negativity of covariance matrices. *Phys. Rev. Lett.* **98**, 060501-060504 (2007)
6. Greenberger, D.M., Horne, M.A., Shimony, A., Zeilinger, A.: Bell’s theorem without inequalities. *Am. J. Phys.* **58**, 1131–1143 (1990)
7. Dicke, R.H.: Coherence in spontaneous radiation processes. *Phys. Rev.* **93**, 99–110 (1954)
8. Majorana, E.: Atomi orientati in campo magnetico variabile. *Nuovo Cimento* **9**, 43–50 (1932)
9. Bloch, F., Rabi, I.I.: Atoms in variable magnetic fields. *Rev. Mod. Phys.* **17**, 237–244 (1945)
10. Penrose, R.: *Shadows of the Mind*. Oxford University Press, Oxford (1994)
11. Mäkelä, H., Messina, A.: N-qubit states as points on the Bloch sphere. *Physica Scripta T* **140**, 014054 (2010)
12. Dennis, M.R.: Canonical representation of spherical functions: Sylvester’s theorem, Maxwell’s multipoles and Majorana’s sphere. *J. Phys. A. Math. Gen.* **37**, 9487–9500 (2004)
13. Swain, J.: The MR of spins and the relation between $SU(\infty)$ and $SDiff(S^2)$, hep-th/arxiv:0405004
14. Dennis, M.R.: Correlations between Maxwell’s multipoles for Gaussian random functions on the sphere. *J. Phys. A. Math.Gen.* **38**, 1653–1658 (2005)
15. Zimba, J.: “Anticoherent” spin states via the Majorana representation. *EJTP* **3**, 143–156 (2006)
16. Bastin, T., Krins, S., Mathonet, P., Godefroid, M., Lamata, L., Solano, E.: Operational families of entanglement classes for symmetric N-qubit states. *Phys. Rev. Lett.* **103**, 070503 (2009)
17. Bastin, T., Mathonet, P., Solano, E.: Operational entanglement families of symmetric mixed N-qubit states, quant-ph/arxiv:1011.1243
18. Usha Devi, A.R., Sudha, Rajagopal, A.K.: Determining the whole pure symmetric N-qubit state from its parts, quant-ph/arXiv:1003.2450
19. Usha Devi, A.R., Sudha, Rajagopal, A.K.: Interconvertibility and irreducibility of permutation symmetric three qubit pure states, quant-ph/arXiv:1002.2820
20. Markham, D.: Entanglement and symmetry in permutation symmetric states, quant-ph/arxiv:1001.0343
21. Aulbach, M., Markham, D., Murao, M.: The maximally entangled symmetric state in terms of the geometric measure. *New J. Phys.* **12**, 073025 (2010)
22. Martin, J., Giraud, O., Braun, P.A., Braun, D., Bastin, T.: Multiqubit symmetric states with high geometric entanglement. *Phys. Rev. A* **81**, 062347 (2010)
23. Aulbach, M., Markham, D., Murao, M.: Geometric entanglement of symmetric states and the Majorana representation, quant-ph/arxiv: 1010.4777
24. Shimony, A.: Degree of entanglement. *Ann. Phys. NY Acad. Sci.* **755**, 675 (1995)
25. Wei, T.C., Ericsson, M., Goldbart, P.M.: Geometric measure of entanglement and applications to bipartite and multipartite quantum states. *Phys. Rev. A* **68**, 042307 (2003)
26. Rose, M.E.: *Elementary Theory of Angular Momentum*. Wiley, New York (1957)
27. Dür, W., Vidal, G., Cirac, J.I.: Three qubits can be entangled in two inequivalent ways. *Phys. Rev. A* **62**, 062314 (2000)
28. Verstraete, F., Dehaene, J., De Moor, B., Verschelde, H.: Four qubits can be entangled in nine different ways. *Phys. Rev. A* **65**, 052112 (2002)
29. Lamata, L., León, J., Salgado, D., Solano, E.: Inductive entanglement classification of four qubits under stochastic local operations and classical communication. *Phys. Rev. A* **75**, 022318 (2007)
30. Maser, A.A., Wiegner, R., Schilling, U., Thiel, C., von Zanthier, J.: A versatile source of polarization-entangled photons, quant-ph/arXiv:0911.5115
31. Kiesel, N., Wieczorek, W., Krins, S., Bastin, T., Weinfurter, H., Solano, E.: Operational multipartite entanglement classes for symmetric photonic qubit states. *Phys. Rev. A* **81**, 032316 (2010)

32. Mathonet, P., Krins, S., Godefroid, M., Lamata, L., Solano, E., Bastin, T.: Entanglement equivalence of N -qubit symmetric states. *Phys. Rev. A* **1**, 052315 (2010)
33. Nielsen, M., Chuang, I.: *Quantum Computation and Quantum Information*. Cambridge University Press, Cambridge (2000)
34. Linden, N., Popescu, S., Wootters, W.K.: Almost every pure state of three qubits is completely determined by its two-particle reduced density matrices. *Phys. Rev. Lett.* **89**, 207901 (2002)
35. Linden, N., Wootters, W.K.: The parts determine the whole in a generic pure quantum state. *Phys. Rev. Lett.* **89**, 277906 (2002)
36. Jones, N.S., Linden, N.: Parts of quantum states. *Phys. Rev. A* **71**, 012324 (2005)
37. Walck, S.N., Lyons, D.W.: Only n -qubit Greenberger-Horne-Zeilinger states are undetermined by their reduced density matrices. *Phys. Rev. Lett.* **100**, 050501 (2008)
38. Walck, S.N., Lyons, D.W.: Only n -qubit Greenberger-Horne-Zeilinger states contain n -partite information. *Phys. Rev. A* **79**, 032326 (2009)
39. Parashar, P., Rana, S.: N -qubit W states are determined by their bipartite marginals. *Phys. Rev. A* **80**, 012319 (2009)
40. Parashar, P., Rana, S.: Reducible correlations in Dicke states. *J. Phys. A: Math. Theor.* **42**, 462003 (2009)
41. Coleman, A.J.: Structure of fermion density matrices. *Rev. Mod. Phys.* **35**, 668 (1963)
42. Colmenero, F., Pérezdel Valle, C., Valdemoro, C.: Approximating q -order reduced density matrices in terms of the lower order ones I general relations. *Phys. Rev. A* **47**, 971–978 (1993)
43. Nakatsuji, H., Yasuda, K.: Direct determination of the quantum-mechanical density matrix using the density equation. *Phys. Rev. Lett.* **76**, 1039 (1996)
44. Yasuda, K., Nakatsuji, H.: Direct determination of the quantum-mechanical density matrix using the density equation II. *Phys. Rev. A* **56**, 2648–2657 (1997)
45. Mazziotti, D.A.: Contracted Schrödinger equation: determining quantum energies and two-particle density matrices without wave functions. *Phys. Rev. A* **57**, 4219–4234 (1998)
46. Mazziotti, D.A.: Pursuit of N -representability for the contracted Schrödinger equation through density-matrix reconstruction. *Phys. Rev. A* **60**, 3618–3626 (1999)
47. Linden, N., Popescu, S., Schumacher, B., Westmoreland, M.: Reversibility of local transformations of multipartite entanglement Preprint quant-ph/9912039
48. Bennett, C.H., Popescu, S., Rohrlich, C., Smolin, J.A., Thapliyal, A.V.: Exact and asymptotic measures of multipartite pure-state entanglement. *Phys. Rev. A* **63**, 012307 (2000)
49. Wootters, W.K.: Entanglement of formation of an arbitrary state of two qubits. *Phys. Rev. Lett.* **80**, 2245 (1998)
50. Hill, S., Wootters, W.K.: Entanglement of a pair of quantum bits. *Phys. Rev. Lett.* **78**, 5022 (1997)
51. Rajagopal, A.K., Rendell, R.: Robust and fragile entanglement of three qubits: relation to permutation symmetry. *Phys. Rev. A* **65**, 032328 (2002)
52. Coffman, V., Kundu, J., Wootters, W.K.: Distributed entanglement. *Phys. Rev. A* **61**, 052306 (2000)
53. Plenio, M., Virmani, S.: An introduction to entanglement measures. *Quantum Inf. Comput.* **7**, 1–51 (2007)
54. Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. *Rev. Mod. Phys.* **81**, 865–942 (2009)
55. Hayashi, M., Markham, D., Murao, M., Owari, M., Virmani, S.: Entanglement of multiparty-stabilizer, symmetric and antisymmetric states. *Phys. Rev. A* **77**, 012104 (2008)
56. Hayashi, M., Markham, D., Murao, M., Owari, M., Virmani, S.: The geometric measure of entanglement for a symmetric pure state with non-negative amplitudes. *J. Math. Phys.* **50**, 122104 (2009)
57. Wei, T.-C., Severini, S.: Matrix permanent and quantum entanglement of permutation invariant states. *J. Math. Phys.* **51**, 092203 (2010)
58. Hübener, R., Kleinmann, M., Wei, T.-C., Guillén, C.G., Gühne, O.: Geometric measure of entanglement for symmetric states. *Phys. Rev. A* **80**, 032324 (2009)
59. Arecchi, F.T., Courtens, E., Gilmore, G., Thomas, H.: Atomic coherent states in quantum optics. *Phys. Rev. A* **6**, 2211 (1972)
60. Ruskai, M.B.: N completeness, N representability, and Geminal expansions. *Phys. Rev. A* **5**, 1336 (1972)