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Effect of Thermal Modulation on the Onset of Convection in Walters B Viscoelastic Fluid-Saturated Porous Medium

I. S. Shivakumara · Jinho Lee · M. S. Malashetty · S. Sureshkumar

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Abstract The linear stability of Walters B viscoelastic fluid-saturated horizontal porous layer is examined theoretically when the walls of the porous layer are subjected to time-periodic temperature modulation. Three types of boundary temperature modulations are considered namely, symmetric, asymmetric, and only the lower wall temperature is modulated while the upper wall is held at constant temperature. A regular perturbation method based on small amplitude of applied temperature field is used to compute the critical values of Rayleigh number and the corresponding wave number. The shift in critical Rayleigh number is calculated as a function of modulation frequency, viscoelastic parameter, and Prandtl number. The effect of all three types of modulations is found to be destabilizing as compared to the unmodulated system. This result is in contrast to the system with other types of fluids. Besides, the influence of physical parameters on the control of convective instability of the system is discussed.

Keywords Thermal modulation · Porous medium · Walters B viscoelastic fluid · Convection

I. S. Shivakumara · J. Lee (✉)
School of Mechanical Engineering, Yonsei University, Seoul 120-749, South Korea
e-mail: jinholee@yonsei.ac.kr

I. S. Shivakumara
Department of Mathematics, UGC-CAS in Fluid Mechanics, Bangalore University, Bangalore 560001, India
e-mail: shivakumarais@gmail.com

M. S. Malashetty
Department of Mathematics, Gulbarga University, Gulbarga 585106, India
e-mail: malashettyms@yahoo.com

S. Sureshkumar
Department of Mathematics, Siddaganga Institute of Technology, Tumkur 572103, India

List of Symbols

A_{h}	Ratio of heat capacities
a	Horizontal wavenumber
c	Specific heat
c_{p}	Specific heat at constant pressure
Da	Darcy number, k/d^2
d	Thickness of the fluid layer
f	Modulation temperature gradient
$\vec{g}$	Gravitational acceleration
l, m	Wave numbers in the x- and y-directions
M	Modified thermal capacity ratio, $A_{\mathrm{h}}/\varepsilon$
p	Pressure
Pr	Modified Prandtl number, $\nu\varepsilon^2/\kappa$
R	Rayleigh number, $\alpha g \Delta T d^3/\nu\kappa$
$\vec{q}$	Velocity
T	Temperature
t	Time
(x, y, z)	Space co-ordinates

Greek Symbols

α	Volumetric expansion coefficient
ε	Porosity of the medium
$\bar{\varepsilon}$	Small amplitude of the thermal modulation
ρ	Density
ω	Frequency
φ	Phase angle
κ	Effective thermal diffusivity
μ	Viscosity
μ_{v}	Viscoelastic constant of Walters B liquid
∇_{h}^2	Horizontal Laplacian operator
Γ_{P}	Elastic parameter, $\mu_{\mathrm{v}}\varepsilon/\rho_0 d^2$

Subscripts/Superscripts

b	Basic state
c	Critical
0	Reference value
$*$	Dimensionless quantity

1 Introduction

Thermal convection in fluid-saturated porous media has attracted researchers in recent decades due to its relevance in a wide range of applications such as geothermal energy utilization, enhanced recovery of petroleum reservoirs, thermal insulation engineering, nuclear waste repository, grain storage, and mantle convection to mention a few. The growing volume of work in this area is well documented in the literature; see, for example, Bear (1988), Ingham and Pop (1998), Vafai (2000), Vafai (2005), and Nield and Bejan (2006).

There has been a growing interest in externally modulated hydrodynamic systems, both theoretically and experimentally. These systems may exhibit novel behavior in response to parametric forcing near a point of instability. Depending on the relative strength and rate of forcing, predictions exist for a variety of responses to the modulation. Among these are the upward or downward shifts of the convective threshold compared to the unmodulated problems. There are many works available in the literature, concerning how a time-periodic boundary temperature affects the onset of Rayleigh–Benard convection. Some of the findings related to this problem have been reviewed by Davis (1976). The studies related to the effect of thermal modulation on the onset of convection in a porous medium have also received equal importance (see, e.g., Nield and Bejan 2006).

The effect of time-dependent wall temperature on the onset of convection in a fluid-saturated porous medium has been studied by Caltagirone (1976) using the Darcy model for the momentum equation. Chhuon and Caltagirone (1976) have studied the stability of a fluid-saturated porous layer where the imposed temperature on the boundary is time-periodic, with a non-zero mean value. They performed experiments and compared their results with those obtained from Floquet theory. Rudraiah and Malashetty (1990b) investigated the stability of a fluid-saturated sparsely packed porous layer subject to time-periodic boundary temperature using the Brinkman model. They recovered the viscous flow results of Venezian (1969), as a special case when the value of the porous parameter tends to zero. Linear stability analysis of the onset of convection induced by a non-uniform time-dependent volumetric heating in a fluid-saturated porous medium has been studied by Nield (1995). Analytical expression that gives upper bounds on an appropriate critical Rayleigh number is derived. The effect of thermal modulation on the convection in a porous medium is studied by Malashetty and Wadi (1999) using the Brinkman model with effective viscosity larger than the fluid viscosity. Further, Malashetty and Basavaraja (2002, 2003, 2004) have examined the single and double diffusive convections in a fluid-saturated anisotropic porous layer subject to time-dependent wall temperature. Bhadauria (2007) has studied the effect of thermal modulation on the onset of convection in a layer of sparsely packed porous medium bounded by rigid boundaries. Recently, Bhadauria (2008) has included the effect of rotation, while Bhadauria and Aalam (2008) have included the effect of magnetic field to study the onset of convection in a porous medium with temperature modulation.

With the growing importance of viscoelastic fluids in modern technology and industries, the investigations of thermal convective instability in such fluids are desirable. In the asthenosphere and the deeper mantle, it is well known now that viscoelastic behavior is an important rheological process. The thermal convective instability in a viscoelastic fluid-saturated porous layer has been studied by several authors in the recent past. Kim et al. (2003) has dealt with the thermal instability driven by buoyancy forces in a horizontal porous layer saturated by a viscoelastic fluid. Yoon et al. (2004) have followed the formulation of Akhatov and Chembarisova (1993) and sought analytically the onset of thermal convection in an isothermally heated porous layer saturated with viscoelastic fluid. Shivakumara et al. (2006) and Malashetty et al. (2006a,b) have discussed, respectively, the effect of local thermal non-equilibrium on the onset of convection in a sparsely and densely packed Oldroyd-B viscoelastic fluid-saturated porous medium. Shivakumara and Sureshkumar (2007) have used linear stability theory to investigate convective instability in a horizontal porous layer saturated with viscoelastic fluid of Oldroyd-B type in the presence of vertical throughflow, and these authors have also extended their previous work to include the effect of quadratic drag in the presence of an additional diffusing component (Shivakumara and Sureshkumar 2008). Thermal stability of a viscoelastic Walters B liquid saturating a porous anisotropic horizontal layer in the presence of a chemical reaction has been discussed by Postelnicu (2007). Chaotic convection of viscoelastic fluid saturating a porous medium has been analyzed by Sheu et al. (2008).

Nonetheless, the studies related to the effect of thermal modulation on the onset of convection in a viscoelastic fluid-saturated porous medium have not received much attention. Chung Liu (2004) has examined the stability of a horizontally extended second-grade fluid layer heated from below subject to temperature modulation at walls. Rudraiah et al. (1990a) have investigated the effect of thermal modulation on the onset of convection in a viscoelastic fluid-saturated porous medium using Oldroyd model, and the effect of anisotropy on the problem has been analyzed by Malashetty et al. (2006a,b).

In the present study, however, the effect of thermal modulation on the onset of convection in a horizontal layer of porous medium saturated with another class of viscoelastic fluids, known as Walters B liquid (Walters 1964), is investigated. The boundary temperature modulation alters the basic temperature distribution from linear to nonlinear which helps in effective control of convective instability. The difficulty in dealing with such instability problems is that one has to solve time-dependent stability equations with variable coefficients, and to our knowledge no work has been initiated for such fluids in this direction. The resulting eigenvalue problem is solved by regular perturbation technique with amplitude of the temperature modulation as a perturbation parameter. In particular, it is shown that the onset of convection can be advanced by a proper tuning of the frequency of the boundary temperature modulation.

2 Mathematical Formulation

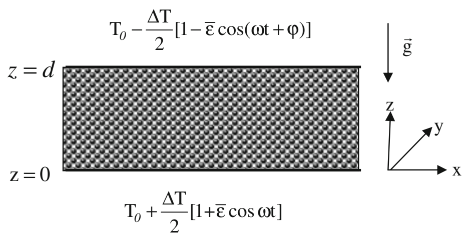
We consider a horizontal layer of Walters B viscoelastic fluid-saturated porous medium of thickness d in the presence of gravity as shown in Fig. 1. The time-dependent temperature of lower and upper surfaces of the porous layer is externally imposed and is given by

$$T = T_0 + \frac{1}{2}\Delta T\,(1 + \bar{\varepsilon}\cos\omega t) \quad \text{at } z = 0 \tag{1}$$

$$T = T_0 - \frac{1}{2}\Delta T\,[1 - \bar{\varepsilon}\cos(\omega t + \varphi)] \quad \text{at } z = d, \tag{2}$$

where $\bar{\varepsilon}$ represents a small amplitude of the thermal modulation, ω the frequency, φ the phase angle, and T_0 is the reference temperature. The time-dependent parts denote the modulation imposed on the adverse thermal gradient caused by the temperatures $T_0 + \Delta T/2$ and $T_0 - \Delta T/2$ at the lower and upper surfaces, respectively. A Cartesian coordinate system (x, y, z) is chosen such that the origin is at the bottom of the porous layer and z-axis is directed vertically upward.

Fig. 1 Physical configuration



The relevant basic equations are:

$$\nabla \cdot \vec{q} = 0 \tag{3}$$

$$\frac{\rho_0}{\varepsilon}\frac{\partial \vec{q}}{\partial t} = -\nabla p + \rho \vec{g} - \frac{1}{k}\left(\mu - \mu_{\mathrm{v}}\frac{\partial}{\partial t}\right)\vec{q} \tag{4}$$

$$A_{\mathrm{h}}\frac{\partial T}{\partial t} + (\vec{q}\cdot\nabla)\,T = \kappa\nabla^2 T \tag{5}$$

$$\rho = \rho_0\left\{1 - \alpha\,(T - T_0)\right\}, \tag{6}$$

where $\vec{q}$ is the velocity, k is the permeability of the porous medium, κ the effective thermal diffusivity, p the pressure, $\vec{g}$ is acceleration due to gravity, T the temperature, $A_{\mathrm{h}} = (\rho_{0\mathrm{c}})_{\mathrm{m}} / \left(\rho_0 c_p\right)_{\mathrm{f}} = \left[(1-\varepsilon)\,(\rho_0 c)_{\mathrm{s}} + \varepsilon\left(\rho_0 c_p\right)_{\mathrm{f}}\right] / \left(\rho_0 c_p\right)_{\mathrm{f}}$ the ratio of heat capacities of the fluid-saturated porous medium to that of the fluid, ε the porosity of the medium, c the specific heat, c_{p} the specific heat at constant pressure, α the volumetric expansion coefficient, μ the viscosity, μ_{v} the viscoelastic constant of Walters B liquid, and ρ_0 is the reference density. The subscripts m, s, and f refer to the porous medium, solid, and fluid, respectively.

2.1 Basic State

The basic state is quiescent and the temperature T_{b}, density ρ_{b}, and the pressure p_{b} satisfy

$$\rho_{\mathrm{b}}\vec{g} + \nabla p_{\mathrm{b}} = 0 \tag{7}$$

$$A_{\mathrm{h}}\frac{\partial T_{\mathrm{b}}}{\partial t} = \kappa\frac{\partial^2 T_{\mathrm{b}}}{\partial z^2}. \tag{8}$$

The solution of Eq. 8 satisfying the thermal conditions given by Eqs. 1 and 2 is

$$T_{\mathrm{b}} = T_1\,(z) + \bar{\varepsilon}T_2\,(z,t), \tag{9}$$

where

$$T_1\,(z) = \frac{\Delta T}{2}\left(1 - \frac{2z}{d}\right) \tag{10}$$

$$T_2\,(z,t) = Re\left[\left\{b\,(\lambda)\,e^{\lambda z/d} + b\,(-\lambda)\,e^{-\lambda z/d}\right\}e^{-i\omega t}\right] \tag{11}$$

with

$$\lambda = (1-i)\left(\frac{A_{\mathrm{h}}\omega d^2}{2\kappa}\right)^{1/2} \tag{12}$$

$$b\,(\lambda) = \left(\frac{\Delta T}{2}\,\frac{e^{-i\varphi} - e^{-\lambda}}{e^{\lambda} - e^{-\lambda}}\right) \tag{13}$$

and Re stands for the real part. The expression for p_{b} and ρ_{b} is not given as they are not explicitly required in the subsequent analysis.

3 Linear Stability Analysis

We give an infinitesimal disturbance to the basic state in the form

$$\vec{q} = \vec{q}\,', \quad T = T_{\mathrm{b}} + \theta', \quad p = p_{\mathrm{b}} + p', \tag{14}$$

where q', θ', and p' represent the perturbed quantities. Substituting Eq. 14 in Eq. 4, eliminating the pressure by operating curl twice, and retaining the vertical component, we get (after ignoring the primes).

$$\left\{\frac{1}{\varepsilon}\frac{\partial}{\partial t}+\frac{\mu}{k\rho_0}\left(1-\frac{\mu_{\mathrm{v}}}{\mu}\frac{\partial}{\partial t}\right)\right\}\nabla^2 w=\alpha g\nabla_{\mathrm{h}}^2\theta, \tag{15}$$

where $\nabla_{\mathrm{h}}^2=\partial^2/\partial x^2+\partial^2/\partial y^2$ is the horizontal Laplacian operator.

Substituting Eq. 14 in Eq. 5 and linearizing, we obtain (after ignoring the primes)

$$A_{\mathrm{h}}\frac{\partial T}{\partial t}=\kappa\nabla^2 T-\frac{\partial T_{\mathrm{b}}}{\partial z}w. \tag{16}$$

Non-dimensionalizing the equations by setting

$$\left(x^*,y^*,z^*\right)=\left(\frac{x}{d},\frac{y}{d},\frac{z}{d}\right),\quad T^*=\frac{T}{\Delta T},\quad w^*=\frac{w}{\kappa/d},\quad t^*=\frac{t}{d^2\varepsilon/\kappa} \tag{17}$$

and substituting in Eqs. 15 and 16, we obtain, respectively,

$$\left[\frac{1}{Pr}\frac{\partial}{\partial t}+Da^{-1}\left(1-\frac{\Gamma_{\mathrm{P}}}{Pr}\frac{\partial}{\partial t}\right)\right]\nabla^2 w-R\nabla_{\mathrm{h}}^2\theta=0 \tag{18}$$

$$\left(M\frac{\partial}{\partial t}-\nabla^2\right)\theta=-\frac{\partial T_{\mathrm{b}}}{\partial z}w, \tag{19}$$

where $R=\alpha g\Delta T d^3/\nu\kappa$ is the Rayleigh number, $Da=k/d^2$ is the Darcy number, $\Gamma_{\mathrm{P}}=\mu_{\mathrm{v}}\varepsilon/\rho_0 d^2$ is the elastic parameter, $Pr=\nu\varepsilon^2/\kappa$ is the modified Prandtl number, and $M=A_{\mathrm{h}}/\varepsilon$ is a modified thermal capacity ratio.

Equations 18 and 19 are to be solved subject to the boundary conditions

$$w=\theta=0\quad \text{at } z=0,1. \tag{20}$$

Eliminating T from Eq. 18 using Eq. 19, we obtain the following equation

$$\left(M\frac{\partial}{\partial t}-\nabla^2\right)\left[\frac{1}{Pr}\frac{\partial}{\partial t}+Da^{-1}\left(1-\frac{\Gamma_{\mathrm{P}}}{Pr}\frac{\partial}{\partial t}\right)\nabla^2 w\right]+R\frac{\partial T_{\mathrm{b}}}{\partial z}\nabla_{\mathrm{h}}^2 w=0. \tag{21}$$

The dimensionless basic temperature gradient is given by

$$\frac{\partial T_{\mathrm{b}}}{\partial z}=-1+\bar{\varepsilon}f. \tag{22}$$

Here, f is the modulation temperature gradient and is given by

$$f=Re\left[\left\{A(\lambda)e^{\lambda z}+A(-\lambda)e^{-\lambda z}\right\}e^{-i\omega t}\right],$$

where

$$A(\lambda)=\left(\frac{\lambda}{2}\frac{e^{-i\varphi}-e^{-\lambda}}{e^{\lambda}-e^{-\lambda}}\right)$$

$$\lambda=(1-i)\left(\frac{M\omega}{2}\right)^{1/2}. \tag{23}$$

4 Method of Solution

The aim of this section is to determine the eigenfunction w and the eigenvalue R of Eq. 21 from the basic temperature gradient given by Eq. 22 that departs from the linear profile $\partial T_b/\partial z = -1$ in modulated system by the quantities of the order $\bar{\varepsilon}$. It follows that the eigenfunction and the eigenvalue of the present problem differ from those associated with usual Darcy–Benard convection by quantities of order $\bar{\varepsilon}$. From Eq. 21, we can also see that when the temperature profile is linear, as far as stationary instability is concerned, the viscoelastic properties of the fluid have no effect on the onset of linear instability. We assume the solution of Eq. 21 in the form

$$(R, w) = (R_0, w_0) + \bar{\varepsilon}\,(R_1, w_1) + \bar{\varepsilon}^2\,(R_2, w_2) + \cdots, \tag{24}$$

where R_0 is the Rayleigh number corresponding to classical Darcy–Benard convection. Substituting Eq. 24 into Eq. 21 and equating different powers of $\bar{\varepsilon}$, we obtain the following system of equations:

$$Lw_0 = 0 \tag{25}$$

$$Lw_1 = R_1\nabla_{\mathrm{h}}^2 w_0 - R_0 f\nabla_{\mathrm{h}}^2 w_0 \tag{26}$$

$$Lw_2 = R_1\nabla_{\mathrm{h}}^2 w_1 + R_2\nabla_{\mathrm{h}}^2 w_0 - R_0 f\nabla_{\mathrm{h}}^2 w_1 - R_1 f\nabla_{\mathrm{h}}^2 w_0, \tag{27}$$

where

$$L = \left(M\frac{\partial}{\partial t} - \nabla^2\right)\left[\frac{1}{Pr}\frac{\partial}{\partial t} + Da^{-1}\left(1 - \frac{\Gamma_{\mathrm{P}}}{Pr}\frac{\partial}{\partial t}\right)\nabla^2\right] - R_0\nabla_{\mathrm{h}}^2.$$

We now assume the marginally stable solutions for (25) in the form

$$w_0^{(n)} = \sin(n\pi z)\exp\left[i\,(lx + my)\right], \quad n = 1, 2, 3, \ldots, \tag{28a}$$

where l and m are the wave numbers in the x- and y-directions, respectively, such that $l^2 + m^2 = a^2$. The corresponding eigenvalues are given by

$$R_0^{(n)} = \frac{Da^{-1}\left(n^2\pi^2 + a^2\right)^2}{a^2}. \tag{28b}$$

For a fixed value of the wave number a, the least eigenvalue occurs at $n = 1$ and is given by

$$R_0 = \frac{Da^{-1}\left(\pi^2 + a^2\right)^2}{a^2}. \tag{29}$$

We note that R_0 attains its critical value, $R_{0\mathrm{c}}$ at $a = a_{\mathrm{c}}$, where

$$R_{0\mathrm{c}} = 4\pi^2 Da^{-1} \tag{30}$$

$$a_{\mathrm{c}} = \pi. \tag{31}$$

The viscoelastic parameter is not appearing in the above expressions and these are the known exact values reported in the literature for a Newtonian fluid-saturated porous layer. Thus, we note that, as far as the steady state is concerned, there is no distinction between the viscoelastic and viscous fluid results. Eq. 26 is inhomogeneous and its solution poses a problem due to the presence of resonance terms. The solvability condition requires that time-independent part of the right-hand side of Eq. 26 should be orthogonal to w_0. The term independent of time on the right-hand side is $R_1\nabla_{\mathrm{h}}^2 w_0$ so that $R_1 = 0$. It follows that all the odd coefficients, i.e., R_1, R_3, ... in Eq. 24, must vanish. If we expand the right-hand side of Eq. 26 in a Fourier series of the form

$$e^{\lambda z}\sin(m\pi z)=\sum_{n=1}^{\infty} g_{nm}(\lambda)\sin(n\pi z) \tag{32}$$

then

$$g_{nm}(x\lambda)=2\int_0^1 e^{\lambda z}\sin(m\pi z)\sin(n\pi z)\,\mathrm{d}z=\frac{-4nm\pi^2\lambda\left[1+(-1)^{n+m+1}e^{\lambda}\right]}{\left[\lambda^2+(n+m)^2\pi^2\right]\left[\lambda^2+(n-m)^2\pi^2\right]}. \tag{33}$$

We thus obtain

$$L\left[\sin(n\pi z)\,e^{-i\omega t}\right]=L(\omega,n)\sin(n\pi z)\,e^{-i\omega t}, \tag{34}$$

where

$$\begin{aligned}L(\omega,n)=&\frac{\omega^2\left(n^2\pi^2+a^2\right)}{Pr}\left(1-Da^{-1}\Gamma_{\mathrm{P}}\right)-Da^{-1}\left(n^2\pi^2+a^2\right)^2+Da^{-1}\left(\pi^2+a^2\right)^2\\&+i\omega\left[Da^{-1}\left(n^2\pi^2+a^2\right)+\frac{\left(n^2\pi^2+a^2\right)^2}{Pr}-\frac{Da^{-1}\Gamma_{\mathrm{P}}\left(n^2\pi^2+a^2\right)^2}{Pr}\right].\end{aligned} \tag{35}$$

From Eq. 26, we have

$$Lw_1=R_0a^2\,\mathrm{Re}\left\{\sum_n\left[A(\lambda)\,g_{n1}(\lambda)+A(-\lambda)\,g_{n1}(-\lambda)\sin(n\pi z)\right]e^{-i\omega t}\right\}. \tag{36}$$

We obtain w_1, by inverting the operator L term by term, in the form

$$w_1=R_0a^2\,\mathrm{Re}\left\{\sum_n\left[\frac{B_n(\lambda)}{L(\omega,n)}\sin(n\pi z)\right]e^{-i\omega t}\right\}, \tag{37}$$

where

$$B_n(\lambda)=A(\lambda)\,g_{n1}(\lambda)+A(-\lambda)\,g_{n1}(-\lambda)\,.$$

The solution of the homogenous equation corresponding to Eq. 36 involves a term proportional to $\sin(\pi z)$. However, addition of such a term to the complete solution of Eq. 36 merely amounts to a renormalization of w_0 because all the terms proportional to $\sin(\pi z)$ can then be grouped to define a new w_0 with corresponding definition for $w_1, w_2, \ldots$. Hence, we can assume that w_0 is orthogonal to all other w_n's. From Eq. 27, we get

$$Lw_2=a^2R_0fw_1-a^2R_2w_0 \tag{38}$$

We do not require the solution of this equation, but merely use it to determine R_2 the first non-zero correction to R. The solvability condition requires that the steady part of the right-hand side is orthogonal to $\sin(\pi z)$. Thus,

$$R_2=2R_0\int_0^1\overline{fw_1}\sin(\pi z)\,\mathrm{d}z, \tag{39}$$

where the upper bar denotes the time average. From Eq. 26, we have

$$\overline{f w_1} \sin(\pi z) = \frac{1}{R_0 a^2} \overline{w_1 L w_1}. \tag{40}$$

Using Eqs. 36 and 37 and finding the time average we obtain $\overline{w_1 L w_1}$, which yields from Eqs. 39 and 40,

$$R_2 = \frac{R_0^2 a^2}{4} \sum \frac{|B_n(\lambda)|^2}{|L(\omega, n)|^2} \left[L(\omega, n) + L^*(\omega, n) \right], \tag{41}$$

where $L^*(\omega, n)$ is the complex conjugate of $L(\omega, n)$. The critical value of R_2, denoted by R_{2c}, is obtained at the wave number given by Eq. 31 for the following three different cases:

Case (i)	Oscillating wall temperature field is symmetric ($\varphi = 0$).
Case (ii)	Oscillating wall temperature field is asymmetric ($\varphi = \pi$).
Case (iii)	Only lower wall temperature is modulated while the upper one is held at constant temperature ($\varphi = -i\infty$).

Case (i) Oscillating wall temperature field is symmetric ($\varphi = 0$)
The oscillating temperature field is symmetric when $\varphi = 0$ and it is found that

$$\begin{aligned} |B_n(\lambda)|^2 &= \frac{16 n^2 \pi^4 \omega^2}{\left[\omega^2 + (n+1)^4 \pi^4\right]\left[\omega^2 + (n-1)^4 \pi^4\right]} \\ &= |b_n|^2 \text{ (say)}, \quad \text{if } n \text{ is even} \\ &= 0, \quad \text{if } n \text{ is odd.} \end{aligned} \tag{42}$$

Then

$$R_{2c} = \frac{R_{0c} a_c^2}{2} \sum_n |b_n|^2 \frac{A}{\left(A^2 + B^2\right)}, \tag{43}$$

where

$$\begin{aligned} A = Re\left[L(\omega, n)\right] = &\frac{\omega^2 \left(n^2\pi^2 + a_c^2\right)}{Pr} - \frac{\omega^2 Da^{-1} \Gamma_p \left(n^2\pi^2 + a_c^2\right)}{Pr} \\ &- Da^{-1}\left(n^2\pi^2 + a_c^2\right)^2 + Da^{-1}\left(\pi^2 + a_c^2\right)^2. \end{aligned} \tag{44a}$$

and

$$B = Im\left[L(\omega, n)\right] = \omega \left[Da^{-1}\left(n^2\pi^2 + a_c^2\right) + \frac{\left(n^2\pi^2 + a_c^2\right)^2}{Pr} - \frac{Da^{-1} \Gamma_p \left(n^2\pi^2 + a_c^2\right)^2}{Pr} \right]. \tag{44b}$$

The summation in Eq. 43 extends over even values of n.

Case (ii) Oscillating wall temperature field is asymmetric ($\varphi = \pi$)

This case is corresponding to out-of-phase temperature modulation with $\varphi = \pi$ and we obtain

$$\begin{aligned} |B_n(\lambda)|^2 &= |b_n|^2 \quad \text{if } n \text{ is odd} \\ &= 0 \quad \text{if } n \text{ is even} \end{aligned}$$

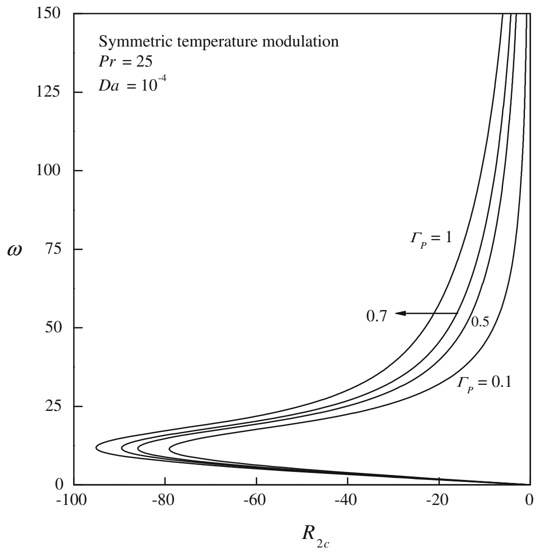


Fig. 2 Variation of R_{2c} with ω for different values of Γ_P

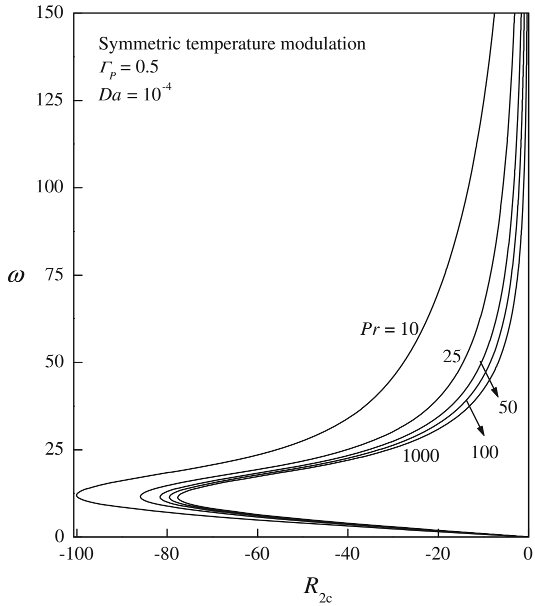


Fig. 3 Variation of R_{2c} with ω for different values of Pr

Then R_{2c} has the same expression as Eq. 43 with the summation extending over odd values of n only.

Case (iii) Only lower wall temperature is modulated while the upper one is held at constant temperature ($\varphi = -i\infty$)**.**

This is the case corresponds to $\varphi = -i\infty$ and we have

$$|B_n(\lambda)|^2 = \frac{|b_n|^2}{4}.$$

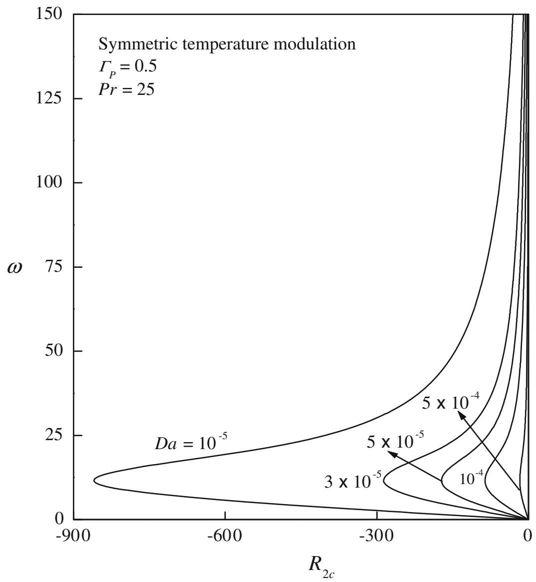


Fig. 4 Variation of R_{2c} with ω for different values of Da

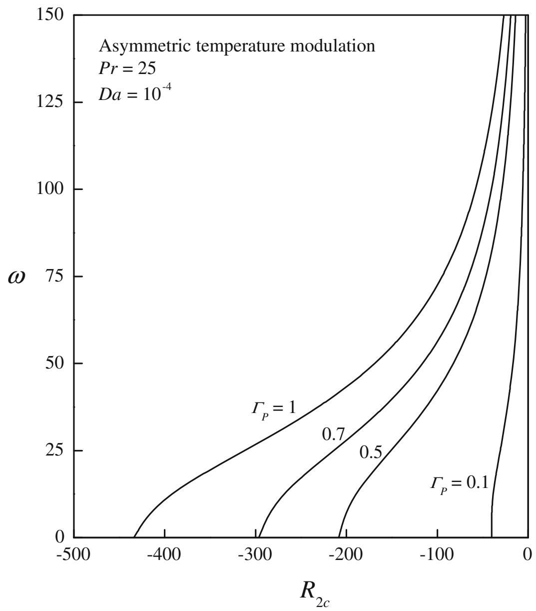


Fig. 5 Variation of R_{2c} with ω for different values of Γ_P

Again R_{2c} is given by Eq. 43 but the summation extends over all values of n.

The variation of R_{2c} with ω for different physical parameters is shown in Figs. 2, 3, 4, 5, 6, 7, 8, 9, and 10 and the results are discussed in the next section.

Fig. 6 Variation of R_{2c} with ω for different values of Pr

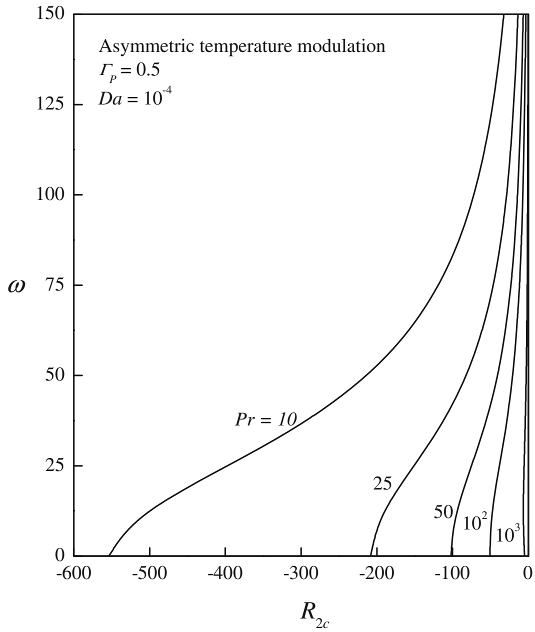
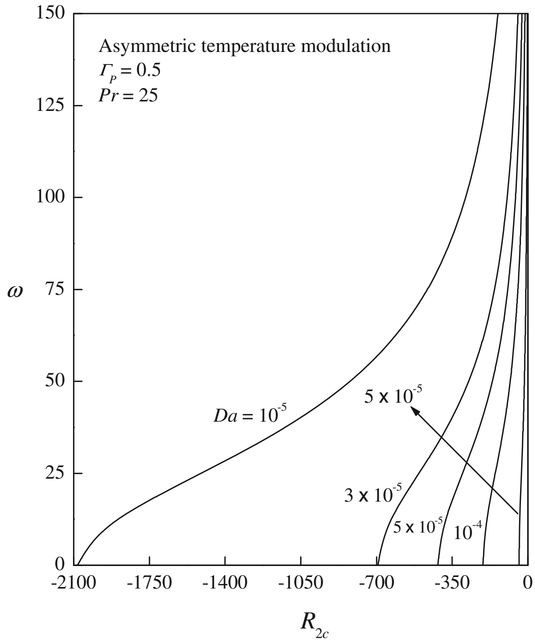


Fig. 7 Variation of R_{2c} with ω for different values of Pr



5 Results and Discussion

The effect of thermal modulation on the onset of convection in a layer of Walters B viscoelastic fluid-saturated porous medium is investigated. A perturbation technique with amplitude of the modulating temperature as a perturbation parameter is used to find the critical thermal

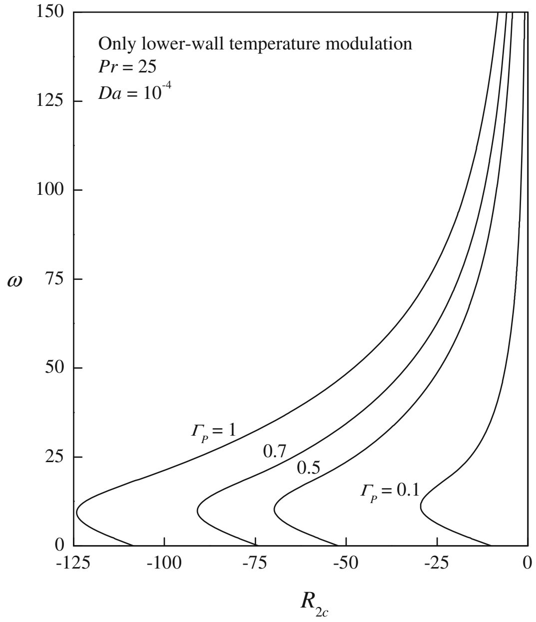


Fig. 8 Variation of R_{2c} with ω for different values of Γ_P

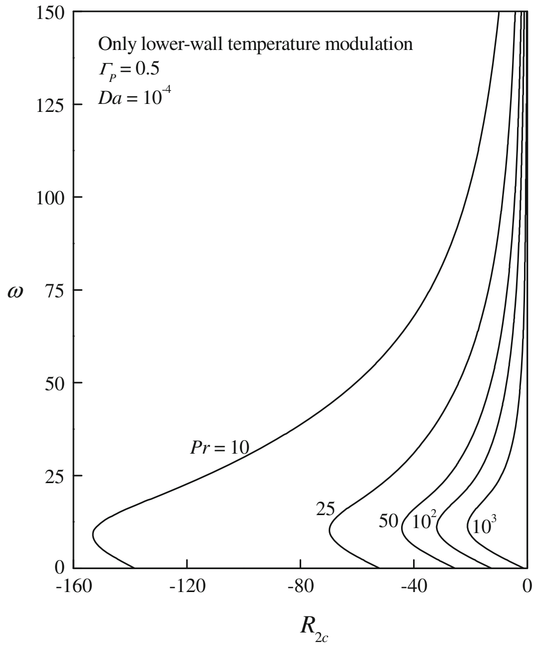


Fig. 9 Variation of R_{2c} with ω for different values of Pr

Rayleigh number as a function of frequency of the modulation, elasticity parameter, Darcy number, and Prandtl number. The stability of the system is characterized by the sign of the correction critical Rayleigh number R_{2c}. A positive and negative R_{2c}, respectively, represents a stabilizing and destabilizing effect of thermal modulation on the system as compared to the unmodulated system.

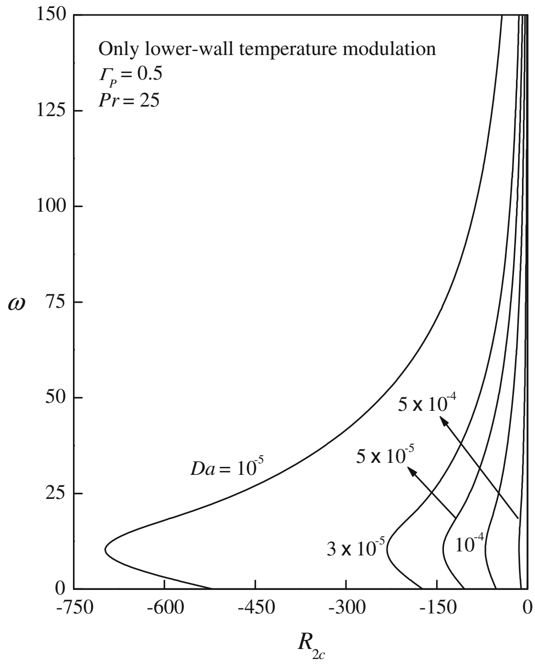


Fig. 10 Variation of R_{2c} with ω for different values of Da

The analytic expression obtained for R_{2c} is computed for various values of physical parameters for the following three cases:

(a) Oscillating wall temperature field is symmetric, i.e., the wall temperatures are modulated in phase, $\varphi = 0$,
(b) Oscillating wall temperature field is asymmetric, i.e., the wall temperatures are modulated out-of-phase modulation, $\varphi = \pi$, and
(c) Only the temperature of the bottom wall is modulated, the upper wall being held at a fixed constant temperature, $\varphi = -i\infty$.

The results obtained for the above cases are depicted in Figs. 2, 3, 4, 5, 6, 7, 8, 9, and 10 as a function of frequency of temperature modulation ω for different values of physical parameters by fixing the value of $M = 1$.

Figure 2 is a plot of the correction Rayleigh number R_{2c} versus ω for different values of elasticity parameter Γ_P when $Pr = 10$ and $Da = 10^{-5}$ for the case of symmetric modulation of the wall temperature. We observe that, in general, R_{2c} is negative over the whole range of frequencies, indicating that the symmetric temperature modulation has a destabilizing effect on the system. That is, in the presence of thermal modulation, convection sets at lower values of Rayleigh number as compared to the unmodulated system. Further, it is noted that as the elasticity parameter Γ_P increases, the magnitude of correction Rayleigh number R_{2c} increases indicating that the effect of elasticity parameter is to advance the onset of convection. Besides, the curves for different values of Γ_P are very close to zero when the modulation frequency is very small. Hence, the modulation has little effect on the stability of the system when ω approaches to zero value. As ω increases, $|R_{2c}|$ increases to its maximum value initially and then starts decreasing with further increase in ω. When ω is very large, all the curves for different Γ_P coalesce and $|R_{2c}|$ approaches to zero. This means that the modulation with large frequency will have no substantial effect on the stability characteristics of the system. This figure also indicates that the peak negative value of R_{2c} increases with an increase in the value of Γ_P.

The results obtained for the case of asymmetric modulation with $Pr = 10$ and $Da = 10^{-5}$ are presented in Fig. 3. We note that the curves of R_{2c} versus ω for different values of elasticity parameter Γ_P do not coalesce as the modulation frequency approaches to zero. Moreover, $|R_{2c}|$ increases monotonically with an increase in the value of ω without attaining any peak value for a fixed value of elasticity parameter Γ_P, and all the curves for different Γ_P coalesce at higher values of ω.

Figure 4 displays the variation of R_{2c} versus ω for different values of Γ_P with $Pr = 10$ and $Da = 10^{-5}$ for the case of only lower wall temperature modulation. Here also we observe that R_{2c} is negative over the whole range of frequencies as noticed in the case of symmetric and asymmetric modulations of the wall temperature. From this figure, it is observed that at low frequencies, $|R_{2c}|$ increases with increasing Γ_P and approaches to zero for large values of frequencies.

The effect of Prandtl number on the correction Rayleigh number R_{2c} with $Da = 10^{-5}$ and $\Gamma_P = 0.1$ for the cases of $\varphi = 0, \pi$, and $-i\infty$ is shown in Figs. 5, 6, and 7, respectively, as a function of ω. We observe that in general, the magnitude of the correction Rayleigh number decreases with increase in the value of the Prandtl number indicating that the Prandtl number has stabilizing effect in the cases of symmetric, asymmetric, and of lower wall temperature modulation.

The variation of R_{2c} as a function of ω for different values of Darcy number Da is shown in Figs. 8, 9, and 10 for symmetric temperature modulation, asymmetric wall temperature modulation, and only lower wall temperature modulation, respectively, when $Pr = 10$ and $\Gamma_P = 0.1$. From the figures, it is evident that the effect of increase in Da has qualitatively similar effect as that of Prandtl number. That is the effect of increasing Darcy number decreases the magnitude of the correction Rayleigh number indicating that the Darcy number reduces the destabilizing effect of modulation.

6 Conclusions

The effect of thermal modulation on the onset of convection in a horizontal layer of porous medium saturated with Walters B viscoelastic fluid is studied using a linear stability analysis. The analytic expression obtained for R_{2c} is computed for various values of physical parameters for the cases of (i) oscillating wall temperature field is symmetric (i.e., the wall temperatures are modulated in phase, $\varphi = 0$), (ii) oscillating wall temperature field is asymmetric (i.e., the wall temperatures are modulated out-of-phase, $\varphi = \pi$), and (iii) only lower wall temperature is modulated and the upper wall being held at a fixed constant temperature (i.e., $\varphi = -i\infty$), and the following conclusions may be drawn:

(1) The effect of all three types of modulation namely, symmetric, asymmetric, and only lower wall temperature modulations is found to be destabilizing as compared to the unmodulated system.
(2) The effect of thermal modulation disappears at large frequencies in all the cases of thermal modulation.
(3) Increase in the value of Pr and Da is to decrease $|R_{2c}|$, while increase in Γ_P is to increase the magnitude of the correction Rayleigh number in all the cases.
(4) The critical correction Rayleigh number $R_{2c} \to 0$ with an increase in ω faster for large values of Da.

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